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Conservation laws for the geodesic equations of the canonical connection on Lie groups in dimensions two and three

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Abstract: In order to characterize the systems of second-order ODEs which admit a regular Lagrangian function, Noether symmetries for the geodesic equations of the canonical linear connection on Lie groups of dimension three or less are obtained, so the characterization of these geodesic equations through their Noether's symmetries Lie Algebras is investigated. The corresponding conservation laws and the first integral for each geodesics are constructed.

Keywords: Systems of second-order ODEs, inverse problem of Lagrangian dynamics, conservations laws, geodesic equations, canonical linear connection on Lie groups.

1. Introduction

Second order ordinary differential equations (ODEs) on finite dimensional manifolds appear in a wide variety of applications in mathematics, physics and engineering. In classical mechanics they are Newtons equations of motion and the Euler-Lagrange equations of a mechanical Lagrangian. The inverse problem of Lagrangian dynamics consists of finding necessary and sufficient conditions for a system of second order ODEs to be the EulerLagrange equations of a regular Lagrangian function and in case they are, to describe all possible such Lagrangians.

The inverse problem of Lagrangian dynamics for the geodesic spray associated to the canonical symmetric linear connection on a Lie group of dimension three or less was solved in [1]. This connection was first introduced by Cartan and J.A. Schouten [2] and its properties were studied in details in [3]. In [1], it was proved that the geodesic equations of this connection are variational for all Lie groups in dimensions two and three. Moreover, an explicit Lagrangians for each group in these dimensions were given. For more details on the inverse problem and the canonical connection we refer the reader to [4,5,1,3].

Our need to characterize those systems of second-order ODEs which admit a regular Lagrangian function indicates the importance of studying these geodesics that were constructed. In this paper, the characterization of these geodesics through their Noether's symmetries Lie Algebras is investigated. The corresponding conservation laws and the first integral for each geodesics are obtained.

The outline of the paper is as follows. In Section 2, we investigate the Noether symmetries and Noether theorem. Lastly, in Sections 3 and 4, the Noether symmetries for all the geodesic equations were derived with the integrals of motion. The classification of the Lie algebras of the Noether symmetries is studied. Concluding and remarks are given in the last section.

2. Noether symmetries and Noether theorem

let us consider the k-th order system of partial differential equations (PDEs) of n independent variables $x = (x_1, x_2, ..., x_n)$ and m dependent variables $u = (u^1, u^2, ..., u^m)$

$$E^\alpha(x, u, u_{(1)}, ..., u_{(k)}) = 0,\ \alpha = 1, ..., m\ , \qquad (2.1)$$

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where $u_{(1)}, u_{(2)}, ..., u_{(k)}$ denote the collections of all first, second,..., k-order partial derivatives,
i.e., $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha)$,...respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x_i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + ..., \; i = 1, ..., n \;, \quad (2.2)$$

in which the summation convention is used.

The following definitions are well-known (see, e.g. [7–10]).

Definition 1(The conserved vector). The n-tuple vector $T = (T^1, T^2, ..., T^n), T^j \in \mathcal{A}, j = 1, ..., n,$ is a *conserved vector* of (2.1) if T^i satisfies

$$D_i T^i \mid_{(2.1)} = 0, \quad (2.3)$$

where $\mathcal{A}$ is the space of *differential functions*.

Definition 2(*The Lie-Bäcklund operator*). The Lie-Bäcklund operator is

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} \; \xi^i, \eta^\alpha \in \mathcal{A} \;, \quad (2.4)$$

where $\mathcal{A}$ is the space of *differential functions*.
The operator (2.4) is an abbreviated form of the infinite formal sum

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} \zeta_{i_1 i_2 \ldots i_s}^\alpha \frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}^\alpha}, \quad (2.5)$$

where the additional coefficients are determined uniquely by the prolongation formulae

$$\begin{array}{l} \zeta_i^\alpha = D_i(W^\alpha) + \xi^j u_{ij}^\alpha, \\ \zeta_{i_1 \ldots i_s}^\alpha = D_{i_1} ... D_{i_s}(W^\alpha) + \xi^j u_{j i_1 \ldots i_s}^\alpha, \; s > 1, \end{array} \quad (2.6)$$

in which W^α is the *Lie characteristic function*

$$W^\alpha = \eta^\alpha - \xi^j u_j^\alpha. \quad (2.7)$$

Definition 3(The Euler-Lagrange operator). The Euler-Lagrange operator for each α, is given by

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} ... D_{i_s} \frac{\partial}{\partial u_{i_1 i_2 \ldots i_s}^\alpha}, \; \alpha = 1, ..., m. \quad (2.8)$$

Definition 4(Lagrangian and Euler-Lagrange equations). If there exists a function $L = L(x, u, u_{(1)}, ..., u_{(l)}) \in A, \; l \leq k$ such that the system (2.1) can be written as $\delta L / \delta u^\alpha = 0$, then L is called a *Lagrangian* of the system (2.1) and the differential equations of the form

$$\frac{\delta L}{\delta u^\alpha} = 0, \; \alpha = 1, ..., m, \quad (2.9)$$

are called *Euler-Lagrange equations*.

Definition 5(The Action Integral). The action integral of a Lagrangian L is given by the following functional

$$J[u] = \int_\Omega L(x, u, u_{(1)}, ..., u_{(k)}) dx, \quad (2.10)$$

where L is defined on a domain Ω in the space $x = (x_1, x_2, ..., x_n)$.

Definition 6. The functional (2.10) is said to be invariant with respect to the group G_r if for all transformations of the group and all functions $u = u(x)$ the following equality is fulfilled irrespective of the choice of the domain of integration

$$\begin{aligned} &\int_\Omega L(x, u, u_{(1)}, ..., u_{(k)}) dx \\ &= \int_{\bar{\Omega}} L(\bar{x}, \bar{u}, \bar{u}_{(1)}, ..., \bar{u}_{(k)}) d\bar{x}, \end{aligned} \quad (2.11)$$

where $\bar{u}$ and $\bar{\Omega}$ are the images of u and Ω, respectively, under the group G_r.

