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On Generalize L-moments Method and Application on Control Chart for Monitoring Fractional, Rates and Proportions Data

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Abstract: This paper proposes GL-moments' method as a generalization to the TL-moments and the L-moments methods. Population GL-moments in terms of the quantile function and shifted Jacobi polynomials were defined. Some relations between GL-moments depending on Jacobi Polynomials were also derived. Furthermore, interpretation of the first two population GL-moments by U-statistics was introduced. Also, we obtained two expressions to estimate the population GL-moments: the first expression, which depends on Downton's estimator, is a nearly unbiased estimator. Finally, to avoid an upper control limit exceeding one in p-chart, we propose a control chart based on generalized linear moments for monitoring fractional, rates and proportions data. Control limits are proposed and simulated average run length experiments show the proposed control charts to be less influenced by extreme observations than their classical counterparts and lead to tighter control limits. An example is given that summarizes the benefits included in the charts.

Keywords: L-moments, TL-moments, Jacobi Polynomials, p-chart

1 Introduction

The estimation problem in statistical inference is accepted as a first topic. A random sample of dimension, n, chosen from a probability distribution where θ is the unknown parameters (shape, location, and scale), is used to estimate the unknown parameters. The standard statistical method is to summarize a probability distribution using the moments of the distribution. In parametric distributions, it is always presumed that the sample moments are equal to those of the fitted distribution. The classical system of moments is not necessarily most fitting. Often it is impossible to determine just how much detail regarding the shape of a distribution is transmitted by its moments. Also, sample moments' numerical values may be somewhat different from those of the probability distribution from which the sample was taken, especially when the sample size is tiny. Moreover, the approximate parameters of the distribution obtained through traditional methods are always less reliable than those produced by other prediction methods, such as the process of maximum likelihood; see, for example, Vogle and Fennessey [1].

Many statistical methods are focused on using linear combinations of order statistics but there has not been established a coherent theory for the classification of probability distributions until Hosking[2] presented L-moments as an alternative to the traditional moments. As an extension of L-moments that depend on assigning zero weight to extreme observations, Elamir and Seheult [3] introduced TL-moments. TL-moments give more robust estimators than L-moments in the presence of outliers. Moreover, population TL-moments may be well defined where the corresponding population L-moments do not exist. Many researchers have developed and used statistical methods based on L-moments and TL-moments: [4,5,6,7,8,9,10,11,12,13,14,15].

The Trimmed L-moments technique was developed as an alternative to the L-moments technique by[3]. The Trimmed L-moments method depends on trimming some observations from both tails of the distribution by assigning these extreme values zero weights. Thus, it is an important method since it gives more robust estimators

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than the L-moments in the presence of outliers. But the authors discussed this method when t (the number of trimmed observations) is an integer. The purpose of this paper is to expand the meaning of t from integer numbers to rational numbers by adding a new approach, which may be called Generalize L-moments (GL-moments), which assigns a minimal weight for the extreme values. This could help to explain better how the distribution continues to evolve. Also, we propose a control chart based on generalized linear moments for monitoring fractional, rates and proportions data. This control chart assumes that the fraction data can be approximated by a normal distribution and proposes new control limits based on GL-moments. The proposed Chart was applied in real study leading to better outcomes when compared with the Shewhart, Ryan[16], Chen[17], Sant'Anna and Caten[18], and Lima-Fiho et al.[19] control charts for monitoring variables of fraction type. This paper is organized as follows. Section 2 presents Generalized Linear-moments (GL-moments) as a generalization of the Trimmed L-moments method. Section 3 provides estimate the parameters using GL-moments of Symmetric Generalized Lambda Distribution and normal distribution. Section 4 presents a literature review about determination of control limits. Section 5 presents the steps required to compute the proposed control limits. Section 6 proposed simulated average run length experiments show the proposed control charts to be less influenced by extreme observations than their classical counterparts, and lead to tighter control limits. Section 7 An example is given that summarizes the benefits included in the charts. In Section 8 concluding remarks are provided.

