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Inclusion and Subordination Properties for Classes of Multivalent Functions Involving Differ-Integral Operator

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Abstract: In this paper, using the linear operator $\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)$ defined by a convolution product [2] we introduced and studied a general class of multivalent functions in the open unit disc introduced by using the concept of the subordination. The main results we obtained deals with inclusion properties between these classes, and with some general subordination properties connected with the mentioned operator. All the results are sharp, the best possible, and are followed by special cases connected with the new defined classes, and other applications in the theory of multivalent and univalent functions.

Keywords: Analytic and univalent functions, Hadamard (convolution) product, differential subordination, Gaussian hypergeometric function, starlike and convex functions

1 Introduction

Let $\mathscr{A}_p$ be the class of analytic multivalent functions in the open unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ with the power series expansion of the form

$$f(z) = z^p + \sum_{k=1+p}^{\infty} a_k z^k,\ z \in \mathbb{D},\ p \in \mathbb{N}, \tag{1}$$

and we set by $\mathscr{A} := \mathscr{A}_1$ the class of all analytic functions in $\mathbb{D}$ normalized with the usual conditions $f(0) = f'(0) - 1 = 0$.

If $f, g \in \mathscr{A}_p$, where f is given by (1) and g is defined by

$$g(z) = z^p + \sum_{k=1+p}^{\infty} b_k z^k,\ z \in \mathbb{D},$$

Hadamard (or convolution) product of the functions f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=1+p}^{\infty} a_k b_k z^k,\ z \in \mathbb{D}.$$

Definition 1.[1] *For two functions f and g analytic in $\mathbb{D}$, we say that the function f is* subordinate *to g, written $f(z) \prec g(z)$, if there exists a function ω, which is analytic in $\mathbb{D}$, satisfying the following conditions $\omega(0) = 0$, $|\omega(z)| < 1$ for all $z \in \mathbb{D}$, such that $f(z) = g(\omega(z))$, $z \in \mathbb{D}$.*

In particular, if the function g is univalent in $\mathbb{D}$, we have the following equivalence:

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{D}) \subset g(\mathbb{D}).$$

Let $\mu > 0$, $a, c \in \mathbb{C}$ such that $\operatorname{Re}(c-a) \geq 0$ and $\operatorname{Re} a \geq -\mu p$, $p \in \mathbb{N}$, $m \in \mathbb{Z}$, $\lambda \geq 0$, and $l > -p$. El-Ashwah and Drbuk [2] introduced the linear operator $\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu) : \mathscr{A}_p \to \mathscr{A}_p$ defined by

$$\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu) f(z) := z^p + \frac{\Gamma(c+\mu p)}{\Gamma(a+\mu p)} \cdot \sum_{k=1+p}^{\infty} \left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m \frac{\Gamma(a+\mu k)}{\Gamma(c+\mu k)} a_k z^k,\ z \in \mathbb{D}, \tag{2}$$

where $f \in \mathscr{A}_p$ is given by (1).

Remark. Note that the operator $\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)$ generalizes several previously studied familiar operators, and we will mention some of the interesting particular cases as follows:

(i) For $m = 0$, $a = \beta$, $c = \beta + 1$ and $\mu = 1$ we obtain the operator J_p^{β} $(\beta > -p)$ studied by Saitoh et al. [3];

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(ii)For $m=0$, $a=\beta$, $c=\alpha+\beta-\gamma+1$ and $\mu=1$ we obtain the operator $\mathfrak{R}_{\beta,p}^{\alpha,\gamma}$ $(\gamma>0,\ \alpha\geq\gamma-1,\ \beta>-p)$ studied by Aouf et al. [4];

(iii)For $m=0$, $a=\beta$, $c=\alpha+\beta$ and $\mu=1$ we obtain the operator $\mathcal{Q}_{\beta,p}^{\alpha}$ $(\alpha\geq 0,\ \beta>-p)$ studied by Liu and Owa [5];

(iv)For $p=1$ and $m=0$ we obtain the operator $\hat{I}(a,c;\mu)$ studied by Raina and Sharma [6];

(v)For $m=0$, $p=1$, $a=\beta$, $c=\alpha+\beta$ and $\mu=1$ we obtain the operator $\mathcal{Q}_{\beta}^{\alpha}$ $(\alpha>0,\ \beta>-1)$ studied by Jung et al. [7];

(vi)For $m=0$, $p=1$, $a=\alpha-1$, $c=\beta-1$ and $\mu=1$ we obtain the operator $L(\alpha,\beta)$ $(\alpha,\ \beta\in\mathbb{C}\backslash\mathbb{Z}_0^-,\ \mathbb{Z}_0^-:=\{0,-1,-2,\dots\})$ studied by Carlson and Shaffer [8];

(vii)For $m=0$, $p=1$, $a=\nu-1$, $c=\nu$ and $\mu=1$ we obtain the operator $I_{\nu,\nu}$ $(\nu>0,\ \nu>-1)$ studied by Choi et al. [9];

(viii)For $m=0$, $p=1$, $a=\alpha$, $c=0$ and $\mu=1$ we obtain the operator D^{α} $(\alpha>-1)$ studied by Ruscheweyh [10];

(ix)For $m=0$, $p=1$, $a=1$, $c=n$ and $\mu=1$ we obtain the operator I_n $(n\in\mathbb{N})$ studied by Noor [11];

(x)For $m=0$, $p=1$, $a=\beta$, $c=\beta+1$ and $\mu=1$ we obtain the integral operator J_{β} $(\beta\in\mathbb{N})$ studied by Bernardi [12];

(xi)For $m=0$, $p=1$, $a=1$, $c=2$ and $\mu=1$ we obtain the integral operator J studied by Libera [13] and Livingston [14];

(xii)For $a=c$ we obtain the operator $\mathscr{I}_p^m(\lambda,l)$ studied by Cătaş [15] (see also [16]);

(xiii)For $a=c$ and $\lambda=1$ we obtain the operator $I_p(m,l)$, studied by Kumar et al. [17];

(xiv)For $a=c$, $\lambda=1$ and $l=0$ we obtain the operator D_p^m studied by Kamali and Orhan [18];

(xv)For $a=c$ and $l=1$ we obtain the operator $D_{\lambda,p}^m$ studied by Aouf et al. [19];

(xvi)For $a=c$ and $m=-n$, $n\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$, we obtain the operator $J_p^n(\lambda,l)$ studied by El-Ashwah and Aouf [20] (see also [21]);

(xvii)For $a=c$, $m=-n$ $(n\in\mathbb{Z})$, $\lambda=1$ and $l=1$ we obtain the operator D_p^n studied by Patel and Sahoo [22];

(xviii)For $a=c$, $p=1$ and $\lambda=1$ we obtain the operator I_l^m studied by Cho and Srivastava [23];

(xix)For $a=c$, $p=1$ and $l=0$ we obtain the operator I_{λ}^m studied by Al-Oboudi [24];

(xx)For $a=c$, $p=1$, $\lambda=1$ and $l=0$ we obtain the operator D^m studied by Sălăgean [25].

It is readily verified from (2) that

$$z\left(\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)\right)'=\frac{a+\mu p}{\mu}\,\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c;\mu)f(z)-\frac{a}{\mu}\,\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z),\tag{3}$$

$$z\left(\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)\right)'=\frac{p+l}{\lambda}\,\mathfrak{d}_{p,m+1}^{\lambda,l}(a,c;\mu)f(z)-\frac{p+l-p\lambda}{\lambda}\,\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z),\tag{4}$$

and

$$z\left(\mathfrak{d}_{p,m}^{\lambda,l}(a,c+1;\mu)f(z)\right)'=\frac{c+\mu p}{\mu}\,\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)-\frac{c}{\mu}\,\mathfrak{d}_{p,m}^{\lambda,l}(a,c+1;\mu)f(z).\tag{5}$$

Using of the operator $\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)$ and the above concept of subordination between two analytic functions, we introduce and investigate the following subclass of $\mathscr{A}_p$ defined as follows:

Definition 2. *A function $f\in\mathscr{A}_p$ is said to be in the class $\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha;A,B)$ if it satisfies the following subordination condition:*

$$\frac{1}{p-\alpha}\left(\frac{z(\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z))'}{\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)}-\alpha\right)\prec\frac{1+Az}{1+Bz},$$

for fixed parameters $A,B\in\mathbb{R}$ with $-1\leq B<A\leq 1$, and $0\leq\alpha<p$.

Remark. We emphasize the next special cases of the subclass $\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha;A,B)$ obtained for appropriate choices of the parameters.

(i)For $A=1$ and $B=-1$, we get

$$\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha;1,-1)=:\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha)$$
$$=\left\{f\in\mathscr{A}_p:\mathrm{Re}\,\frac{z(\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z))'}{\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)}>\alpha,\ z\in\mathbb{D}\right\},$$
$$0\leq\alpha<p;$$

(ii)For $m=0$, $a=c$, $A=1$ and $B=-1$, the class $\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha;A,B)$ reduces to the class $S_p^*(\alpha)$ $(0\leq\alpha<p)$ which was studied by Patel and Thakare [26];

(iii)For $m=1$, $a=c$, $\lambda=1$, $l=0$, $A=1$ and $B=-1$, the class $\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha;A,B)$ reduces to the class $\mathscr{K}_p(\alpha)$ $(0\leq\alpha<p)$ which studied by Owa [27].