Lemma 1.[10] The functional (2.10) is invariant with respect to the group G_r with the *Lie-Bäcklund operator* X of the form (2.5) if and only if the following equalities hold

$$W^\alpha \delta L / \delta u^\alpha + D_i(N^i L) \equiv XL + L \, D_i \xi^i = D_i B^i, \quad (2.12)$$

where

$$\begin{aligned} N^i &= \xi^i + W^\alpha \frac{\delta}{\delta u_i^\alpha} \\ &+ \sum_{s \geq 1} D_{i_1} ... D_{i_s}(W^\alpha) \frac{\delta}{\delta u_{i i_1 i_2 \ldots i_s}^\alpha}, \; i = 1, ..., n. \end{aligned} \quad (2.13)$$

Theorem 1(Noether's Theorem). [10] Let the functional (2.10) be invariant with respect to the group G_r with the *Lie-Bäcklund operator* X of the form (2.5). Then the Euler-Lagrange equations (2.9) have r linearly independent conservation laws $D_i T^i = 0$, where

$$T^i = B^i - N^i L, \; i = 1, ..., n. \quad (2.14)$$

3. The two-dimensional Lie algebra

In this section there are two Lie algebras to consider, namely, the non-abelian, $g_{2,non-abelian}$, and the abelian one. In both cases the geodesic equations of the canonical connection and their corresponding non-singular Lagrangians are given. The Lie algebras of the Noether's symmetries for both cases are classified and the integrals of motions are obtained. For simplicity we shall denote the derivatives $(\dot{x}, \dot{y}, \dot{z})$ of x, y and z w.r.t. the independent variable t by (x_1, y_1, z_1). As a final remark, since the calculations are very lengthy, we will only do the non-abelian 2-dimensional case (3.1) in details and for the rest of the cases we will list the Noether's symmetries.

3.1. $[e_1, e_2] = e_1$ *(The Lie algebra of the affine group on the real line* $g_{2,non-abelian}$ *)*

In local coordinates (x, y), the geodesic equations are given by

$$\begin{array}{c} \ddot{x} = \dot{x}\dot{y} \\ \ddot{y} = 0 \end{array} \tag{3.1}$$

System (3.1) has a Lagrangian [1]

$$L = e^{-y}\frac{{x_1}^2}{2y_1} + \frac{{y_1}^2}{2} \tag{3.2}$$

for which the Noether symmetry

$$X = \xi_1\partial_t + \eta_1\partial_x + \eta_2\partial_y \tag{3.3}$$

satisfies (2.12), viz.

$$\begin{array}{l} \eta_2{x_1}^2y_1 + \frac{\partial}{\partial t}\eta_2{x_1}^2 - 2\,x_1y_1\frac{\partial}{\partial t}\eta_1 - 2\,{x_1}^2y_1\frac{\partial}{\partial x}\eta_1 \\ -2\,x_1{y_1}^2\frac{\partial}{\partial y}\eta_1 + y_1\frac{\partial}{\partial y}\eta_2{x_1}^2 + 2\,{y_1}^3\frac{\partial}{\partial y}B_1\mathrm{e}^y \\ +2\,\frac{\partial}{\partial t}B_1{y_1}^2\mathrm{e}^y - 2\,\frac{\partial}{\partial t}\eta_2{y_1}^3\mathrm{e}^y - 2\,{y_1}^4\frac{\partial}{\partial y}\eta_2\mathrm{e}^y \\ +{y_1}^4\frac{\partial}{\partial t}\xi_1\mathrm{e}^y + {y_1}^5\frac{\partial}{\partial y}\xi_1\mathrm{e}^y + 2\,x_1\frac{\partial}{\partial x}B_1{y_1}^2\mathrm{e}^y \\ +{x_1}^3\frac{\partial}{\partial x}\eta_2 - 2\,x_1\frac{\partial}{\partial x}\eta_2{y_1}^3\mathrm{e}^y + {y_1}^4x_1\frac{\partial}{\partial x}\xi_1\mathrm{e}^y = 0. \end{array} \tag{3.4}$$

Now comparing coefficients of the derivatives of x and y, we obtain the following overdetermined linear system.

$$\begin{array}{l} \frac{\partial}{\partial t}\eta_1 = 0,\ \eta_2 - 2\,\frac{\partial}{\partial x}\eta_1 + \frac{\partial}{\partial y}\eta_2 = 0, \\ \frac{\partial}{\partial t}\eta_2 = 0,\ 2\frac{\partial}{\partial y}\eta_2 - \frac{\partial}{\partial t}\xi_1 = 0, \\ \frac{\partial}{\partial x}\eta_2 = 0,\ \frac{\partial}{\partial y}\eta_1 - \frac{\partial}{\partial x}B_1\mathrm{e}^y = 0, \\ \frac{\partial}{\partial x}\eta_2 = 0,\ \frac{\partial}{\partial t}\eta_2 - \frac{\partial}{\partial y}B_1 = 0, \\ \frac{\partial}{\partial x}\xi_1 = 0,\ \frac{\partial}{\partial t}B_1 = 0, \\ \frac{\partial}{\partial y}\xi_1 = 0. \end{array} \tag{3.5}$$

Solving this system gives rise to

$$\begin{array}{l} \xi_1 = c_1,\ \eta_1 = \frac{1}{2}\,c_3\,x + c_4\,\mathrm{e}^y + c_2, \\ \eta_2 = c_3,\ B_1 = c_4\,x + c_5. \end{array} \tag{3.6}$$

At this stage we construct the Noether symmetries corresponding to each of the constant involved. These are total of four generators given by

$$\begin{array}{l} X_1 = \frac{\partial}{\partial t},\ X_3 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}, \\ X_2 = \frac{\partial}{\partial x},\ X_4 = e^y\frac{\partial}{\partial x}. \end{array} \tag{3.7}$$

The nonzero commutators for the Lie algebra arising from the 4 Noether symmetries (3.7) are given by

$$[X_2, X_3] = X_2, \quad [X_3, X_4] = X_4. \tag{3.8}$$

One can see that under a change of basis, namely,

$$e_1 = X_2,\ e_2 = X_3,\ e_3 = X_4, \tag{3.9}$$

the Lie algebra $\mathfrak{g}$ of the Noether symmetries is decomposable as a direct sum of the form

$$\mathfrak{g} = \langle e_1, e_2, e_3\rangle \oplus \mathbb{R} \tag{3.10}$$

where $\langle e_1, e_2, e_3\rangle$ is the three-dimensional solvable Lie algebra given by the non-zero brackets $[e_1, e_2] = e_1, [e_2, e_3] = e_3$.