2 A Generalized Linear Moments

The Trimmed L-moments method depends on trimming some observations from both tails of the distribution by assigning these extreme values to zero weights. Thus, it is an essential method since it gives more robust estimators than the L-moments in outliers' presence. Nevertheless, the authors discussed this method when t (the number of trimmed observations) is an integer. This section aims to extend the value of t from integer numbers to rational numbers by introducing a new linear method of estimation that may be called Generalized Linear-moments (GL-moments) as a generalization of the Trimmed L-moments method.

2.1 L-moments and TL-moments

Hosking [2] introduced L-moments, as an alternative to the conventional moments. The L in L-moments emphasizes the construction of L-moments from linear combinations of order statistics. The main advantage of L-moments over conventional moments is that L-moments, being linear functions of the random sample observations, suffer less from the effects of sampling variability, so that, L-moments are more robust than conventional moments.

Elamir and Seheult[3] introduced TL-moments as an extension of L-moments that depend on giving zero weight to extreme observations. TL-moments give more robust estimators than L-moments in the presence of outliers. Moreover, population TL-moments may be well defined where the corresponding population L-moments do not exist. Also, they discussed TL-moments when t (number of trimmed observations) is an integer number.

Definition 2.1. Let X be a real-valued random variable with distribution function F and quantile function $x(F)$, and let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the order statistics of a random sample of size n drawn from the distribution of X. Thus, the rth TL-moment $\Psi_r^{(t_1,t_2)}$ is given as

$$\Psi_r^{(t_1,t_2)} = r^{-1}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}E\left(X_{r+t_1-j:r+t_1+t_2}\right),\quad r=1,2,\dots,\ t_1,t_2=0,1,2,\dots, \tag{1}$$

where $t_1$and $t_2$are the amounts of lower and upper trimming.

In the symmetric case, where $t_1 = t_2 = t$, (1) can be re-written as

$$\Psi_r^{(t)} = r^{-1}\sum_{j=0}^{r-1}(-1)^j\binom{r-1}{j}E\left(X_{r+t-j:r+2t}\right),\quad r=1,2,\dots,\ t=0,1,2,\dots \tag{2}$$

Note that, the TL-moments $\Psi_r^{(t_1,t_2)}$are equal to the L-moments when $t_1 = t_2 = 0$.

The first two Sample TL-moments by using (2) are given as

$$\psi_1^{(t)} = \hat{E}\left(X_{t+1:2t+1}\right),\quad \psi_2^{(t)} = \hat{E}\left(X_{t+2:2t+2} - X_{t+1:2t+2}\right),$$

when $t = 1$ the first two sample TL-moments are

$$\psi_1^{(1)} = \hat{E}\left(X_{2:3}\right), \psi_2^{(1)} = \frac{1}{2}\hat{E}\left(X_{3:4} - X_{2:4}\right),$$

while the alternative expressions for the first two sample TL-moments, when $t = 1$, are given by

$$\psi_1^{(1)} = \sum_{i=2}^{n-1}\left[\frac{\binom{i-1}{1}\binom{n-i}{1}}{\binom{n}{3}}\right]x_{i:n},$$

$$\psi_2^{(1)} = \frac{1}{2}\sum_{i=2}^{n-1}\left[\frac{\binom{i-1}{2}\binom{n-i}{1}-\binom{i-1}{1}\binom{n-i}{2}}{\binom{n}{4}}\right]x_{i:n}.$$

2.2 Population GL-moments

We can define the population GL-moments in terms of Jacobi polynomials. The Jacobi polynomials of degree n is defined by

$$H_n^{(\alpha,\omega)}(X) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n+\alpha}{k}\binom{n+\omega}{n-k}(X-1)^{n-k}\times (X+1)^k, \alpha,\omega \geq -1,\ n=1,2,\ldots \quad (3)$$