In this paper we obtain some inclusion theorems for the class $\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha;A,B)$ with respect to variations of the parameters m, a and c. Also, we establish subordination properties for the class $\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha;A,B)$ and find several sufficient conditions under which subordination results of the form

$$\left(\frac{\varsigma\mathfrak{d}_{p,m+1}^{\lambda,l}(a,c,\mu)f(z)+\eta\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)f(z)}{z^p(\varsigma+\eta)}\right)^k\prec q(z)$$

hold for suitable univalent function q. Several other special cases of the main results are obtained.

2 Preliminary Results

In order to establish our main results we shall make use of the following lemmas. The first lemma is a special case of Corollary 3.2 of [28]:

Lemma 1. *[29, Lemma 2, p. 323] If $-1\leq B<A\leq 1$, $\beta>0$, and the complex number γ is constrained by*

$$\mathrm{Re}\,\gamma\geq-\frac{\beta(1-A)}{1-B},$$

then the differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=\frac{1+Az}{1+Bz},\ z\in\mathbb{D},$$

has a univalent solution given by

$$q(z)=\begin{cases}\dfrac{z^{\beta+\gamma}(1+Bz)^{\frac{\beta(A-B)}{B}}}{\beta\int_0^z t^{\beta+\gamma-1}(1+Bt)^{\frac{\beta(A-B)}{B}}dt}-\dfrac{\gamma}{\beta}, & \text{if } B\neq 0,\\ \dfrac{z^{\beta+\gamma}e^{\beta Az}}{\beta\int_0^z t^{\beta+\gamma-1}e^{\beta At}dt}-\dfrac{\gamma}{\beta}, & \text{if } B=0.\end{cases}$$

If φ is regular in $\mathbb{D}$ and satisfies the differential subordination

$$\varphi(z)+\frac{z\varphi'(z)}{\beta\varphi(z)+\gamma}\prec\frac{1+Az}{1+Bz},$$

then $\varphi(z)\prec q(z)\prec\dfrac{1+Az}{1+Bz}$, and q is the best dominant of the above subordination.

Lemma 2.*[30, Lemma 2, p. 288] Let ν be a positive measure on the interval $[0,1]$. Let $h(z,t)$ be a complex valued function defined on $\mathbb{D}\times[0,1]$ such that, for each $t\in[0,1]$, $h(\cdot,t)$ is analytic in $\mathbb{D}$, and for each $z\in\mathbb{D}$, $h(z,\cdot)$ is ν-integrable on $[0,1]$. In addition, suppose that $\operatorname{Re} h(z,t)>0$, $h(-r,t)$ is real and*

$$\operatorname{Re}\frac{1}{h(z,t)}\geq\frac{1}{h(-r,t)},\ |z|\leq r<1,\ t\in[0,1].$$

If H is defined by

$$H(z)=\int_0^1 h(z,t)d\nu(t),$$

then

$$\operatorname{Re}\frac{1}{H(z)}\geq\frac{1}{H(-r)},\ |z|\leq r<1.$$

For any complex numbers a, b and c ($c\notin\mathbb{Z}_0^-:=\{0,-1,-2,\ldots\}$), the *Gaussian hypergeometric function* is defined by

$${}_2F_1(a,b;c;z):=1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\ldots.$$

Lemma 3.*[31, Ch. 14] For any complex numbers a, b, c ($c\notin\mathbb{Z}_0^-$), we have:*

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt=\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z),$$

if $\operatorname{Re} c>\operatorname{Re} b>0$, (6)

$${}_2F_1(a,b;c;z)=(1-z)^{-a}\,{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right),$$

for $z\in\mathbb{C}\setminus(1,\infty)$, (7)

$$(b+1)\,{}_2F_1(1,b;b+1;z)=(b+1)+bz\,{}_2F_1(1,b+1;b+2;z). \tag{8}$$

Lemma 4.*[32, Theorem 3.4h, p. 132] Let q be univalent in the unit disc $\mathbb{D}$ and let Φ and θ be analytic in a domain $\triangle$ containing $q(\mathbb{D})$ with $\Phi(w)\neq 0$ when $w\in q(\mathbb{D})$. Set $Q(z)=zq'(z)\Phi(q(z))$ and $h(z)=\theta(q(z))+Q(z)$. Suppose that*

(i)Q is starlike univalent in $\mathbb{D}$;

(ii)$\operatorname{Re}\dfrac{zh'(z)}{Q(z)}>0$, $z\in\mathbb{D}$.

If p is analytic in $\mathbb{D}$ with $p(0)=q(0)$, $p(\mathbb{D})\subseteq\triangle$ and

$$\theta(p(z))+zp'(z)\Phi(p(z))\prec\theta(q(z))+zq'(z)\Phi(q(z)),$$

then

$$p(z)\prec q(z),$$

and q is the best dominant.

Lemma 5.*[33, Lemma 2.2, p. 3] Let q be a convex univalent function in $\mathbb{D}$ and let $\varpi\in\mathbb{C}$, $\vartheta\in\mathbb{C}^*:=\mathbb{C}\setminus\{0\}$ with*

$$\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right)>\max\left\{0;-\operatorname{Re}\frac{\varpi}{\vartheta}\right\},\ z\in\mathbb{D}.$$

If p is analytic in $\mathbb{D}$ with $p(0)=q(0)$ and

$$\varpi p(z)+\vartheta zp'(z)\prec\varpi q(z)+\vartheta zq'(z),$$

then

$$p(z)\prec q(z),$$

and q is the best dominant.

3 Inclusion results

Unless otherwise mentioned, we shall assume throughout the paper that $m\in\mathbb{Z}$, $\lambda\geq 0$, $l>-p$, $\mu>0$, $a,c\in\mathbb{R}$, $c-a\geq 0$, $a\geq-\mu p$, $p\in\mathbb{N}$, $-1\leq B<A\leq 1$ and $0\leq\alpha<p$.

The first inclusion theorem with respect to the parameter a is given by the next result:

Theorem 1.*(i) If $f\in\mathbf{S}_{p,m}^{\lambda,l}(a+1,c,\mu;\alpha;A,B)$ such that $\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)\neq 0$ for all $z\in\dot{\mathbb{D}}:=\mathbb{D}\setminus\{0\}$, and*

$$\left(\frac{a}{\mu}+\alpha\right)(1-B)\geq-(p-\alpha)(1-A),$$

then

$$\frac{1}{p-\alpha}\left(\frac{z(\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z))'}{\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)}-\alpha\right)\prec\frac{1}{p-\alpha}.$$
$$\left[\frac{1}{Q_1(z)}-\left(\frac{a}{\mu}+\alpha\right)\right]=:q_1(z)\prec\frac{1+Az}{1+Bz}, \tag{9}$$

where

$$Q_1(z):=\begin{cases}\displaystyle\int_0^1 t^{\frac{a}{\mu}+p-1}\left(\frac{1+Bzt}{1+Bz}\right)^{(p-\alpha)\frac{A-B}{B}}dt, & \text{if } B\neq 0,\\ \displaystyle\int_0^1 t^{\frac{a}{\mu}+p-1}e^{(p-\alpha)A(t-1)z}dt, & \text{if } B=0,\end{cases} \tag{10}$$

and q_1 is the best dominant of (9). Therefore,

$$f\in\mathbf{S}_{p,m}^{\lambda,l}(a+1,c,\mu;\alpha;A,B)\text{ such that }\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)\neq 0,$$
$$z\in\dot{\mathbb{D}}\Rightarrow f\in\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha;A,B).$$

(ii) Also, if the extra constraints

$-1\leq B<0$,

$$\frac{a}{\mu}+p+1\geq(p-\alpha)\frac{B-A}{B},$$

are satisfied, then

$$\frac{1}{p-\alpha}\operatorname{Re}\left(\frac{z(\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z))'}{\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)}-\alpha\right)>\wp_1,\ z\in\mathbb{D},\qquad(11)$$

where

$$\wp_1:=\frac{1}{p-\alpha}\left[\frac{\frac{a}{\mu}+p}{{}_2F_1\left(1,(p-\alpha)\frac{B-A}{B};\frac{a}{\mu}+p+1;\frac{B}{B-1}\right)}-\left(\frac{a}{\mu}+\alpha\right)\right].$$

The bound $\wp_1$ is the best possible.

*Proof.*For the function $f\in\mathbf{S}_{p,m}^{\lambda,l}(a+1,c,\mu;\alpha;A,B)$ let define

$$\varphi(z):=\frac{1}{p-\alpha}\left(\frac{z(\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z))'}{\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)}-\alpha\right),\ z\in\mathbb{D}.\qquad(12)$$

Then, φ is analytic in $\mathbb{D}$ and $\varphi(0)=1$. Using the identity (3) in (12) it follows that

$$(p-\alpha)\varphi(z)+\frac{a}{\mu}+\alpha=\frac{a+\mu p}{\mu}\cdot\frac{\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c;\mu)f(z)}{\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)}.\qquad(13)$$

Next, using the logarithmic differentiation of the both sides of (13) with respect to z and multiplying by z, we get

$$\varphi(z)+\frac{z\varphi'(z)}{(p-\alpha)\varphi(z)+\frac{a}{\mu}+\alpha}=\frac{1}{p-\alpha}\left(\frac{z(\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c;\mu)f(z))'}{\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c;\mu)f(z)}-\alpha\right)\prec\frac{1+Az}{1+Bz}.$$

Therefore, using Lemma 1 for $\beta:=p-\alpha$ and $\gamma:=\frac{a}{\mu}+\alpha$, we obtain that

$$\varphi(z)\prec q_1(z)\prec\frac{1+Az}{1+Bz},$$

where q_1 is given by (9) and it is the best dominant, and the proof of the item (i) is complete.