To find the conservation laws corresponding to the above Noether symmetries, we use the formula given by (2.14).

$$\begin{array}{l} T^1 = {y_1}^2,\ T^2 = \frac{\mathrm{e}^{-y}x_1}{y_1}, \\ T^3 = \frac{\left(x_1y_1x - {x_1}^2 + 2\,{y_1}^3\mathrm{e}^y\right)\mathrm{e}^{-y}}{2{y_1}^2},\ T^4 = \frac{y_1x - x_1}{y_1}. \end{array} \tag{3.11}$$

3.2. $[e_1, e_2] = 0$ *(Abelian Lie algebra)*

In local coordinates (x, y), the geodesic equations are given by

$$\begin{array}{c} \ddot{x} = 0 \\ \ddot{y} = 0 \end{array} \tag{3.12}$$

System (3.12) has a Lagrangian [1]

$$L = {x_1}^2 + {y_1}^2 \tag{3.13}$$

for which the Noether symmetry

$$X = \xi_1\partial_t + \eta_1\partial_x + \eta_2\partial_y \tag{3.14}$$

satisfies (2.12), can be estimated easily as.

$$\begin{array}{l} X_1 = \frac{\partial}{\partial t},\ X_3 = \frac{\partial}{\partial y},\ X_5 = t\frac{\partial}{\partial y}, \\ X_7 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \\ X_2 = \frac{\partial}{\partial x},\ X_4 = t\frac{\partial}{\partial x},\ X_6 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \\ X_8 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y}. \end{array} \tag{3.15}$$

with the gauge term $B_1\left(t, x, y\right) = c_8\left(x^2 + y^2\right) + 2c_4\,x + 2c_5\,y + c_9$.

The nonzero commutators for the Lie algebra arising from the 8 Noether symmetries (3.15) are given by

$$\begin{array}{lll} [X_1, X_4] = X_2, & [X_1, X_5] = X_3, & [X_1, X_7] = 2\,X_1, \\ [X_1, X_8] = X_7, & [X_2, X_6] = -X_3, & [X_2, X_7] = X_2, \\ [X_2, X_8] = X_4, & [X_3, X_6] = X_2, & [X_3, X_7] = X_3, \\ [X_3, X_8] = X_5, & [X_4, X_6] = -X_5, & [X_4, X_7] = -X_4, \\ [X_5, X_6] = X_4, & [X_5, X_7] = -X_5, & [X_7, X_8] = 2\,X_8. \end{array} \tag{3.16}$$

One can see that under a change of basis, namely,

$$\begin{array}{l} e_1 = X_2, e_2 = X_3, e_3 = X_4, \\ e_4 = X_5, e_5 = X_6, \\ e_6 = X_1, e_7 = -X_8, e_8 = -X_7. \end{array} \tag{3.17}$$

The Lie algebra $\mathfrak{g}$ of the Noether symmetries is a semi-direct sum of the form

$$\mathfrak{g} = \langle e_1, e_2, e_3, e_4, e_5\rangle \oplus sl(2, \mathbb{R}) \tag{3.18}$$

where $\langle e_1, e_2, e_3, e_4, e_5\rangle$ is the five-dimensional solvable Lie algebra given by the non-zero brackets $[e_1, e_5] = -e_2$, $[e_2, e_5] = e_1, [e_3, e_5] = -e_4, [e_4, e_5] = e_3$.

And the conservation laws corresponding to the above Noether symmetries are

$$
\begin{array}{l}
T^1 = {x_1}^2 + {y_1}^2,\ T^2 = x_1,\ T^3 = y_1,\\
T^4 =\\
x - x_1 t,\ T^5 = y - y_1 t,\ T^6 = x_1 y - y_1 x,\\
T^7 = \left({x_1}^2 + {y_1}^2\right) t - x_1 x - y_1 y,\\
T^8 = \left({x_1}^2 + {y_1}^2\right) t^2 - 2\left(y_1 y + x_1 x\right) t + x^2 + y^2.
\end{array}
\tag{3.19}
$$

4. The three-dimensional Lie algebra

In this section we consider the list of the three-dimensional Lie algebras described in Jacobson's classification [6]. Following the convention in [1], the brackets of the Lie algebras (4.1-4.6) are given by

$$
[e_1, e_2] = 0, [e_1, e_3] = a\, e_1 + c\, e_2, [e_2, e_3] = b\, e_1 + d\, e_2, \tag{4.1}
$$

and their corresponding geodesic equations take the form

$$
\begin{array}{l}
\ddot{x} = (a\dot{x} + b\dot{y})\dot{z}\\
\ddot{y} = (c\dot{x} + d\dot{y})\dot{z}\\
\ddot{z} = 0
\end{array}
\tag{4.2}
$$

For the algebra 4.7, it is a direct sum of the two-dimensional non-abelian algebra with the reals, and so we only add $\ddot{z} = 0$ to the system of the geodesic equation for the two-dimensional case, and for the algebras in 4.8, the connection is flat and we consider a quadratic Lagrangian. For all cases, we only list the Noether symmetries, classify their Lie algebras and obtain the integrals of motion.