(see, Abramowitz and Stegun[19]). The Jacobi polynomials in (3) form a complete orthogonal system on the interval $[-1,1]$ with respect to the weight function $(1-X)^{\alpha}(1+X)^{\omega}$. The shifted Jacobi polynomials can be obtained via putting $X = 2F-1,\ \ 0 \leq F \leq 1$ as

$$H_n^{*(\alpha,\omega)}(F) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n+\alpha}{k}\binom{n+\omega}{n-k}F^k\times (1-F)^{n-k},\ \ \alpha,\omega \geq -1,\ n=1,2,\ldots \quad (4)$$

The shifted Jacobi polynomials in (4) are orthogonal on $[0,1]$ with respect to the weight function$F^{\alpha}(1-F)^{\omega}$. When, $\alpha = \omega = 0$,$\mathrm{H}_n^{*(0,0)}(F)$reduces to the shifted Legendre polynomials

$$H_n^{*}(F) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n}{n-k}F^k\times (1-F)^{n-k},\ \ \alpha,\omega \geq -1,\ n=1,2,\ldots, \quad (5)$$

which are, also, orthogonal on the interval [0,1] with constant weight function; (see, Lanczos [21]). Note that, the shifted Legendre polynomials may be re-written as

$$H_n^{*}(F) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{n+k}{k}F^k, n=1,2,\ldots \quad (6)$$

Hosking [2], introduced an alternative expression for the L-moments, which depends on shifted Legendre polynomials and the quantile function $x(F) = F^{-1}(x)$, as

$$\Psi_{r+1} = \int_0^1 x(F)H_n^{*}(F)\ dF,\quad r=0,1,2,\ldots \quad (7)$$

where $\mathrm{H}_n^{*}(F)$ is the shifted Legendre polynomials that is given in (6).

Hosking [6] introduced an analogous result for trimmed L-moments which is given by

$$\Psi_{r+1}^{(t_1,t_2)} = \frac{r!\,(r+t_1+t_2+1)!}{(r+1)\,(r+t_1)!\,(r+t_2)!}\int_0^1 x(F)F^{t_1}\times (1-F)^{t_2}H_n^{*(t_{21},t_1)}(F)\ \ dF,\quad r=0,1,2,\ldots\ ,\ t_1,t_2=0,1,2,\ldots \quad (8)$$

where, $H_n^{*(t_{21},t_1)}(F)$is a shifted Jacobi polynomial introduced in (4).

Since $H_n^{*(t_{21},t_1)}(F)$ is defined for t_1,t_2 real numbers, where $t_1,t_2 > -1$, which include rational numbers, we can extend the TL-moments, defined in (8), to GL-moments by replacing $t_1,t_2 = 0,1,\ldots$ integer numbers, to $g_1,g_2 \geq 0$, positive rational numbers, as follows

$$\Psi_{r+1}^{(g_1,g_2)} = \frac{r!\Gamma(r+g_1+g_2+2)}{(r+1)\Gamma(r+g_1+1)\Gamma(r+g_2+1)}\times\int_0^1 x(F)F^{g_1}\ (1-F)^{g_2}H_n^{*(g_2,g_1)}(F)\ dF$$

$$= \frac{r!\Gamma(r+g_1+g_2+2)}{(r+1)\Gamma(r+g_1+1)\Gamma(r+g_2+1)}\times E\left[x(F)\ F^{g_1}(1-F)^{g_2}H_n^{*(g_2,g_1)}(F)\right],\quad r=0,1,2,\ldots\ ,\ g_1,g_2\geq 0, \quad (9)$$

$$H_n^{*(g_2,g_1)}(F) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r+g_2}{j}\binom{r+g_1}{r-j}\times F^j(1-F)^{r-j} = \sum_{j=0}^{r}(-1)^{r-j}\times\frac{\Gamma(r+g_1+1)\Gamma(r+g_2+1)}{j!\,(r-j)!\Gamma(r-j+g_2+1)\Gamma(g_1+j+1)}\times F^j(1-F)^{r-j},$$

under the condition that the integral in (9) is defined.