To prove the inequality (11), from the above subordination we get

$$\begin{aligned}\frac{1}{p-\alpha}\operatorname{Re}\left(\frac{z(\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z))'}{\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)}-\alpha\right)&>\inf\{\operatorname{Re}q_1(z):z\in\mathbb{D}\}\\&=\inf\left\{\frac{1}{p-\alpha}\operatorname{Re}\left[\frac{1}{Q_1(z)}-\left(\frac{a}{\mu}+\alpha\right)\right]:z\in\mathbb{D}\right\}\\&=\frac{1}{p-\alpha}\left[\inf\left\{\operatorname{Re}\frac{1}{Q_1(z)}:z\in\mathbb{D}\right\}-\left(\frac{a}{\mu}+\alpha\right)\right].\end{aligned}\qquad(14)$$

Now, we need to find $\inf\left\{\operatorname{Re}\frac{1}{Q_1(z)}:z\in\mathbb{D}\right\}$. Since $B\neq0$, from (10) we have

$$Q_1(z)=(1+Bz)^{\zeta}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1+Btz)^{-\zeta}dt,$$

where $\zeta:=-(p-\alpha)\frac{A-B}{B}$, $\beta:=\frac{a}{\mu}+p$ and $\gamma:=\beta+1$. Using (6) and (7) of Lemma 3 and the assumption $\gamma>\beta>0$, the above relation yields

$$Q_1(z)=\frac{\Gamma(\beta)}{\Gamma(\gamma)}\,{}_2F_1\left(1,\zeta;\gamma,\frac{Bz}{Bz+1}\right).\qquad(15)$$

Also, the condition $\frac{a}{\mu}+p+1>(p-\alpha)\frac{B-A}{B}$ with $-1\le B<0$ implies that $\gamma>\zeta>0$. Using again (6) of Lemma 3, from (15) we have

$$Q_1(z)=\int_0^1 h(z,t)d\nu(t),$$

where $h(z,t)=\frac{Bz+1}{(1-t)Bz+1}$ and

$$d\nu(t)=\frac{\Gamma(\beta)}{\Gamma(\zeta)\Gamma(\gamma-\zeta)}t^{\zeta-1}(1-t)^{\gamma-\zeta-1}dt$$

is a positive measure on $[0,1]$. We note that $\operatorname{Re}h(z,t)>0$, and $h(-r,t)$ is real and

$$\operatorname{Re}\frac{1}{h(z,t)}=\operatorname{Re}\frac{(1-t)Bz+1}{Bz+1}\ge\frac{1-(1-t)Br}{1-Br}=\frac{1}{h(-r,t)}$$

for $0\le r<1$ and $t\in[0,1]$. Therefore, from Lemma 2 we deduce that

$$\operatorname{Re}\frac{1}{Q_1(z)}\ge\frac{1}{Q_1(-r)},\ |z|\le r<1,$$

and by letting $r\to1^-$ we get

$$\begin{aligned}\inf\left\{\operatorname{Re}\frac{1}{Q_1(z)}:z\in\mathbb{D}\right\}&=\frac{1}{Q_1(-1)}\\&=\frac{\frac{a}{\mu}+p}{{}_2F_1\left(1,(p-\alpha)\frac{B-A}{B};\frac{a}{\mu}+p+1;\frac{B}{B-1}\right)}.\end{aligned}\qquad(16)$$

Taking

$$A\to B-\frac{B}{p-\alpha}\left(\frac{a}{\mu}+p+1\right)$$

for the case $\frac{a}{\mu}+p+1=(p-\alpha)\frac{B-A}{B}$, in view of (14) the inequality (11) follows from (16)

The result is the best possible as the function q_1 is the best dominant of (9).

Taking $A=1$ and $B=-1$ in Theorem 1 we obtain the next result:

Corollary 1.*(i) If*

$$\max\left\{0;-\frac{a}{\mu}\right\}\le\alpha<p,\qquad(17)$$

then

$$f\in\mathbf{S}_{p,m}^{\lambda,l}(a+1,c,\mu;\alpha)\ \textit{such that}\ \mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)\neq0,\ z\in\dot{\mathbb{D}}\Rightarrow f\in\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha).$$

(ii) If $f\in\mathbf{S}_{p,m}^{\lambda,l}(a+1,c,\mu;\alpha)$ such that $\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)\neq0$, $z\in\dot{\mathbb{D}}$, and in addition to (17) assume that

$$\max\left\{0;\frac{1}{2}\left(p-1-\frac{a}{\mu}\right)\right\}\le\alpha<p.$$

Then

$$\frac{1}{p-\alpha}\operatorname{Re}\left(\frac{z(\eth_{p,m}^{\lambda,l}(a,c;\mu)f(z))'}{\eth_{p,m}^{\lambda,l}(a,c;\mu)f(z)}-\alpha\right)>\nu_1,\ z\in\mathbb{D},$$

where

$$\nu_1:=\frac{1}{p-\alpha}\left[\frac{\frac{a}{\mu}+p}{{}_2F_1\left(1,2(p-\alpha);\frac{a}{\mu}+p+1;\frac{1}{2}\right)}-\left(\frac{a}{\mu}+\alpha\right)\right],$$

and the bound ν_1 is the best possible.

In the next theorems we give the inclusions regarding to the parameter m and c, respectively.

Theorem 2.*(i) If $f\in\mathbf{S}_{p,m+1}^{\lambda,l}(a,c,\mu;\alpha;A,B)$ such that $\eth_{p,m}^{\lambda,l}(a,c;\mu)f(z)\neq 0$ for all $z\in\dot{\mathbb{D}}$, and*

$$\left(\frac{p+l}{\lambda}-p+\alpha\right)(1-B)\geq-(p-\alpha)(1-A),$$

then

$$\begin{aligned}&\frac{1}{p-\alpha}\left(\frac{z(\eth_{p,m}^{\lambda,l}(a,c;\mu)f(z))'}{\eth_{p,m}^{\lambda,l}(a,c;\mu)f(z)}-\alpha\right)\\&\quad\prec\frac{1}{p-\alpha}\left[\frac{1}{Q_2(z)}-\left(\frac{p+l}{\lambda}-p+\alpha\right)\right]\\&\quad=:q_2(z)\prec\frac{1+Az}{1+Bz},\end{aligned}\tag{18}$$

where

$$Q_2(z):=\begin{cases}\displaystyle\int_0^1 t^{\frac{p+l}{\lambda}-1}\left(\frac{1+Bzt}{1+Bz}\right)^{(p-\alpha)\frac{A-B}{B}}dt, & \text{if } B\neq 0,\\ \displaystyle\int_0^1 t^{\frac{p+l}{\lambda}-1}e^{(p-\alpha)A(t-1)z}dt, & \text{if } B=0,\end{cases}\tag{19}$$

and q_2 is the best dominant of (18). Therefore,

$$f\in\mathbf{S}_{p,m+1}^{\lambda,l}(a,c,\mu;\alpha;A,B)\quad\text{such that}\quad \eth_{p,m}^{\lambda,l}(a,c;\mu)f(z)\neq 0,\ z\in\dot{\mathbb{D}}\Rightarrow f\in\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha;A,B).$$

(ii) Also, if the extra constraints

$$-1\leq B<0,$$
$$\frac{p+l}{\lambda}+1\geq(p-\alpha)\frac{B-A}{B},$$

are satisfied, then

$$\frac{1}{p-\alpha}\operatorname{Re}\left(\frac{z(\eth_{p,m}^{\lambda,l}(a,c;\mu)f(z))'}{\eth_{p,m}^{\lambda,l}(a,c;\mu)f(z)}-\alpha\right)\geq\wp_2,\ z\in\mathbb{D},\tag{20}$$

where

$$\wp_2:=\frac{1}{p-\alpha}\left[\frac{\frac{p+l}{\lambda}}{{}_2F_1\left(1,(p-\alpha)\frac{B-A}{B};\frac{p+l}{\lambda}+1;\frac{B}{B-1}\right)}-\left(\frac{p+l}{\lambda}-p+\alpha\right)\right].$$

The bound $\wp_2$ is the best possible.

*Proof.*If $f\in\mathbf{S}_{p,m+1}^{\lambda,l}(a,c,\mu;\alpha;A,B)$, let define the function

$$\varphi(z):=\frac{1}{p-\alpha}\left(\frac{z(\eth_{p,m}^{\lambda,l}(a,c;\mu)f(z))'}{\eth_{p,m}^{\lambda,l}(a,c;\mu)f(z)}-\alpha\right),\ z\in\mathbb{D}.\tag{21}$$

Hence, φ is analytic in $\mathbb{D}$ and $\varphi(0)=1$. From the relation (4), the above definition formula (21) yields

$$(p-\alpha)\varphi(z)+\frac{p+l}{\lambda}-p+\alpha=\frac{p+l}{\lambda}\cdot\frac{\eth_{p,m+1}^{\lambda,l}(a,c;\mu)f(z)}{\eth_{p,m}^{\lambda,l}(a,c;\mu)f(z)}.\tag{22}$$

Using the logarithmic differentiation of both sides of (22) with respect to z and multiplying by z, we have

$$\begin{aligned}&\varphi(z)+\frac{z\varphi'(z)}{(p-\alpha)\varphi(z)+\frac{p+l}{\lambda}-p+\alpha}\\&=\frac{1}{p-\alpha}\left(\frac{z(\eth_{p,m+1}^{\lambda,l}(a,c;\mu)f(z))'}{\eth_{p,m+1}^{\lambda,l}(a,c;\mu)f(z)}-\alpha\right)\prec\frac{1+Az}{1+Bz}.\end{aligned}$$

From Lemma 1 for $\beta:=p-\alpha$ and $\gamma:=\dfrac{p+l}{\lambda}-p+\alpha$ we get

$$\varphi(z)\prec q_2(z)\prec\frac{1+Az}{1+Bz},$$

where q_2 is given by (18) and it is the best dominant. Thus, the proof of the item (i) of Theorem 2 is complete.