4.1. $[e_1, e_3] = a\, e_1, [e_2, e_3] = d\, e_2$ *where* $(a + d) \neq 0,\ ad(a - d) \neq 0$

In local coordinates (x, y, z), the geodesic equations are given by

$$
\begin{array}{l}
\ddot{x} = a\dot{x}\dot{z}\\
\ddot{y} = d\dot{y}\dot{z}\\
\ddot{z} = 0
\end{array}
\tag{4.3}
$$

System (4.3) has a Lagrangian [1]

$$
L = \frac{\mathrm{e}^{-a\,z}{x_1}^2 + \mathrm{e}^{-d\,z}{y_1}^2}{z_1} + {z_1}^2 \tag{4.4}
$$

for which the Noether symmetry

$$
X = \xi_1 \partial_t + \eta_1 \partial_x + \eta_2 \partial_y + \eta_3 \partial_z \tag{4.5}
$$

satisfies (2.12), can be estimated easily as.

$$
\begin{array}{l}
X_1 = \frac{\partial}{\partial t},\ X_4 = ax\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z},\\
X_2 = \frac{\partial}{\partial x},\ X_5 = e^{az}\frac{\partial}{\partial x},\\
X_3 = \frac{\partial}{\partial y},\ X_6 = e^{dz}\frac{\partial}{\partial y},
\end{array}
\tag{4.6}
$$

with the gauge term $B_1(t, x, y, z) = 2\, c_5\, a\, x + 2\, c_6\, d\, y + c_7$.
The nonzero commutators for the Lie algebra arising from the 6 Noether symmetries (4.6) are given by

$$
\begin{array}{l}
[X_2, X_4] = a\, X_2, [X_3, X_4] = d\, X_3, [X_4, X_5] = a\, X_5,\\
[X_4, X_6] = d\, X_6.
\end{array}
\tag{4.7}
$$

One can see that under a change of basis, namely,

$$
\begin{array}{l}
e_1 = X_2,\ e_2 = X_3,\\
e_3 = X_4,\ e_4 = X_5,\\
e_5 = X_6,\ e_6 = X_1.
\end{array}
\tag{4.8}
$$

the Lie algebra $\mathfrak{g}$ of the Noether symmetries is decomposable as a direct sum of the form

$$
\mathfrak{g} = \langle e_1, e_2, e_3, e_4, e_5 \rangle \oplus \mathbb{R} \tag{4.9}
$$

where $\langle e_1, e_2, e_3, e_4, e_5 \rangle$ is the five-dimensional solvable Lie algebra given by the non-zero brackets $[e_1, e_3] = ae_1$, $[e_2, e_3] = de_2, [e_3, e_4] = ae_4, [e_3, e_5] = d\, e_5$.

And the conservation laws corresponding to the above Noether symmetries are

$$
\begin{array}{l}
T^1 = {z_1}^2,\ T^2 = \frac{\mathrm{e}^{-a\,z}x_1}{z_1},\ T^3 = \frac{\mathrm{e}^{-d\,z}y_1}{z_1},\\
T^4 = \frac{\mathrm{e}^{-z(a+d)}\left(2\,{z_1}^3\mathrm{e}^{z(a+d)} + y_1(yz_1 d - y_1)\mathrm{e}^{a\,z} + x_1\mathrm{e}^{d\,z}(xz_1a - x_1)\right)}{{z_1}^2},\\
T^5 = \frac{xz_1a - x_1}{z_1 a},\ T^6 = \frac{yz_1d - y_1}{d\,z_1}.
\end{array}
\tag{4.10}
$$

4.2. $[e_1, e_3] = a\, e_1 - b\, e_2, [e_2, e_3] = b\, e_1 + a\, e_2$ *where* $a \neq 0, b \neq 0,\ a^2 + b^2 = 1$

In local coordinates (x, y, z), the geodesic equations are given by

$$
\begin{array}{l}
\ddot{x} = (a\dot{x} + b\dot{y})\dot{z}\\
\ddot{y} = (-b\dot{x} + a\dot{y})\dot{z}\\
\ddot{z} = 0
\end{array}
\tag{4.11}
$$

System (4.11) has a Lagrangian [1]

$$
L = \frac{\mathrm{e}^{-a\,z}\left(\left({y_1}^2 - {x_1}^2\right)\cos(bz) + 2\,x_1 y_1 \sin(bz)\right)}{2z_1} + {z_1}^3 \tag{4.12}
$$

for which the Noether symmetry

$$
X = \xi_1 \partial_t + \eta_1 \partial_x + \eta_2 \partial_y + \eta_3 \partial_z \tag{4.13}
$$

satisfies (2.12), can be estimated easily as.

$$
\begin{array}{l}
X_1 = \frac{\partial}{\partial t},\\
X_2 = \frac{\partial}{\partial x},\\
X_3 = \frac{\partial}{\partial y},\\
X_4 = (a\,x + b\,y)\frac{\partial}{\partial x} + (a\,y - b\,x)\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z},\\
X_5 = (b\cos(b\,z) - a\,\sin(b\,z))\,\mathrm{e}^{a\,z}\frac{\partial}{\partial x}\\
\quad - (a\,\cos(b\,z) + b\sin(b\,z))\,\mathrm{e}^{a\,z}\frac{\partial}{\partial y},\\
X_6 = (a\,\cos(b\,z) + b\sin(b\,z))\,\mathrm{e}^{a\,z}\frac{\partial}{\partial x}\\
\quad + (b\cos(b\,z) - a\,\sin(b\,z))\,\mathrm{e}^{a\,z}\frac{\partial}{\partial y}.
\end{array}
\tag{4.14}
$$

with the gauge term $B_1(t,x,y,z) = c_6\,x + c_5\,y + c_7$.
The nonzero commutators for the Lie algebra arising from the 6 Noether symmetries (4.14) are given by

$$\begin{array}{ll}[X_2, X_4] = a\,X_2 - b\,X_3, & [X_3, X_4] = b\,X_2 + a\,X_3,\\ [X_4, X_5] = a\,X_5 - b\,X_6, & [X_4, X_6] = b\,X_5 + a\,X_6.\end{array} \tag{4.15}$$