In the symmetric case, $g_1 = g_2 = g$, the expression in (9) can be re-written as

$$\Psi_{r+1}^{(g)} = \frac{\Gamma(r+1)\ \Gamma(r+2g+2)}{(r+1)\left[\Gamma(r+g+1)\right]^2}\times E\left[x(F)\ F^{g}(1-F)^{g}H_n^{*(g)}(F)\right]\ ,\ g\geq 0, \quad (10)$$

where

$$H_n^{*(g)}(F) = \sum_{j=0}^{r}(-1)^{r-j}\binom{r+g}{j}\binom{r+g}{r-j}F^j(1-F)^{r-j} = \sum_{j=0}^{r}(-1)^{r-j}\frac{\left[\Gamma(r+g+1)\right]^2}{j!\,(r-j)!\Gamma(r-j+g+1)\Gamma(g+j+1)}\times F^j(1-F)^{r-j}.$$

Note that, the expression of the GL-moments in (9) can be re-written as follows.

$$\Psi_{r+1}^{(g_1,g_2)} = (r+1)^{-1}\sum_{j=0}^{r}(-1)^j\binom{r}{j} \times \frac{\Gamma(r+g_1+g_2+2)}{\Gamma(r+g_1-j+1)\Gamma(g_2+j+1)} \times E\left[x(F)\,F^{r+g_1-j}(1-F)^{g_2+j}\right], \quad r=0,1,2,\ldots,\ g_1,g_2\geq 0. \tag{11}$$

Also, In the symmetric case, $g_1=g_2=g$, the expression in (10) can be re-written as

$$\Psi_{r+1}^{(g)} = (r+1)^{-1}\sum_{j=0}^{r}(-1)^j\binom{r}{j} \times \frac{\Gamma(r+2g+2)}{\Gamma(r+g-j+1)\Gamma(g+j+1)} \times \int_0^1 x(F)F^{r+g-j}(1-F)^{g+j}\,dF$$

$$\Psi_{r+1}^{(g)} = (r+1)^{-1}\sum_{j=0}^{r}(-1)^j\binom{r}{j} \times \frac{\Gamma(r+2g+2)}{\Gamma(r+g-j+1)\Gamma(g+j+1)} \times E\left[x(F)\ F^{r+g-j}(1-F)^{g+j}\right], \quad r=0,1,2,\ldots,\ g_1,g_2\geq 0. \tag{12}$$

Note that, the expression of the GL-moments in (11) and (12)) are analogous to the expression of the TL-moments, by replacing $t_1,t_2=0,1,\ldots$ by $g_1,g_2\geq 0$.

2.3 The First Two Population GL-moments

The general expressions for the first Two GL-moments in the general and symmetric case are given as follows: **The general expressions for the first Two GL-moments, in the general case, in (9), are**

$$\Psi_1^{(g_1,g_2)} = \frac{\Gamma(g_1+g_2+2)}{\Gamma(g_1+1)\Gamma(g_2+1)}\int_0^1 x(F)F^{g_1}(1-F)^{g_2}\,dF$$

$$\Psi_2^{(g_1,g_2)} = \frac{\Gamma(g_1+g_2+3)}{2\,\Gamma(g_1+1)\Gamma(g_2+2)}\int_0^1 x(F)F^{g_1}(1-F)^{g_2} \times\left[\left(\frac{g_1+g_2+2}{g_1+1}\right)F-1\right]dF.$$

Illustrative Examples:

The first Two GL-moments for each of the following choices of g_1 and g_2

$g_1=0,\ g_2=0.5$, $g_1=0.5,\ g_2=0$, $g_1=0.5,\ g_2=1.5$ *and* $g_1=1.5,\ g_2=0.5$ are:

(1) $g_1=0,\ g_2=0.5$:

$$\Psi_1^{(0,0.5)} = \frac{\Gamma(2.5)}{\Gamma(1.5)}\int_0^1 x(F)(1-F)^{0.5}dF = \frac{\Gamma(2.5)}{\Gamma(1.5)}\,E\left[x(F)\,(1-F)^{0.5}\right],$$

$$\Psi_2^{(0,0.5)} = \frac{\Gamma(3.5)}{2\,\Gamma(2.5)}\int_0^1 x(F)(1-F)^{0.5}\,[2.5F-1]\,dF = \frac{\Gamma(3.5)}{2\Gamma(2.5)}\,E\left[x(F)\,(1-F)^{0.5}\,(2.5F-1)\right],$$

(2) $g_1=0.5,\ g_2=0$:

$$\Psi_1^{(0.5,0)} = \frac{\Gamma(2.5)}{\Gamma(1.5)}\int_0^1 x(F)F^{0.5}\,dF = \frac{\Gamma(2.5)}{\Gamma(1.5)}\,E\left[x(F)\ F^{0.5}\right],$$

$$\Psi_2^{(0.5,0)} = \frac{\Gamma(3.5)}{2\,\Gamma(2.5)}\int_0^1 x(F)F^{0.5}\left[\frac{2.5}{1.5}F-1\right]dF = \frac{\Gamma(3.5)}{2\Gamma(2.5)}\,E\left[x(F)\ F^{0.5}\left(\frac{2.5}{1.5}F-1\right)\right],$$

(3) $g_1=0.5,\ g_2=1.5$:

$$\Psi_1^{(0.5,1.5)} = \frac{\Gamma(4)}{\Gamma(1.5)\Gamma(2.5)}\int_0^1 x(F)F^{0.5}\ (1-F)^{1.5}dF = \frac{\Gamma(4)}{\Gamma(1.5)\Gamma(2.5)}\,E\left[x(F)\ F^{0.5}(1-F)^{1.5}\right],$$

$$\Psi_2^{(0.5,0)} = \frac{\Gamma(5)}{2\Gamma(1.5)\Gamma(3.5)} \times\int_0^1 x(F)F^{0.5}(1-F)^{1.5}\left[\frac{4}{1.5}F-1\right]dF = \frac{\Gamma(5)}{2\Gamma(1.5)\Gamma(3.5)} \times E\left[x(F)\,F^{0.5}\,(1-F)^{1.5}\left(\frac{4}{1.5}F-1\right)\right],$$

(4) $g_1=1.5,\ g_2=0.5$:

$$\Psi_1^{(1.5,0.5)} = \frac{\Gamma(4)}{\Gamma(1.5)\Gamma(2.5)}\int_0^1 x(F)F^{1.5}\ (1-F)^{0.5}dF = \frac{\Gamma(4)}{\Gamma(1.5)\Gamma(2.5)}\,E\left[x(F)\ F^{1.5}(1-F)^{0.5}\right],$$

$$\Psi_2^{(1.5,0.5)} = \frac{\Gamma(5)}{2\Gamma(2.5)\Gamma(2.5)} \times \int_0^1 x(F)F^{1.5}(1-F)^{0.5}\left[\frac{4}{2.5}F-1\right]dF = \frac{\Gamma(5)}{2\Gamma(2.5)\Gamma(2.5)} \times E\left[x(F)\ F^{1.5}(1-F)^{0.5}\left(\frac{4}{2.5}F-1\right)\right].$$

The general expressions for the first Two GL-moments, in the symmetric case, in (10), are

$$\Psi_1^{(g)} = \frac{\Gamma(2g+2)}{\Gamma(g+1)\Gamma(g+1)}\int_0^1 x(F)F^g(1-F)^g\,dF,$$

$$\Psi_2^{(g)} = \frac{\Gamma(2g+3)}{2\,\Gamma(g+1)\Gamma(g+2)} \times \int_0^1 x(F)F^g(1-F)^g\ [2F-1]\,dF.$$

Illustrative Examples:

As a simple example, where $g=0$, $g=0.5$ *and* $g=1.5$ the first four GL-moments are:

(1) $g=0$**:**

$$\Psi_1^{(0)} = \int_0^1 x(F)dF = E[x(F)],$$

$$\Psi_2^{(0)} = \int_0^1 x(F)\ [2F-1]\,dF = E[x(F)\,[2F-1]].$$

(2) $g=0.5$:

$$\Psi_1^{(0.5)} = \frac{\Gamma(3)}{\Gamma^2(1.5)}\int_0^1 x(F)F^{0.5}(1-F)^{0.5}dF = \frac{\Gamma(3)}{\Gamma^2(1.5)}E\left[x(F)\ F^{0.5}\ (1-F)^{0.5}\right],$$

$$\Psi_2^{(0.5)} = \frac{\Gamma(3)}{2\,\Gamma(1.5)\Gamma(2.5)} \times \int_0^1 x(F)F^{0.5}(1-F)^{0.5}[2F-1]\,dF = \frac{\Gamma(3)}{2\,\Gamma(1.5)\Gamma(2.5)} \times E\left[x(F)F^{0.5}(1-F)^{0.5}\ [2F-1]\right].$$

(3) $g=1.5$:

$$\Psi_1^{(1.5)} = \frac{\Gamma(5)}{\Gamma^2(2.5)}\int_0^1 x(F)F^{1.5}(1-F)^{1.5}\,dF = \frac{\Gamma(5)}{\Gamma^2(2.5)}E\left[x(F)\ F^{1.5}\ (1-F)^{1.5}\right],$$

$$\Psi_1^{(1.5)} = \frac{\Gamma(5)}{\Gamma^2(2.5)}\int_0^1 x(F)F^{1.5}(1-F)^{1.5}\,dF = \frac{\Gamma(5)}{\Gamma^2(2.5)}E\left[x(F)\ F^{1.5}\ (1-F)^{1.5}\right],$$

$$\Psi_2^{(1.5)} = \frac{\Gamma(6)}{2\,\Gamma(2.5)\Gamma(3.5)} \times \int_0^1 x(F)F^{1.5}(1-F)^{1.5}[2F-1]\,dF = \frac{\Gamma(6)}{2\,\Gamma(2.5)\Gamma(3.5)} \times E\left[x(F)F^{1.5}(1-F)^{1.5}\ [2F-1]\right].$$

2.4 Relations between GL-moments

In this section we use some recurrence relations of Jacobi polynomials to derive corresponding relations between GL-moments with different rational degrees.

Relation 1

For $r=0,1,2,\ldots$ and $g_1,g_2 \geq 0$

$$\Psi_{r+1}^{(g_1,g_2)} = \frac{(r+1)(2r+g_1+g_2+2)\Psi_{r+1}^{(g_1,g_2+1)}}{(r+1)(r+g_1+g_2+2)} + \frac{(r+2)(r+g_1+1)\Psi_{r+2}^{(g_1,g_2)}}{(r+1)(r+g_1+g_2+2)}, \tag{13}$$

thus

$$\Psi_{r+1}^{(g_1,g_2+1)} = \frac{(r+1)(r+g_1+g_2+2)\Psi_{r+1}^{(g_1,g_2)}}{(r+1)(2r+g_1+g_2+2)} - \frac{(r+2)(r+g_1+1)\Psi_{r+2}^{(g_1,g_2)}}{(r+1)(2r+g_1+g_2+2)}.$$