To prove the inequality (20), from the previous subordination we get

$$\begin{aligned}&\frac{1}{p-\alpha}\operatorname{Re}\left(\frac{z(\eth_{p,m}^{\lambda,l}(a,c;\mu)f(z))'}{\eth_{p,m}^{\lambda,l}(a,c;\mu)f(z)}-\alpha\right)>\inf\{\operatorname{Re}q_2(z):z\in\mathbb{D}\}\\&=\inf\left\{\frac{1}{p-\alpha}\operatorname{Re}\left[\frac{1}{Q_2(z)}-\left(\frac{p+l}{\lambda}-p+\alpha\right)\right]:z\in\mathbb{D}\right\}\\&=\frac{1}{p-\alpha}\left[\inf\left\{\operatorname{Re}\frac{1}{Q_2(z)}:z\in\mathbb{D}\right\}-\left(\frac{p+l-p\lambda}{\lambda}+\alpha\right)\right].\end{aligned}\tag{23}$$

Now, we will find $\inf\left\{\operatorname{Re}\dfrac{1}{Q_2(z)}:z\in\mathbb{D}\right\}$. Since $B\neq 0$, from (19) we have

$$Q_2(z)=(1+Bz)^{\zeta}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1+Btz)^{-\zeta}dt,$$

where $\zeta:=-(p-\alpha)\dfrac{A-B}{B}$, $\beta:=\dfrac{p+l}{\lambda}$ and $\gamma:=\beta+1$. Since $\gamma>\beta>0$, using (6) and (7) of Lemma 3 we obtain

$$Q_2(z)=\frac{\Gamma(\beta)}{\Gamma(\gamma)}\,{}_2F_1\left(1,\zeta;\gamma,\frac{Bz}{Bz+1}\right).\tag{24}$$

Also, the conditions $\left(\dfrac{p+l}{\lambda}+1\right)>(p-\alpha)\dfrac{B-A}{B}$ and $-1\leq B<0$ implies that $\gamma>\zeta>0$. Thus, using again (6) of Lemma 3, the relation (24) leads to

$$Q_2(z)=\int_0^1 h(z,t)d\nu(t),$$

where $h(z,t) = \dfrac{Bz+1}{(1-t)Bz+1}$ and

$$d\nu(t) = \frac{\Gamma(\beta)}{\Gamma(\zeta)\Gamma(\gamma-\zeta)} t^{\zeta-1}(1-t)^{\gamma-\delta-1}dt$$

is a positive measure on $[0,1]$. We note that $\mathrm{Re}\, h(z,t) > 0$, $h(-r,t)$ is real and

$$\mathrm{Re}\,\frac{1}{h(z,t)} = \mathrm{Re}\,\frac{(1-t)Bz+1}{Bz+1} \geq \frac{1-(1-t)Br}{1-Br} = \frac{1}{h(-r,t)}$$

for $0 \leq r < 1$ and $t \in [0,1]$. Consequently, from Lemma 2 we have

$$\mathrm{Re}\,\frac{1}{Q_2(z)} \geq \frac{1}{Q_2(-r)},\ |z| \leq r < 1,$$

and by letting $r \to 1^-$ we get

$$\inf\left\{\mathrm{Re}\,\frac{1}{Q_2(z)} : z \in \mathbb{D}\right\} = \frac{1}{Q_2(-1)}$$
$$= \frac{\frac{p+l}{\lambda}}{{}_2F_1\left(1,(p-\alpha)\frac{B-A}{B};\frac{p+l}{\lambda}+1;\frac{B}{B-1}\right)}. \quad (25)$$

Taking

$$A \to B - \frac{B}{p-\alpha}\left(\frac{p+l}{\lambda}+1\right)$$

for the case $\left(\frac{p+l}{\lambda}+1\right) = (p-\alpha)\frac{B-A}{B}$, in view of (23) the inequality (20) follows from (25)

The result is the best possible as the function q_2 is the best dominant of (18).

For $A = 1$ and $B = -1$ the above theorem reduces to the next special case:

Corollary 2.*(i) If*

$$\max\left\{0; p - \frac{p+l}{\lambda}\right\} \leq \alpha < p, \quad (26)$$

then

$$f \in \mathbf{S}^{\lambda,l}_{p,m+1}(a,c,\mu;\alpha) \quad \text{such that} \quad \eth^{\lambda,l}_{p,m}(a,c;\mu)f(z) \neq 0,$$
$$z \in \dot{\mathbb{D}} \Rightarrow f \in \mathbf{S}^{\lambda,l}_{p,m}(a,c,\mu;\alpha).$$

(ii) If $f \in \mathbf{S}^{\lambda,l}_{p,m+1}(a,c,\mu;\alpha)$ such that $\eth^{\lambda,l}_{p,m}(a,c;\mu)f(z) \neq 0$, $z \in \dot{\mathbb{D}}$, and in addition to (26) assume that

$$\max\left\{0; p - \frac{1}{2}\left(\frac{p+l}{\lambda}+1\right)\right\} \leq \alpha < p.$$

Then

$$\frac{1}{p-\alpha}\mathrm{Re}\left(\frac{z(\eth^{\lambda,l}_{p,m}(a,c;\mu)f(z))'}{\eth^{\lambda,l}_{p,m}(a,c;\mu)f(z)} - \alpha\right) > \nu_2,\ z \in \mathbb{D},$$

where

$$\nu_2 := \frac{1}{p-\alpha}\left[\frac{\frac{p+l}{\lambda}}{{}_2F_1\left(1,2(p-\alpha);\frac{p+l}{\lambda}+1;\frac{1}{2}\right)} - \left(\frac{p+l}{\lambda} - p + \alpha\right)\right],$$

and the bound ν_2 is the best possible.

Remark.(i) Putting $m = 0$, $a = c$, $l = 0$ and $\lambda = 1$ in Corollary 2 we obtain the result due to Patel et al. [29, Corollary 1];

(ii) For $p = 1$, $m = 0$, $a = c$, $l = 0$ and $\lambda = 1$, Corollary 2 reduces to the result of MacGregor [34].

Theorem 3.*(i) If $f \in \mathbf{S}^{\lambda,l}_{p,m}(a,c,\mu;\alpha;A,B)$ such that $\eth^{\lambda,l}_{p,m}(a,c+1;\mu)f(z) \neq 0$ for all $z \in \dot{\mathbb{D}}$, and*

$$\left(\frac{c}{\mu}+\alpha\right)(1-B) \geq -(p-\alpha)(1-A),$$

then

$$\frac{1}{p-\alpha}\left(\frac{z(\eth^{\lambda,l}_{p,m}(a,c+1;\mu)f(z))'}{\eth^{\lambda,l}_{p,m}(a,c+1;\mu)f(z)} - \alpha\right) \prec \frac{1}{p-\alpha}\left[\frac{1}{Q_3(z)} - \left(\frac{c}{\mu}+\alpha\right)\right] =: q_3(z) \prec \frac{1+Az}{1+Bz}, \quad (27)$$

where

$$Q_3(z) = \begin{cases} \displaystyle\int_0^1 t^{\frac{c}{\mu}+p-1}\left(\frac{1+Bzt}{1+Bz}\right)^{(p-\alpha)\frac{A-B}{B}}dt, & \text{if } B \neq 0, \\ \displaystyle\int_0^1 t^{\frac{c}{\mu}+p-1}e^{(p-\alpha)A(t-1)z}dt, & \text{if } B = 0, \end{cases} \quad (28)$$

and q_3 is the best dominant of (27). Therefore,

$$f \in \mathbf{S}^{\lambda,l}_{p,m}(a,c,\mu;\alpha;A,B) \text{ such that } \eth^{\lambda,l}_{p,m}(a,c+1;\mu)f(z) \neq 0,$$
$$z \in \dot{\mathbb{D}} \Rightarrow f \in \mathbf{S}^{\lambda,l}_{p,m}(a,c+1,\mu;\alpha;A,B).$$

(ii) Also, if the extra constraints

$$-1 \leq B < 0,$$
$$\frac{c}{\mu}+p+1 \geq (p-\alpha)\frac{B-A}{B},$$

are satisfied, then

$$\frac{1}{p-\alpha}\mathrm{Re}\left(\frac{z(\eth^{\lambda,l}_{p,m}(a,c+1;\mu)f(z))'}{\eth^{\lambda,l}_{p,m}(a,c+1;\mu)f(z)} - \alpha\right) > \wp_3,\ z \in \mathbb{D}, \quad (29)$$

where

$$\wp_3 := \frac{1}{p-\alpha}\left[\frac{\frac{c}{\mu}+p}{{}_2F_1\left(1,(p-\alpha)\frac{B-A}{B};\frac{c}{\mu}+p+1;\frac{B}{B-1}\right)} - \left(\frac{c}{\mu}+\alpha\right)\right].$$

The bound on $\wp_3$ is the best possible.