One can see that under a change of basis, namely,

$$e_1 = X_2,\, e_2 = X_3,\, e_3 = X_4,\, e_4 = X_5,\, e_5 = X_6,\, e_6 = X_1, \tag{4.16}$$

the Lie algebra $\mathfrak{g}$ of the Noether symmetries is decomposable as a direct sum of the form

$$\mathfrak{g} = \langle e_1, e_2, e_3, e_4, e_5\rangle \oplus \mathbb{R} \tag{4.17}$$

where $\langle e_1, e_2, e_3, e_4, e_5\rangle$ is the five-dimensional solvable Lie algebra given by the non-zero brackets
$[e_1, e_3] = ae_1 - be_2, [e_2, e_3] = be_1 + ae_2,$
$[e_3, e_4] = ae_4 - be_5, [e_3, e_5] = be_4 + ae_5.$

And the conservation laws corresponding to the above Noether symmetries are

$$\begin{array}{l}T^1 = {z_1}^3,\\ T^2 = \frac{(x_1\cos(b\,z) - y_1\sin(b\,z))\mathrm{e}^{-a\,z}}{z_1},\\ T^3 = \frac{(y_1\cos(b\,z) + x_1\sin(b\,z))\mathrm{e}^{-a\,z}}{z_1},\\ T^4 = \left((-y_1y + xx_1)\,az_1 - {x_1}^2 + {y_1}^2\right)\cos(b\,z)\,\mathrm{e}^{-a\,z}{z_1}^{-2}\\ +\left((-x_1y - y_1x)\,az_1 + 2\,x_1y_1\right)\sin(b\,z)\,\mathrm{e}^{-a\,z}{z_1}^{-2}\\ +(x_1y + y_1x)\,b\cos(b\,z)\,\mathrm{e}^{-a\,z}{z_1}^{-1} - 6\,{z_1}^2\\ +(-y_1y + xx_1)\,b\sin(b\,z)\,\mathrm{e}^{-a\,z}{z_1}^{-1},\\ T^5 = \frac{y\,z_1 - b\,x_1 - a\,y_1}{z_1}\\ T^6 = \frac{x\,z_1 + b\,y_1 - a\,x_1}{z_1}.\end{array} \tag{4.18}$$

4.3. $[e_1, e_3] = e_1, [e_2, e_3] = e_1 + e_2$

In local coordinates (x, y, z), the geodesic equations are given by

$$\begin{array}{l}\ddot{x} = (\dot{x} + \dot{y})\dot{z}\\ \ddot{y} = \dot{y}\dot{z}\\ \ddot{z} = 0\end{array} \tag{4.19}$$

System (4.19) has a Lagrangian [1]

$$L = x_1(\ln y_1 - \ln z_1 - z) + \frac{\mathrm{e}^{-z}{y_1}^2}{z_1} + yz_1 + {z_1}^2 \tag{4.20}$$

for which the Noether symmetry

$$X = \xi_1\partial_t + \eta_1\partial_x + \eta_2\partial_y + \eta_3\partial_z \tag{4.21}$$

satisfies (2.12), can be estimated easily as.

$$\begin{array}{l}X_1 = \frac{\partial}{\partial t},\\ X_2 = \frac{\partial}{\partial x},\\ X_3 = \frac{\partial}{\partial y},\end{array} \tag{4.22}$$

with the gauge term $B_1(t,x,y,z) = c_3\,z + c_4$.
The commutators for the Lie algebra arising from the 3 Noether symmetries (4.22) are all zeros, so the Lie algebra $\mathfrak{g}$ of the Noether symmetries is the three-dimensional abelian Lie algebra.

And the conservation laws corresponding to the above Noether symmetries are

$$T^1 = {z_1}^2,\, T^2 = \ln y_1 - \ln z_1 - z,\, T^3 = z - \frac{x_1}{y_1} - 2\,\frac{y_1}{z_1}\mathrm{e}^{-z}. \tag{4.23}$$

4.4. $[e_1, e_3] = -e_2,\ [e_2, e_3] = e_1$ *(The Euclidean Group on the plane $E(2)$)*

In local coordinates (x, y, z), the geodesic equations are given by

$$\begin{array}{l}\ddot{x} = \dot{y}\dot{z}\\ \ddot{y} = -\dot{x}\dot{z}\\ \ddot{z} = 0\end{array} \tag{4.24}$$

System (4.24) has a Lagrangian [1]

$$L = xy_1 - yx_1 + \frac{{y_1}^2 + {x_1}^2}{z_1} + {z_1}^2 \tag{4.25}$$

for which the Noether symmetry

$$X = \xi_1\partial_t + \eta_1\partial_x + \eta_2\partial_y + \eta_3\partial_z \tag{4.26}$$

satisfies (2.12), can be estimated easily as.

$$\begin{array}{l}X_1 = \frac{\partial}{\partial t},\ X_5 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\\ X_2 = \frac{\partial}{\partial x},\ X_6 = \sin z\frac{\partial}{\partial x} + \cos z\frac{\partial}{\partial y},\\ X_3 = \frac{\partial}{\partial y},\ X_7 = \cos z\frac{\partial}{\partial x} - \sin z\frac{\partial}{\partial y},\\ X_4 = \frac{\partial}{\partial z},\end{array} \tag{4.27}$$

with the gauge term $B_1(t,x,y,z) = (x\,c_6 - y\,c_7)\cos(z) + (-x\,c_7 - y\,c_6)\sin(z) - x\,c_3 + c_2\,y + c_8$.
The nonzero commutators for the Lie algebra arising from the 7 Noether symmetries (4.27) are given by

$$\begin{array}{lll}[X_2, X_5] = -X_3, & [X_3, X_5] = X_2, & [X_4, X_6] = X_7,\\ [X_4, X_7] = -X_6, & [X_5, X_6] = -X_7, & [X_5, X_7] = X_6.\end{array} \tag{4.28}$$

One can see that under a change of basis, namely,

$$\begin{array}{l}e_1 = X_2,\ e_2 = X_3,\ e_3 = X_4 + X_5,\ e_4 = X_4,\ e_5 = X_6,\\ e_6 = X_7, e_7 = X_1.\end{array} \tag{4.29}$$

The Lie algebra of the Noether symmetries is decomposable as a direct sum of two copies of the Euclidean algebra and the reals $(E(2) \oplus E(2) \oplus \mathbb{R})$.