Proof. From [19], and [21] we find

$$H_n^{(\alpha+1,\omega)}(x) = \frac{2(n+\alpha+1)H_n^{(\alpha,\omega)}(x)}{(1-x)(2n+\alpha+\omega+2)} + \frac{-2(n+1)H_{n+1}^{(\alpha,\omega)}(x)}{(1-x)(2n+\alpha+\omega+2)}$$

where $n=0,1,2,\ldots$ and $\alpha,\omega \geq 0$.
This can be expressed in terms of shifted Jacobi polynomials as

$$H_r^{*(\alpha+1,\omega)}(F) = \frac{(r+\alpha+1)H_r^{(\alpha,\omega)}(F)}{(1-F)(2r+\alpha+\omega+2)} + \frac{-(r+1)H_{r+1}^{(\alpha,\omega)}(F)}{(1-F)(2r+\alpha+\omega+2)},$$

thus,

$$H_r^{*(\alpha,\omega)}(F) = \frac{(1-F)(2r+\alpha+\omega+2)H_r^{*(\alpha+1,\omega)}(F)}{(r+\alpha+1)} - \frac{(r+1)H_{r+1}^{*(\alpha,\omega)}(F)}{(r+\alpha+1)}$$

A direct substitution of this expression in (9) with replacing α,ω by g_1,g_2 respectively, in shifted Jacobi polynomials, gives relation 1.

Relation 2
For $r=0,1,2,\ldots$ and $g_1,g_2 \geq 0$

$$\Psi_{r+1}^{(g_1,g_2)} = \frac{(r+1)(2r+g_1+g_2+2)\Psi_{r+1}^{(g_1+1,g_2)}}{(r+1)(r+g_1+g_2+2)} - \frac{(r+2)(r+g_2+1)\Psi_{r+2}^{(g_1,g_2)}}{(r+1)(r+g_1+g_2+2)}, \quad (14)$$

thus,

$$\Psi_{r+1}^{(g_1+1,g_2)} = \frac{(r+1)(r+g_1+g_2+2)\Psi_{r+1}^{(g_1,g_2)}}{(r+1)(2r+g_1+g_2+2)} - \frac{(r+2)(r+g_2+1)\Psi_{r+2}^{(g_1,g_2)}}{(r+1)(2r+g_1+g_2+2)}.$$

Proof:
From [20], and [22] we find

$$H_n^{(\alpha,\omega+1)}(x) = \frac{2(n+\omega+1)H_n^{(\alpha,\omega)}(x)}{(1+x)(2n+\alpha+\omega+2)} + \frac{-2(n+1)H_{n+1}^{(\alpha,\omega)}(x)}{(1+x)(2n+\alpha+\omega+2)},$$

where $n=0,1,2,\ldots$ and $\alpha,\omega \geq 0$ this can be expressed in terms of shifted Jacobi polynomials as

$$H_r^{*(\alpha,\omega+1)}(F) = \frac{(r+\omega+1)H_r^{*(\alpha,\omega)}(F)}{(F)(2r+\alpha+\omega+2)} + \frac{(r+\omega+1)H_r^{*(\alpha,\omega)}(F)+(r+1)H_{r+1}^{*(\alpha,\omega)}(F)}{(F)(2r+\alpha+\omega+2)},$$

this can be expressed in terms of shifted Jacobi polynomials as

$$H_r^{*(\alpha,\omega)}(F) = \frac{(F)(2r+\alpha+\omega+2)H_r^{*(\alpha,\omega+1)}(F)}{(r+\omega+1)} - \frac{(r+1)H_{r+1}^{*(\alpha,\omega)}(F)}{(r+\omega+1)}.$$

A direct substitution of this expression in (9) gives relation 2.

For example, when $g_1=g_2=0.5$ we obtain the following relations

$$\Psi_{r+1}^{(1.5,0.5)} = \frac{(r+3)}{(2r+3)}\Psi_{r+1}^{(0.5,0.5)} + \frac{(r+2)(r+1.5)}{(r+1)(2r+3)}\Psi_{r+2}^{(0.5,0.5)},$$

and

$$\Psi_{r+1}^{(0.5,1.5)} = \frac{(r+3)}{(2r+3)}\Psi_{r+1}^{(0.5,0.5)} + \frac{(r+2)(r+1.5)}{(r+1)(2r+3)}\Psi_{r+2}^{(0.5,0.5)},$$

adding both results gives

$$\Psi_{r+1}^{(1.5,0.5)} + \Psi_{r+1}^{(0.5,1.5)} = \frac{2(r+3)}{(2r+3)}\Psi_{r+1}^{(0.5,0.5)},$$