*Proof.*Let $f \in \mathbf{S}^{\lambda,l}_{p,m}(a,c,\mu;\alpha;A,B)$ and define the function

$$\varphi(z) = \frac{1}{p-\alpha}\left(\frac{z(\eth^{\lambda,l}_{p,m}(a,c+1;\mu)f(z))'}{\eth^{\lambda,l}_{p,m}(a,c+1;\mu)f(z)} - \alpha\right),\ z \in \mathbb{D}. \quad (30)$$

Then, φ is analytic in $\mathbb{D}$ with $\varphi(0) = 1$, and using the identity (5) in (30) we get

$$(p-\alpha)\varphi(z) + \frac{c}{\mu} + \alpha = \frac{c+\mu p}{\mu}\cdot\frac{\eth^{\lambda,l}_{p,m}(a,c;\mu)f(z)}{\eth^{\lambda,l}_{p,m}(a,c+1;\mu)f(z)}. \quad (31)$$

By logarithmical differentiation of both sides of (31) with respect to z and multiplying by z, it follows that

$$\varphi(z)+\frac{z\varphi'(z)}{(p-\alpha)\varphi(z)+\frac{c}{\mu}+\alpha}=\frac{1}{p-\alpha}\left(\frac{z(\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z))'}{\mathfrak{d}_{p,m}^{\lambda,l}(a,c;\mu)f(z)}-\alpha\right)\prec\frac{1+Az}{1+Bz}.$$

Therefore, from Lemma 1 with $\beta:=p-\alpha$ and $\gamma:=\frac{c}{\mu}+\alpha$ we get

$$\varphi(z)\prec q_3(z)\prec\frac{1+Az}{1+Bz},$$

where q_3 is given by (27) and it is the best dominant.

To prove the inequality (29) of (ii), from the above subordination we have

$$\begin{aligned}\frac{1}{p-\alpha}\operatorname{Re}&\left(\frac{z(\mathfrak{d}_{p,m}^{\lambda,l}(a,c+1;\mu)f(z))'}{\mathfrak{d}_{p,m}^{\lambda,l}(a,c+1;\mu)f(z)}-\alpha\right)\\&>\inf\{\operatorname{Re}q_3(z):z\in\mathbb{D}\}\\&=\inf\left\{\frac{1}{p-\alpha}\operatorname{Re}\left[\frac{1}{Q_3(z)}-\left(\frac{c}{\mu}+\alpha\right)\right]:z\in\mathbb{D}\right\}\\&=\frac{1}{p-\alpha}\left[\inf\left\{\operatorname{Re}\frac{1}{Q_3(z)}:z\in\mathbb{D}\right\}-\left(\frac{c}{\mu}+\alpha\right)\right].\end{aligned}\tag{32}$$

To find the value $\inf\left\{\operatorname{Re}\frac{1}{Q_2(z)}:z\in\mathbb{D}\right\}$, since $B\neq0$ from (28) we have

$$Q_3(z)=(1+Bz)^{\zeta}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1+Btz)^{-\zeta}dt,$$

where $\zeta:=(p-\alpha)\frac{B-A}{B}$, $\beta:=\frac{c}{\mu}+p$ and $\gamma:=\beta+1$. Since $\gamma>\beta>0$, using (6) and (7) of Lemma 3 we obtain

$$Q_3(z)=\frac{\Gamma(\beta)}{\Gamma(\gamma)}\,{}_2F_1\left(1,\zeta;\gamma,\frac{Bz}{Bz+1}\right).\tag{33}$$

The conditions $\frac{c}{\mu}+p+1>(p-\alpha)\frac{B-A}{B}$ and $-1\leq B<0$ implies that $\gamma>\zeta>0$. Thus, using again (6) of Lemma 3, the relation (33) gives

$$Q_3(z)=\int_0^1 h(z,t)d\nu(t),$$

where $h(z,t)=\frac{Bz+1}{(1-t)Bz+1}$ and

$$d\nu(t)=\frac{\Gamma(\beta)}{\Gamma(\delta)\Gamma(\gamma-\zeta)}t^{\zeta-1}(1-t)^{\gamma-\zeta-1}dt$$

is a positive measure on $[0,1]$. Since $\operatorname{Re}h(z,t)>0$, $h(-r,t)$ is real and

$$\operatorname{Re}\frac{1}{h(z,t)}=\operatorname{Re}\frac{(1-t)Bz+1}{Bz+1}\geq\frac{1-(1-t)Br}{1-Br}=\frac{1}{h(-r,t)}$$

for $0\leq r<1$ and $t\in[0,1]$, from Lemma 2 we deduce that

$$\operatorname{Re}\frac{1}{Q_3(z)}\geq\frac{1}{Q_3(-r)},\ |z|\leq r<1.$$

Letting $r\to1^-$ we get

$$\begin{aligned}\inf\left\{\operatorname{Re}\frac{1}{Q_3(z)}:z\in\mathbb{D}\right\}&=\frac{1}{Q_3(-1)}\\&=\frac{\frac{c}{\mu}+p}{{}_2F_1\left(1,(p-\alpha)\frac{B-A}{B};\frac{c}{\mu}+p+1;\frac{B}{B-1}\right)}.\end{aligned}\tag{34}$$

Taking

$$A\to B-\frac{B}{p-\alpha}\left(\frac{c}{\mu}+p+1\right)$$

for the case $\frac{c}{\mu}+p+1>(p-\alpha)\frac{B-A}{B}$, in view of (32) the inequality (29) follows from (34).

The result is the best possible as the function q_3 is the best dominant of (27).

For $A=1$ and $B=-1$ Theorem 3 reduces to the next special case:

Corollary 3.*(i) If*

$$\max\left\{0;-\frac{c}{\mu}\right\}\leq\alpha<p,\tag{35}$$

then

$$f\in\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha)\text{ such that }\mathfrak{d}_{p,m}^{\lambda,l}(a,c+1;\mu)f(z)\neq0,\ z\in\dot{\mathbb{D}}\Rightarrow f\in\mathbf{S}_{p,m}^{\lambda,l}(a,c+1,\mu;\alpha).$$

(ii) If $f\in\mathbf{S}_{p,m}^{\lambda,l}(a,c,\mu;\alpha)$ such that $\mathfrak{d}_{p,m}^{\lambda,l}(a,c+1;\mu)f(z)\neq0$, $z\in\dot{\mathbb{D}}$, and in addition to (35) assume that

$$\max\left\{0;\frac{1}{2}\left(p-1-\frac{c}{\mu}\right)\right\}\leq\alpha<p.$$

Then

$$\frac{1}{p-\alpha}\operatorname{Re}\left(\frac{z(\mathfrak{d}_{p,m}^{\lambda,l}(a,c+1;\mu)f(z))'}{\mathfrak{d}_{p,m}^{\lambda,l}(a,c+1;\mu)f(z)}-\alpha\right)>\nu_3,\ z\in\mathbb{D},$$

where

$$\nu_3=\frac{1}{p-\alpha}\left[\frac{\frac{c}{\mu}+p}{{}_2F_1\left(1,2(p-\alpha);\frac{c}{\mu}+p+1;\frac{1}{2}\right)}-\left(\frac{c}{\mu}+\alpha\right)\right],$$

and the bound ν_3 is the best possible.

4 Subordination results

In this section, for a given function q we find sufficient conditions such that the subordinations

$$\frac{\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)f(z)}{z^p}\prec q(z)$$

and

$$\left(\frac{\varsigma\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)+\eta\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)f(z)}{z^p(\varsigma+\eta)}\right)^k\prec q(z)$$

hold.

Theorem 4. *Let q be a convex univalent function in $\mathbb{D}$ with $q(0) = 1$, and let $\rho \in \mathbb{C}^* := \mathbb{C}\setminus\{0\}$. Suppose that q and $f \in \mathscr{A}_p$ satisfy any one of the next pairs of conditions:*

$$\begin{aligned}&\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\frac{p(a+\mu p)}{\mu}\cdot \operatorname{Re}\frac{1}{\rho}\right\},\ z\in\mathbb{D},\\ &\frac{\rho}{p}\cdot\frac{\eth^{\lambda,l}_{p,m}(a+1,c,\mu)f(z)}{z^p}+\frac{p-\rho}{p}\cdot\frac{\eth^{\lambda,l}_{p,m}(a,c,\mu)f(z)}{z^p}\\ &\prec q(z)+\frac{\rho\mu}{p(a+\mu p)}zq'(z),\end{aligned} \tag{36}$$

or

$$\begin{aligned}&\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\frac{p(p+l)}{\lambda}\cdot \operatorname{Re}\frac{1}{\rho}\right\},\ \text{with } \lambda>0,\\ &\frac{\rho}{p}\cdot\frac{\eth^{\lambda,l}_{p,m+1}(a,c,\mu)f(z)}{z^p}+\frac{p-\rho}{p}\cdot\frac{\eth^{\lambda,l}_{p,m}(a,c,\mu)f(z)}{z^p}\\ &\prec q(z)+\frac{\rho\lambda}{p(p+l)}zq'(z),\end{aligned} \tag{37}$$

or

$$\begin{aligned}&\operatorname{Re}\left(1+\frac{zq''(z)}{q'(z)}\right) > \max\left\{0; -\frac{p(c-1+\mu p)}{\mu}\cdot \operatorname{Re}\frac{1}{\rho}\right\},\ z\in\mathbb{D},\\ &\frac{\rho}{p}\cdot\frac{\eth^{\lambda,l}_{p,m}(a,c-1,\mu)f(z)}{z^p}+\frac{p-\rho}{p}\cdot\frac{\eth^{\lambda,l}_{p,m}(a,c,\mu)f(z)}{z^p}\\ &\prec q(z)+\frac{\rho\mu}{p(c-1+\mu p)}zq'(z),\end{aligned} \tag{38}$$

$c-1-a \geq 0.$

Then

$$\frac{\eth^{\lambda,l}_{p,m}(a,c,\mu)f(z)}{z^p} \prec q(z), \tag{39}$$

where q is the best dominant of (36), (37) and (38), respectively.