And the conservation laws corresponding to the above Noether symmetries are

$$\begin{array}{l}T^1 = {z_1}^2,\ T^2 = y - \frac{x_1}{z_1},\ T^3 = x + \frac{y_1}{z_1},\\ T^4 = \frac{{y_1}^2}{{z_1}^2} + \frac{{x_1}^2}{{z_1}^2} - 2\,z_1,\ T^5 = y^2 + x^2 + 2\,\frac{xy_1}{z_1} - 2\,\frac{yx_1}{z_1},\\ T^6 = \frac{\sin(z)x_1 + \cos(z)y_1}{z_1},\ T^7 = \frac{\cos(z)x_1 - \sin(z)y_1}{z_1}.\end{array} \tag{4.30}$$

4.5. $[e_1, e_3] = e_1, [e_2, e_3] = e_2$

In local coordinates (x, y, z), the geodesic equations are given by

$$\begin{aligned} \ddot{x} &= \dot{x}\dot{z} \\ \ddot{y} &= \dot{y}\dot{z} \\ \ddot{z} &= 0 \end{aligned} \tag{4.31}$$

System (4.31) has a Lagrangian [1]

$$L = \frac{2\mathrm{e}^{-z} y_1 x_1}{z_1} + {z_1}^2 \tag{4.32}$$

for which the Noether symmetry

$$X = \xi_1 \partial_t + \eta_1 \partial_x + \eta_2 \partial_y + \eta_3 \partial_z \tag{4.33}$$

satisfies (2.12), can be estimated easily as.

$$\begin{aligned} &X_1 = \tfrac{\partial}{\partial t},\ X_4 = y\tfrac{\partial}{\partial y} + \tfrac{\partial}{\partial z},\quad X_6 = e^z\tfrac{\partial}{\partial y}, \\ &X_2 = \tfrac{\partial}{\partial x},\ X_5 = x\tfrac{\partial}{\partial x} - y\tfrac{\partial}{\partial y},\ X_7 = e^z\tfrac{\partial}{\partial x}, \\ &X_3 = \tfrac{\partial}{\partial y}, \end{aligned} \tag{4.34}$$

with the gauge term $B_1(t, x, y, z) = 2c_6x + 2c_7y + c_8$.
The nonzero commutators for the Lie algebra arising from the 7 Noether symmetries (4.34) are given by

$$\begin{aligned} &[X_2, X_5] = X_2, \quad [X_3, X_4] = X_3, \quad [X_3, X_5] = -X_3, \\ &[X_4, X_7] = X_7, \quad [X_5, X_6] = X_6, \quad [X_5, X_7] = -X_7. \end{aligned} \tag{4.35}$$

One can see that under a change of basis, namely,

$$\begin{aligned} &e_1 = X_2,\ e_2 = X_4 + X_5,\ e_3 = X_6,\ e_4 = X_3,\ e_5 = X_4, \\ &e_6 = X_7,\ e_7 = X_1, \end{aligned} \tag{4.36}$$

the Lie algebra $\mathfrak{g}$ of the Noether symmetries is decomposable as a direct sum of the form

$$\mathfrak{g} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4, e_5, e_6 \rangle \oplus \mathbb{R} \tag{4.37}$$

where $\langle e_4, e_5, e_6 \rangle$ is a copy of $\langle e_1, e_2, e_3 \rangle$ given by the non-zero brackets $[e_1, e_3] = e_1, [e_2, e_3] = -e_2$.

And the conservation laws corresponding to the above Noether symmetries are

$$\begin{aligned} &T^1 = {z_1}^2,\ T^2 = \tfrac{y_1}{z_1}\mathrm{e}^{-z},\ T^3 = \tfrac{x_1}{z_1}\mathrm{e}^{-z}, \\ &T^4 = z_1 + \tfrac{x_1(z_1y - y_1)}{{z_1}^2}\mathrm{e}^{-z},\ T^5 = \tfrac{y_1x - x_1y}{z_1}\mathrm{e}^{-z}, \\ &T^6 = \tfrac{z_1x - x_1}{z_1},\ T^7 = \tfrac{z_1y - y_1}{z_1}. \end{aligned} \tag{4.38}$$

4.6. $[e_1, e_3] = e_1, [e_2, e_3] = -e_2$

In local coordinates (x, y, z), the geodesic equations are given by

$$\begin{aligned} \ddot{x} &= \dot{x}\dot{z} \\ \ddot{y} &= -\dot{y}\dot{z} \\ \ddot{z} &= 0 \end{aligned} \tag{4.39}$$

System (4.39) has a Lagrangian [1]

$$L = \frac{\mathrm{e}^{-z}{x_1}^2 + \mathrm{e}^{z}{y_1}^2}{z_1} + {z_1}^2 \tag{4.40}$$

for which the Noether symmetry

$$X = \xi_1 \partial_t + \eta_1 \partial_x + \eta_2 \partial_y + \eta_3 \partial_z \tag{4.41}$$

satisfies (2.12), can be estimated easily as.

$$\begin{aligned} &X_1 = \tfrac{\partial}{\partial t},\ X_4 = \mathrm{e}^z\tfrac{\partial}{\partial x},\quad X_6 = y\mathrm{e}^z\tfrac{\partial}{\partial x} - x\mathrm{e}^{-z}\tfrac{\partial}{\partial y}, \\ &X_2 = \tfrac{\partial}{\partial x},\ X_5 = \mathrm{e}^{-z}\tfrac{\partial}{\partial y},\ X_7 = x\tfrac{\partial}{\partial x} - y\tfrac{\partial}{\partial y} + 2\tfrac{\partial}{\partial z}, \\ &X_3 = \tfrac{\partial}{\partial y}, \end{aligned} \tag{4.42}$$

with the gauge term $B_1(t, x, y, z) = (2c_6\, y + 2c_4)\, x - 2c_5\, y + c_8$.
The nonzero commutators for the Lie algebra arising from the 7 Noether symmetries (4.42) are given by

$$\begin{aligned} &[X_2, X_6] = -X_5, \quad [X_2, X_7] = X_2, \quad [X_3, X_6] = X_4, \\ &[X_3, X_7] = -X_3, \quad [X_4, X_6] = -X_3, \quad [X_4, X_7] = -X_4, \\ &[X_5, X_6] = X_2, \quad [X_5, X_7] = X_5. \end{aligned} \tag{4.43}$$

One can see that under a change of basis, namely,

$$\begin{aligned} &e_1 = X_2 - i\,X_5,\ e_2 = X_3 - i\,X_4,\ e_3 = \tfrac{(X_7 + i\,X_6)}{2}, \\ &e_4 = X_2 + i\,X_5,\ e_5 = X_3 + i\,X_4,\ e_6 = \tfrac{(X_7 - i\,X_6)}{2}, \\ &e_7 = X_1, \end{aligned} \tag{4.44}$$

the Lie algebra $\mathfrak{g}$ of the Noether symmetries is decomposable over the complex numbers as a direct sum of the form

$$\mathfrak{g} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4, e_5, e_6 \rangle \oplus \mathbb{R} \tag{4.45}$$

where $\langle e_4, e_5, e_6 \rangle$ is a copy of $\langle e_1, e_2, e_3 \rangle$ given by the non-zero brackets $[e_1, e_3] = e_1, [e_2, e_3] = -e_2$.

And the conservation laws corresponding to the above Noether symmetries are

$$\begin{aligned} &T^1 = {z_1}^2,\ T^2 = \tfrac{x_1}{z_1}\mathrm{e}^{-z},\ T^3 = \tfrac{y_1}{z_1}\mathrm{e}^{z}, \\ &T^4 = x - \tfrac{x_1}{z_1},\ T^5 = y + \tfrac{y_1}{z_1},\ T^6 = xy - \tfrac{x_1y}{z_1} + \tfrac{y_1x}{z_1}, \\ &T^7 = \tfrac{y_1(z_1y + y_1)}{{z_1}^2}\mathrm{e}^{z} + \tfrac{x_1(x_1 - z_1x)}{{z_1}^2}\mathrm{e}^{-z} - 2z_1. \end{aligned} \tag{4.46}$$

4.7. $[e_1, e_2] = e_1$ ($g_{2,non-abelian} \oplus \mathbb{R}$)

In local coordinates (x, y, z), the geodesic equations are given by

$$\begin{aligned} \ddot{x} &= \dot{x}\dot{y} \\ \ddot{y} &= 0 \\ \ddot{z} &= 0 \end{aligned} \tag{4.47}$$

System (4.47) has a Lagrangian [1]

$$L = e^{-y}\frac{{x_1}^2}{2y_1} + \frac{{y_1}^2}{2} + \frac{{z_1}^2}{2} \tag{4.48}$$

for which the Noether symmetry

$$X = \xi_1 \partial_t + \eta_1 \partial_x + \eta_2 \partial_y + \eta_3 \partial_z \tag{4.49}$$

satisfies (2.12), can be estimated easily as.

$$\begin{array}{l} X_1 = \frac{\partial}{\partial t},\ X_4 = e^y \frac{\partial}{\partial x}, \\ X_2 = \frac{\partial}{\partial x},\ X_5 = t\frac{\partial}{\partial z}, \\ X_3 = \frac{\partial}{\partial z},\ X_6 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial y}, \end{array} \quad (4.50)$$

with the gauge term $B_1(t,x,y) = c_4\, x + c_5\, z + c_7$.
The nonzero commutators for the Lie algebra arising from the 6 Noether symmetries (4.50) are given by

$$[X_1, X_5] = X_3, \quad [X_2, X_6] = X_2, \quad [X_4, X_6] = -X_4. \quad (4.51)$$

One can see that under a change of basis, namely,

$$e_1 = X_2,\ e_2 = X_6,\ e_3 = X_4,\ e_4 = X_1,\ e_5 = X_3,\ e_6 = X_5, \quad (4.52)$$

the Lie algebra $\mathfrak{g}$ of the Noether symmetries is decomposable as a direct sum of the form

$$\mathfrak{g} = \langle e_1, e_2, e_3\rangle \oplus H_3 \quad (4.53)$$

where $\langle e_1, e_2, e_3\rangle$ is the three-dimensional solvable Lie algebra given by the non-zero brackets
$[e_1, e_2] = e_1, [e_2, e_3] = e_3$ and H_3 is the three dimensional Heisenberg Lie algebra.

And the conservation laws corresponding to the above Noether symmetries are

$$\begin{array}{l} T^1 = {y_1}^2 + {z_1}^2,\ T^2 = \frac{x_1}{y_1}\mathrm{e}^{-y},\ T^3 = z_1, \\ T^4 = x - \frac{x_1}{y_1},\ T^5 = z - z_1 t,\ T^6 = \frac{x_1(xy_1 - x_1)}{{y_1}^2}\mathrm{e}^{-y} + 2\,y_1. \end{array} \quad (4.54)$$

4.8. $[e_1, e_2] = 0, [e_1, e_3] = 0, [e_2, e_3] = 0$ *(Abelian Lie algebra)* *OR* $[e_1, e_2] = e_3$ *(Heisenberg Lie algebra)* *OR* $[e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_2, e_3] = e_1$ *(so(3))* *OR* $[e_1, e_2] = e_3, [e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2$ *(sl(2,$\mathbb{R}$))*

In local coordinates (x, y, z), the geodesic equations are given by

$$\begin{array}{l} \ddot{x} = 0 \\ \ddot{y} = 0 \\ \ddot{z} = 0 \end{array} \quad (4.55)$$

System (4.55) has a Lagrangian [1]

$$L = {x_1}^2 + {y_1}^2 + {z_1}^2 \quad (4.56)$$

for which the Noether symmetry

$$X = \xi_1\partial_t + \eta_1\partial_x + \eta_2\partial_y + \eta_3\partial_z \quad (4.57)$$

satisfies (2.12), can be estimated easily as.