Proof. Let the analytic function h be given by

$$h(z) := \frac{\eth^{\lambda,l}_{p,m}(a,c,\mu)f(z)}{z^p},\ z\in\mathbb{D}. \tag{40}$$

Differentiating (40) with respect to z and using the identities (3)–(5) we obtain, respectively

$$\frac{\eth^{\lambda,l}_{p,m}(a+1,c,\mu)f(z)}{z^p} = h(z)+\frac{\mu}{a+\mu p}zh'(z),$$

$$\frac{\eth^{\lambda,l}_{p,m+1}(a,c,\mu)f(z)}{z^p} = h(z)+\frac{\lambda}{p+l}zh'(z),$$

and

$$\frac{\eth^{\lambda,l}_{p,m}(a,c-1,\mu)f(z)}{z^p} = h(z)+\frac{\mu}{c-1+\mu p}zh'(z).$$

From the above three identities we get that the subordination conditions (36), (37) and (38) are, respectively, equivalent to

$$h(z)+\frac{\rho\mu}{p(a+\mu p)}zh'(z) \prec q(z)+\frac{\rho\mu}{p(a+\mu p)}zq'(z), \tag{41}$$

$$h(z)+\frac{\rho\lambda}{p(p+l)}zh'(z) \prec q(z)+\frac{\rho\lambda}{p(p+l)}zq'(z), \tag{42}$$

and

$$h(z)+\frac{\rho\mu}{p(c-1+\mu p)}zh'(z) \prec q(z)+\frac{\rho\mu}{p(c-1+\mu p)}zq'(z). \tag{43}$$

Using Lemma 5 to each of the subordinations (41)–(43) with suitable choices of ϖ and ϑ, we get the conclusion (39) of Theorem 4.

For the special case $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, Theorem 4 leads to the following three results:

Corollary 4. *Suppose that*

$$\frac{p(a+\mu p)}{\mu}\cdot\operatorname{Re}\frac{1}{\rho} > \frac{|B|-1}{|B|+1},\ \rho\in\mathbb{C}^*.$$

If $f \in \mathscr{A}_p$ satisfies the subordination condition

$$\begin{aligned}&\frac{\rho}{p}\cdot\frac{\eth^{\lambda,l}_{p,m}(a+1,c,\mu)f(z)}{z^p}+\frac{p-\rho}{p}\cdot\frac{\eth^{\lambda,l}_{p,m}(a,c,\mu)f(z)}{z^p}\\ &\prec \frac{\rho\mu}{p(a+\mu p)}\frac{(A-B)z}{(1+Bz)^2}+\frac{1+Az}{1+Bz},\end{aligned} \tag{44}$$

then (39) holds, and $\frac{1+Az}{1+Bz}$ is the best dominant of (44).

Corollary 5. *Suppose that $\lambda > 0$ and*

$$\frac{p(p+l)}{\lambda}\cdot\operatorname{Re}\frac{1}{\rho} > \frac{|B|-1}{|B|+1},\ \rho\in\mathbb{C}^*.$$

If $f \in \mathscr{A}_p$ satisfies the subordination condition

$$\begin{aligned}&\frac{\rho}{p}\cdot\frac{\eth^{\lambda,l}_{p,m+1}(a,c,\mu)f(z)}{z^p}+\frac{p-\rho}{p}\cdot\frac{\eth^{\lambda,l}_{p,m}(a,c,\mu)f(z)}{z^p}\\ &\prec \frac{\rho\lambda}{p(p+l)}\frac{(A-B)z}{(1+Bz)^2}+\frac{1+Az}{1+Bz},\end{aligned} \tag{45}$$

then (39) holds, and $\frac{1+Az}{1+Bz}$ is the best dominant of (45).

Corollary 6. *Suppose that $c-1-a \geq 0$ and*

$$\frac{p(c-1+\mu p)}{\mu}\cdot\operatorname{Re}\frac{1}{\rho} > \frac{|B|-1}{|B|+1},\ \rho\in\mathbb{C}^*.$$

If $f \in \mathscr{A}_p$ satisfies the subordination condition

$$\begin{aligned}&\frac{\rho}{p}\cdot\frac{\eth^{\lambda,l}_{p,m}(a,c-1,\mu)f(z)}{z^p}+\frac{p-\rho}{p}\cdot\frac{\eth^{\lambda,l}_{p,m}(a,c,\mu)f(z)}{z^p}\\ &\prec \frac{\rho\mu}{p(c-1+\mu p)}\frac{(A-B)z}{(1+Bz)^2}+\frac{1+Az}{1+Bz},\end{aligned} \tag{46}$$

then (39) holds, and $\frac{1+Az}{1+Bz}$ is the best dominant of (46).

Taking $p = A = 1$ and $B = -1$ in Corollaries 4–6 we get the following special cases:

Example 1.(i) If
$$\frac{a+\mu}{\mu}\cdot \operatorname{Re}\rho > 0,$$
and $f \in \mathscr{A}$ satisfies the subordination
$$\frac{\rho}{z}\mathfrak{d}_m^{\lambda,l}(a+1,c,\mu)f(z)+\frac{1-\rho}{z}\mathfrak{d}_m^{\lambda,l}(a,c,\mu)f(z) \prec \frac{2\rho\mu z}{(a+\mu)(1-z)^2}+\frac{1+z}{1-z}, \tag{47}$$
then
$$\frac{\mathfrak{d}_m^{\lambda,l}(a,c,\mu)f(z)}{z} \prec \frac{1+z}{1-z}, \tag{48}$$
and $\frac{1+z}{1-z}$ is the best dominant of (47).

(ii) If
$$\frac{1+l}{\lambda}\cdot \operatorname{Re}\rho > 0,\ \lambda > 0,$$
and $f \in \mathscr{A}$ satisfies the subordination
$$\frac{\rho}{z}\mathfrak{d}_{m+1}^{\lambda,l}(a,c,\mu)f(z)+\frac{1-\rho}{z}\mathfrak{d}_m^{\lambda,l}(a,c,\mu)f(z) \prec \frac{2\rho\lambda z}{(1+l)(1-z)^2}+\frac{1+z}{1-z}, \tag{49}$$
then (48) holds, and $\frac{1+z}{1-z}$ is the best dominant of (49).

(iii) If
$$c-1-a \geq 0,\ \frac{c-1+\mu}{\mu}\cdot \operatorname{Re}\rho > 0,$$
and $f \in \mathscr{A}$ satisfies the subordination
$$\frac{\rho}{z}\mathfrak{d}_m^{\lambda,l}(a,c-1,\mu)f(z)+\frac{1-\rho}{z}\mathfrak{d}_m^{\lambda,l}(a,c,\mu)f(z) \prec \frac{2\rho\mu z}{(c-1+\mu)(1-z)^2}+\frac{1+z}{1-z}, \tag{50}$$
then (48) holds, and $\frac{1+z}{1-z}$ is the best dominant of (50).

Remark.(i) Letting $a=c=m=0$ and $\mu=1$ in Example 1 (i), or $a=c$, $m=0$, $\lambda=1$ and $l=0$ in Example 1 (ii) we get the next result:

If $\operatorname{Re}\rho > 0$ and $f \in \mathscr{A}$ satisfies the subordination
$$\rho f'(z)+(1-\rho)\frac{f(z)}{z} \prec \frac{2\rho z}{(1-z)^2}+\frac{1+z}{1-z}, \tag{51}$$
then
$$\frac{f(z)}{z} \prec \frac{1+z}{1-z},$$
and $\frac{1+z}{1-z}$ is the best dominant of (51).

(ii) Letting $a=\mu=1$, $c=2$ and $m=0$ in Example 1 (iii) we get the next result:

If $\operatorname{Re}\rho > 0$ and $f \in \mathscr{A}$ satisfies the subordination
$$\rho\frac{f(z)}{z}+(1-\rho)\frac{2}{z^2}\int_0^z f(t)dt \prec \frac{\rho z}{(1-z)^2}+\frac{1+z}{1-z}, \tag{52}$$
then
$$\frac{2}{z^2}\int_0^z f(t)dt \prec \frac{1+z}{1-z},$$
and $\frac{1+z}{1-z}$ is the best dominant of (52).

(iii) Letting $c=\mu=1$ and $m=a=0$ in Example 1 (iii) we get the next result:

If $\operatorname{Re}\rho > 0$ and $f \in \mathscr{A}$ satisfies the subordination
$$\rho\frac{f(z)}{z}+(1-\rho)\frac{1}{z}\int_0^z \frac{f(t)}{t}dt \prec \frac{2\rho z}{(1-z)^2}+\frac{1+z}{1-z}, \tag{53}$$
then
$$\frac{1}{z}\int_0^z \frac{f(t)}{t}dt \prec \frac{1+z}{1-z},$$
and $\frac{1+z}{1-z}$ is the best dominant of (53).

Theorem 5.*Let q be a univalent function in $\mathbb{D}$ with $q(0)=1$, such that*
$$\operatorname{Re}\left(\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+1\right) > 0,\ z \in \mathbb{D}. \tag{54}$$
Suppose that $f \in \mathscr{A}_p$ satisfy the condition
$$\frac{\varsigma\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)+\eta\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)f(z)}{z^p(\varsigma+\eta)} \neq 0,$$
where $\varsigma, \eta \in \mathbb{C}$ with $\varsigma+\eta \neq 0$.

If
$$k\left(\frac{\varsigma z\left(\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)\right)'+\eta z\left(\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)f(z)\right)'}{\varsigma\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)+\eta\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)f(z)}-p\right) \prec \frac{zq'(z)}{q(z)}, \tag{55}$$
then
$$\left(\frac{\varsigma\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)+\eta\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)f(z)}{z^p(\varsigma+\eta)}\right)^k \prec q(z), \tag{56}$$
with $k \in \mathbb{C}^*$, and q is the best dominant of (55). (The power is the principal one, that is $\log 1 = 0$.)

*Proof.*To prove our result we will use Lemma 4 for
$$\Phi(\omega) := \frac{1}{\omega},\ \theta(\omega) := 0,\ Q(z) := zq'(z)\Phi(q(z)) = \frac{zq'(z)}{q(z)},$$
$$h(z) := Q(z),\ \omega \in \mathbb{C},\ z \in \mathbb{D}.$$
Since $Q'(0)=q'(0)\neq 0$, from the assumption (54) we have that the function Q is a starlike univalent in $\mathbb{D}$, and
$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left(1+\frac{zq''(z)}{q(z)}-\frac{zq'(z)}{q(z)}\right) > 0,\ z \in \mathbb{D}.$$
Thus, both of the assumptions of this lemma are satisfied.

Next, let the function p be given by

$$p(z) := \left(\frac{\varsigma \mathfrak{d}^{\lambda,l}_{p,m}(a+1,c,\mu)f(z) + \eta \mathfrak{d}^{\lambda,l}_{p,m}(a,c,\mu)f(z)}{z^p(\varsigma+\eta)} \right)^k, \quad z \in \mathbb{D}. \quad (57)$$

Then, the function p is analytic in $\mathbb{D}$, $p(0) = q(0) = 1$, and

$$\frac{zp'(z)}{p(z)} = k\left(\frac{\varsigma z \left(\mathfrak{d}^{\lambda,l}_{p,m}(a+1,c,\mu)f(z)\right)' + \eta z \left(\mathfrak{d}^{\lambda,l}_{p,m}(a,c,\mu)f(z)\right)'}{\varsigma \mathfrak{d}^{\lambda,l}_{p,m}(a+1,c,\mu)f(z) + \eta \mathfrak{d}^{\lambda,l}_{p,m}(a,c,\mu)f(z)} - p \right). \quad (58)$$

From (58), the assumption (55) could be written as

$$\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)},$$

that is equivalent to

$$\theta(p(z)) + zp'(z)\Phi(p(z)) \prec \theta(q(z)) + zq'(z)\Phi(q(z)).$$

Therefore, by Lemma 4 we conclude that $p(z) \prec q(z)$, that is (56), and q is the best dominant of (55).

For $\varsigma = 0$, $\eta = 1$ and $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$, it is easy to check that

$$\mathrm{Re}\left(\frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + 1 \right) = 1 - \mathrm{Re}\left(\frac{Az}{1+Az} + \frac{Bz}{1+Bz} \right)$$
$$> 1 - \left(\frac{|A|}{1+|A|} + \frac{|B|}{1+|B|} \right) = \frac{1-|A||B|}{(1+|A|)(1+|B|)} > 0, \ z \in \mathbb{D}.$$

Since the assumption(54) is satisfied, from Theorem 5 we get the next result:

Corollary 7. *Suppose that $f \in \mathscr{A}_p$ satisfy the condition*

$$\frac{\mathfrak{d}^{\lambda,l}_{p,m}(a,c,\mu)f(z)}{z^p} \neq 0, \ z \in \mathbb{D}.$$

If

$$k\left(\frac{z\left(\mathfrak{d}^{\lambda,l}_{p,m}(a,c,\mu)f(z)\right)'}{\mathfrak{d}^{\lambda,l}_{p,m}(a,c,\mu)f(z)} - p \right) \prec \frac{(A-B)z}{(1+Bz)(1+Az)}, \quad (59)$$

then

$$\left(\frac{\mathfrak{d}^{\lambda,l}_{p,m}(a,c,\mu)f(z)}{z^p} \right)^k \prec \frac{1+Az}{1+Bz},$$

with $k \in \mathbb{C}^*$, and $\frac{1+Az}{1+Bz}$ is the best dominant of (59). (The power is the principle one.)

For $A = p = 1$ and $B = -1$ Corollary 7 reduces to the next special case:

Corollary 8. *Suppose that $f \in \mathscr{A}$ satisfy the condition*

$$\frac{\mathfrak{d}^{\lambda,l}_{m}(a,c,\mu)f(z)}{z} \neq 0, \ z \in \mathbb{D}.$$

If

$$k\left(\frac{z\left(\mathfrak{d}^{\lambda,l}_{m}(a,c,\mu)f(z)\right)'}{\mathfrak{d}^{\lambda,l}_{m}(a,c,\mu)f(z)} - 1 \right) \prec \frac{2z}{1-z^2}, \quad (60)$$

then

$$\left(\frac{\mathfrak{d}^{\lambda,l}_{m}(a,c,\mu)f(z)}{z} \right)^k \prec \frac{1+z}{1-z},$$

with $k \in \mathbb{C}^*$, and $\frac{1+z}{1-z}$ is the best dominant of (60). (The power is the principle one.)

For $a = c$, $m = 0$ and $\mu = k = 1$ Corollary 8 leads to the next example:

Example 2. Suppose that $f \in \mathscr{A}$ satisfy the condition

$$\frac{f(z)}{z} \neq 0, \ z \in \mathbb{D}.$$

If

$$\frac{zf'(z)}{f(z)} - 1 \prec \frac{2z}{1-z^2}, \quad (61)$$

then

$$\frac{f(z)}{z} \prec \frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant of (61).

Corollary 9. *If $f \in \mathscr{A}$ is a starlike univalent function of order α $(0 \le \alpha < 1)$ in $\mathbb{D}$, and $\beta \in (0,1]$, then*

$$\left(\frac{f(z)}{z} \right)^{\frac{\beta}{1-\alpha}} \prec (1-z)^{-2\beta}. \quad (62)$$

The function $(1-z)^{-2\beta}$ is the best dominant. (The power is the principle one.)

Proof. Since $f \in \mathscr{A}$ is a starlike univalent function of order α $(0 \le \alpha < 1)$, it follows that f is univalent in $\mathbb{D}$, hence $\frac{f(z)}{z} \neq 0$ for all $z \in \mathbb{D}$. Denoting $q(z) = (1-z)^{-2\beta}$ $(0 < \beta \le 1)$, the assumption

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z},$$

is equivalent to

$$k\left(\frac{zf'(z)}{f(z)} - 1 \right) \prec \frac{2\beta z}{1-z} = \frac{zq'(z)}{q(z)}.$$

If we let

$$\varphi : \mathbb{D} \to \Delta := \left\{ w \in \mathbb{C} : \mathrm{Re}\, w > \frac{1}{2} \right\}, \ \varphi(z) := \frac{1}{1-z},$$
$$\psi : \Delta \to \Omega := \left\{ w = u + iv \in \mathbb{C} : v^2 > \frac{1}{4} - u \right\}, \ \psi(z) := z^2,$$
$$\chi : \Omega \to \mathbb{C}, \ \chi(z) := z^\beta, \ 0 < \beta \le 1,$$

then

$$q(z) = (1-z)^{-2\beta} = (\chi \circ \psi \circ \varphi)(z),\ z \in \mathbb{D}.$$

It's easy to prove that φ, ψ and χ are univalent functions on their definition domains, hence q is univalent in $\mathbb{D}$.

Putting $\varsigma = m = 0$, $a = c$, $\mu = \eta = p = 1$ and $q(z) = (1-z)^{-2\beta}$ in Theorem 5 we get the conclusion (62).