$$\begin{array}{l} X_1 = \frac{\partial}{\partial t},\ X_6 = t\frac{\partial}{\partial y}, \\ X_2 = \frac{\partial}{\partial x},\ X_7 = t\frac{\partial}{\partial z}, \\ X_3 = \frac{\partial}{\partial y},\ X_8 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \\ X_4 = \frac{\partial}{\partial z},\ \ X_9 = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, \\ X_5 = t\frac{\partial}{\partial x},\ X_{10} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}, \\ X_{11} = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \\ X_{12} = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y} + tz\frac{\partial}{\partial z}, \end{array} \quad (4.58)$$

with the gauge term $B_1(t,x,y) = c_{12}\left(x^2 + y^2 + z^2\right) + 2c_5\, x + 2c_6\, y + 2c_7\, z + c_{13}$.
The nonzero commutators for the Lie algebra arising from the 12 Noether symmetries (4.58) are given by

$$\begin{array}{lll} [X_1, X_5] = X_2, & [X_1, X_6] = X_3, & [X_1, X_7] = X_4, \\ [X_1, X_{11}] = 2\,X_1, & [X_1, X_{12}] = X_{11}, & [X_2, X_8] = -X_3, \\ [X_2, X_9] - X_4, & [X_2, X_{11}] = X_2, & [X_2, X_{12}] = X_5, \\ [X_3, X_8] = X_2, & [X_3, X_{10}] = -X_4, & [X_3, X_{11}] = X_3, \\ [X_3, X_{12}] = X_6, & [X_4, X_9] = X_2, & [X_4, X_{10}] = X_3, \\ [X_4, X_{11}] = X_4, & [X_4, X_{12}] = X_7, & [X_5, X_8] = -X_6, \\ [X_5, X_9] = -X_7, & [X_5, X_{11}] = -X_5, & [X_6, X_8] = X_5, \\ [X_6, X_{10}] = -X_7, & [X_6, X_{11}] = -X_6, & [X_7, X_9] = X_5, \\ [X_7, X_{10}] = X_6, & [X_7, X_{11}] = -X_7, & [X_8, X_9] = X_{10}, \\ [X_8, X_{10}] = -X_9, & [X_9, X_{10}] = X_8, & [X_{11}, X_{12}] = 2\,X_{12}. \end{array} \quad (4.59)$$

One can see that under a change of basis, namely,

$$\begin{array}{l} e_1 = X_2,\ e_2 = X_3,\ e_3 = X_4,\ e_4 = X_5,\ e_5 = X_6,\ e_6 = X_7, \\ e_7 = X_1,\ e_8 = X_{11},\ e_9 = X_{12},\ e_{10} = X_8,\ e_{11} = X_9,\ e_{12} = X_{10}, \end{array} \quad (4.60)$$

the Lie algebra $\mathfrak{g}$ of the Noether symmetries is a semi-direct sum of the six-dimensional abelian solvable Lie algebra $\langle e_1, e_2, e_3, e_4, e_5, e_6\rangle$ and a six-dimensional semi-simple Lie algebra $\langle e_7, e_8, e_9, e_{10}, e_{11}, e_{12}\rangle$, where the latter is a direct sum of $sl(2,\mathbb{R})$ and $so(3)$.

Finally, the conservation laws corresponding to the above Noether symmetries are

$$\begin{array}{l} T^1 = {x_1}^2 + {y_1}^2 + {z_1}^2,\ T^2 = x_1,\ T^3 = y_1,\ T^4 = z_1, \\ T^5 = x - x_1 t,\ T^6 = y - y_1 t,\ T^7 = z - z_1 t,\ T^8 = x_1 y - y_1 x, \\ T^9 = x_1 z - z_1 x,\ T^{10} = y_1 z - z_1 y, \\ T^{11} = \left({x_1}^2 + {y_1}^2 + {z_1}^2\right)t - (x_1 x + y_1 y + z_1 z), \\ T^{12} = \left({x_1}^2 + {y_1}^2 + {z_1}^2\right)t^2 - 2\,(x_1 x + y_1 y + z_1 z)\,t + x^2 + y^2 + z^2. \end{array} \quad (4.61)$$

5. Conclusion

The geodesic equations of the canonical connection on Lie groups in dimensions two and three admit the invariance of a variational principle under time translation $\frac{\partial}{\partial t}$ which gives rise to the conservation of energy and invariance under translations in the x-direction $\frac{\partial}{\partial x}$ which implies conservation of linear momentum. Additional conservation laws for each case are given. A summary of the Noether's symmetries Lie Algebras for the geodesic equations of the canonical connection on Lie groups in dimensions two and three is given below:

I. In dimensions two:

1. Direct sum of three-dimensional solvable Lie algebra and $\mathbb{R}$.

2. Semi-direct sum of five-dimensional solvable Lie algebra and $sl(2,\mathbb{R})$.

II. In dimensions three:

1. The three-dimensional abelian Lie algebra.
2. Direct sum of five-dimensional solvable Lie algebra and $\mathbb{R}$.
3. Direct sum of three-dimensional solvable Lie algebra and the three dimensional Heisenberg Lie algebra.
4. Direct sum of two copies of the Euclidean algebra and the reals $\mathfrak{g} = E(2) \oplus E(2) \oplus \mathbb{R}$.
5. Direct sum of the form $\mathfrak{g} = \langle e_1, e_2, e_3 \rangle \oplus \langle e_4, e_5, e_6 \rangle \oplus \mathbb{R}$ where $\langle e_4, e_5, e_6 \rangle$ is a copy of $\langle e_1, e_2, e_3 \rangle$ given by the non-zero brackets $[e_1, e_3] = e_1, [e_2, e_3] = -e_2$.
6. Semi-direct sum of the six-dimensional abelian solvable Lie algebra and a six-dimensional semi-simple Lie algebra, where the latter is a direct sum of $sl(2,\mathbb{R})$ and $so(3)$.

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