Taking $\varsigma = 1$, $\eta = 0$ and $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 5 we obtain:

Corollary 10. *Suppose that $f \in \mathscr{A}_p$ satisfy the condition*

$$\frac{\eth_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)}{z^p} \neq 0,\ z \in \mathbb{D}.$$

If

$$k\left(\frac{z\left(\eth_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)\right)'}{\eth_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)} - p\right) \prec \frac{z(A-B)}{(1+Bz)(1+Az)}, \quad (63)$$

then

$$\left(\frac{\eth_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)}{z^p}\right)^k \prec \frac{1+Az}{1+Bz},$$

and $\dfrac{1+Az}{1+Bz}$ *is the best dominant of* (63). *(The power is the principle one.)*

For $A = p = 1$ and $B = -1$ Corollary 10 leads to the next special case:

Corollary 11. *Suppose that $f \in \mathscr{A}$ satisfy the condition*

$$\frac{\eth_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)}{z} \neq 0,\ z \in \mathbb{D}.$$

If

$$k\left(\frac{z\left(\eth_{m}^{\lambda,l}(a+1,c,\mu)f(z)\right)'}{\eth_{m}^{\lambda,l}(a+1,c,\mu)f(z)} - 1\right) \prec \frac{2z}{1-z^2}, \quad (64)$$

then

$$\left(\frac{\eth_{m}^{\lambda,l}(a+1,c,\mu)f(z)}{z}\right)^k \prec \frac{1+z}{1-z},$$

and $\dfrac{1+z}{1-z}$ *is the best dominant of* (64). *(The power is the principle one.)*

If we set $a = c = m = 0$ and $k = 1$ in Corollary 11, since $k = 1$ the assumption $f'(z) \neq 0$, $z \in \mathbb{D}$ could be omitted, thus we obtain the next example:

Example 3. If $f \in \mathscr{A}$ and

$$\frac{zf''(z)}{f'(z)} \prec \frac{2z}{1-z^2}, \quad (65)$$

then

$$f'(z) \prec \frac{1+z}{1-z},$$

and $\dfrac{1+z}{1-z}$ is the best dominant of (65).

Theorem 6. *Let q be a univalent function in $\mathbb{D}$ with $q(0) = 1$, and $\delta \in \mathbb{C}$ such that*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\{0; -\operatorname{Re}\delta\},\ z \in \mathbb{D}. \quad (66)$$

Suppose that $f \in \mathscr{A}_p$ satisfy the condition

$$\frac{\varsigma\eth_{p,m}^{\lambda,l}(a+1,c,\mu)f(z) + \eta\eth_{p,m}^{\lambda,l}(a,c,\mu)f(z)}{z^p(\varsigma+\eta)} \neq 0,\ z \in \mathbb{D},$$

where $\varsigma, \eta \in \mathbb{C}$ with $\varsigma + \eta \neq 0$. Set

$$\Lambda(z) := \left(\frac{\varsigma\eth_{p,m}^{\lambda,l}(a+1,c,\mu)f(z) + \eta\eth_{p,m}^{\lambda,l}(a,c,\mu)f(z)}{z^p(\varsigma+\eta)}\right)^k \cdot \left[\delta + k\left(\frac{\varsigma z\left(\eth_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)\right)' + \eta z\left(\eth_{p,m}^{\lambda,l}(a,c,\mu)f(z)\right)'}{\varsigma\eth_{p,m}^{\lambda,l}(a+1,c,\mu)f(z) + \eta\eth_{p,m}^{\lambda,l}(a,c,\mu)f(z)} - p\right)\right].$$

If

$$\Lambda(z) \prec \delta q(z) + zq'(z), \quad (67)$$

then

$$\left(\frac{\varsigma\eth_{p,m}^{\lambda,l}(a+1,c,\mu)f(z) + \eta\eth_{p,m}^{\lambda,l}(a,c,\mu)f(z)}{z^p(\varsigma+\eta)}\right)^k \prec q(z), \quad (68)$$

with $k \in \mathbb{C}^*$, and q is the best dominant of (67). *(All the powers are the principal ones.)*

Proof. The proof of this theorem is similar to that of Theorem 5. If p is defined as in (57), using (58) we have

$$zp'(z) = kp(z) \cdot \left(\frac{\varsigma z\left(\eth_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)\right)' + \eta z\left(\eth_{p,m}^{\lambda,l}(a,c,\mu)f(z)\right)'}{\varsigma\eth_{p,m}^{\lambda,l}(a+1,c,\mu)f(z) + \eta\eth_{p,m}^{\lambda,l}(a,c,\mu)f(z)} - p\right). \quad (69)$$

Set

$$\theta(\omega) := \delta\omega,\ \Phi(\omega) := 1,\ Q(z) := zq'(z),\ \omega \in \mathbb{C},\ z \in \mathbb{D},$$

and

$$h(z) := Q(z) + \theta(q(z)) = \delta q(z) + zq'(z),\ z \in \mathbb{D}.$$

Since $Q'(0) = q'(0) \neq 0$, from (66) it follows that Q is a starlike univalent function in $\mathbb{D}$, and

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left(\delta + 1 + \frac{zq''(z)}{q'(z)}\right) > 0,\ z \in \mathbb{D}.$$

From (57) and (69) we have

$$\theta(p(z)) + zp'(z)\Phi(p(z)) = \delta p(z) + zp'(z) = \Lambda(z),$$

hence the assumption (67) is equivalent to

$$\theta(p(z)) + zp'(z)\Phi(p(z)) \prec \theta(q(z)) + zq'(z)\Phi(q(z)).$$

Therefore, from Lemma 4 it follows that

$$p(z) \prec q(z),$$

and q is the best dominant of (67), that is the assertion in (68) holds.

Taking $\varsigma = 0$, $\eta = 1$ and $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 6 we get:

Corollary 12. *Suppose that $f \in \mathscr{A}_p$ satisfy the condition*

$$\frac{\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)f(z)}{z^p} \neq 0,\ z \in \mathbb{D}.$$

If

$$\left(\frac{\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)f(z)}{z^p}\right)^k \cdot \left[\delta + k\left(\frac{z\left(\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)f(z)\right)'}{\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)f(z)} - p\right)\right] \prec \delta\frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Bz)^2}, \tag{70}$$

then

$$\left(\frac{\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)f(z)}{z^p}\right)^k \prec \frac{1+Az}{1+Bz},$$

with $k \in \mathbb{C}^*$, $\delta := \dfrac{|B|-1}{1+|B|}$, and $\dfrac{1+Az}{1+Bz}$ is the best dominant of (70). (All the powers are the principal ones.)

For $p = \mu = A = 1$, $m = 0$, $a = c$, $B = -1$ and $\delta = \dfrac{|B|-1}{1+|B|} = 0$, Corollary 12 leads to the next special case:

Corollary 13. *Suppose that $f \in \mathscr{A}$ satisfy the condition*

$$\frac{f(z)}{z} \neq 0,\ z \in \mathbb{D}.$$

If

$$k\left(\frac{f(z)}{z}\right)^k\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{2z}{(1-z)^2}, \tag{71}$$

then

$$\left(\frac{f(z)}{z}\right)^k \prec \frac{1+z}{1-z},$$

with $k \in \mathbb{C}^*$, and $\dfrac{1+z}{1-z}$ is the best dominant of (71). (All the powers are the principal ones.)

Taking $k = 1$ in Corollary 13, the assumption $\dfrac{f(z)}{z} \neq 0, z \in \mathbb{D}$ could be omitted, and we obtain the following example:

Example 4. If $f \in \mathscr{A}$ and

$$f'(z) - \frac{f(z)}{z} \prec \frac{2z}{(1-z)^2}, \tag{72}$$

then

$$\frac{f(z)}{z} \prec \frac{1+z}{1-z},$$

and $\dfrac{1+z}{1-z}$ is the best dominant of (72).

Putting $\varsigma = 1$, $\eta = 0$ and $q(z) = \dfrac{1+Az}{1+Bz}$ $(-1 \le B < A \le 1)$ in Theorem 6 we obtain:

Corollary 14. *Suppose that $f \in \mathscr{A}_p$ satisfy the condition*

$$\frac{\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)}{z^p} \neq 0,\ z \in \mathbb{D}.$$

If

$$\left(\frac{\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)}{z^p}\right)^k \cdot \left[\delta + k\left(\frac{z\left(\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)\right)'}{\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)} - p\right)\right] \prec \delta\frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Bz)^2}, \tag{73}$$

then

$$\left(\frac{\mathfrak{d}_{p,m}^{\lambda,l}(a+1,c,\mu)f(z)}{z^p}\right)^k \prec \frac{1+Az}{1+Bz},$$

with $k \in \mathbb{C}^*$, $\delta := \dfrac{|B|-1}{1+|B|}$, and $\dfrac{1+Az}{1+Bz}$ is the best dominant of (73). (All the powers are the principal ones.)

If we take $k = p = A = 1$, $m = a = c = 0$, $B = -1$ and $\delta = \dfrac{|B|-1}{1+|B|} = 0$ in Corollary 14, the assumption $f'(z) \neq 0$, $z \in \mathbb{D}$ could be omitted, and we obtain the following example:

Example 5. If $f \in \mathscr{A}$ and

$$zf''(z) \prec \frac{2z}{(1-z)^2}, \tag{74}$$

then

$$f'(z) \prec \frac{1+z}{1-z},$$

and $\dfrac{1+z}{1-z}$ is the best dominant of (74).

5 Conclusion

The above results give an interesting approach for the study of many multivalent classes previously defined by different authors, because these classes extend and generalize a lot of those defined and studied by several renowned specialists in this field of interest. Moreover, the general subordination theorems yield us to some interesting special cases that were further used to determine new results connected with the classes we introduced. Our main results are followed by some particular and special cases that could be used for the future studies in the theory of multivalent, and also univalent functions.

The investigation tools used in the paper allowed us to find exclusively the best results (i.e. best dominants of the subordinations, and best bounds for the inequalities), that means it could not be improved under the given assumptions.

All of these strength points of the results of the paper consist in the facts that the linear operator $\mathfrak{d}_{p,m}^{\lambda,l}(a,c,\mu)$ defined by [2] has a very general form like it could be seen in Remark 1 (i)-(xx), extending many other earlier studied, and moreover, the methods we are using in the proofs are considered between the most efficient ones in the subordination theory [28,30].

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