Subtracting the second from the first gives

$$\Psi_{r+1}^{(1.5,0.5)} - \Psi_{r+1}^{(0.5,1.5)} = \frac{2(r+2)(r+1.5)}{(r+1)(2r+3)}\Psi_{r+1}^{(0.5,0.5)}.$$

These relations also give the relationship between L-moments and TL-moments. For example, when $g_1=g_2=0$ we obtain the following relations

$$\Psi_{r+1}^{(1,0)} = \frac{(r+2)}{(2r+2)}\left[\Psi_{r+1}^{(0,0)} + \Psi_{r+2}^{(0,0)}\right],$$

and

$$\Psi_{r+1}^{(0,1)} = \frac{(r+2)}{(2r+2)}\left[\Psi_{r+1}^{(0,0)} + \Psi_{r+2}^{(0,0)}\right],$$

adding both results gives

$$\Psi_{r+1}^{(1,0)} + \Psi_{r+1}^{(0,1)} = \frac{(r+2)}{(r+1)}\Psi_{r+1}^{(0,0)},$$

subtracting the second from the first gives

$$\Psi_{r+1}^{(1,0)} - \Psi_{r+1}^{(0,1)} = \frac{(r+2)}{(r+1)}\Psi_{r+2}^{(0,0)}.$$

Different relations could be obtained for different rationales.

2.5 Representation of Population GL- moments as U-statistics

We may represent GL-moments as linear functions of the U-statistics $\chi_{p:n}$ represented by [22], where

$$E(X_{p+1:n} - X_{p:n}) = \binom{n}{p}\int_x F_x^p(1-F_x)^{n-p}dx = \chi_{p:n}, \quad (15)$$

hence, we have

$$\int_x F_x^p (1-F_x)^{n-p} dx = \frac{\chi_{p:n}}{\binom{n}{p}}. \quad (16)$$

To re-express the GL-moments in terms of U-statistics we use the following integral, for the shifted Jacobi polynomials,

$$\int_0^1 F^{\omega}(1-F)^{\alpha} H_r^{*(\alpha,\omega)}(F)\ dF = \frac{F^{\omega+1}(1-F)^{\alpha+1}\ H_{r-1}^{*(\alpha+1,\omega+1)}}{r}, \quad (17)$$

(see, [19]). If the quantile function $x(F)$ is differentiable, we can integrate (9) by parts using (17) to obtain

$$\Psi_{r+1}^{(g_1,g_2)} = \frac{(r-1)!\Gamma(r+g_1+g_2+2)}{(r+1)\Gamma(r+g_1+1)\Gamma(r+g_2+1)} \times \int_0^1 x'(F) F^{g_1+1}(1-F)^{g_2+1} H_n^{*(g_2+1,g_1+1)}(F)\ dF$$
$$= \frac{(r-1)!\Gamma(r+g_1+g_2+2)}{(r+1)\Gamma(r+g_1+1)\Gamma(r+g_2+1)} \times E\left[x'(F)\ F^{g_1+1}(1-F)^{g_2+1} H_{r-1}^{*(g_2+1,g_1+1)}(F)\right] \quad (18)$$

where $x'(F) = 1/f(x)$ is the $1st$ derivative of the quantile function, we may write the GL-moments as a function of U-statistics for$r = 1,2,3,\ldots$ using (18) as follows.

$$\Psi_{r+1}^{(g_1,g_2)} = \frac{(r-1)!\Gamma(r+g_1+g_2+2)}{(r+1)\Gamma(r+g_1+1)\Gamma(r+g_2+1)} \times \int_x F^{g_1+1}(1-F)^{g_2+1} H_{r-1}^{*(g_2+1,g_1+1)}(F)\ dx$$
$$= \frac{1}{(r+1)} \sum_{j=0}^{r-1} (-1)^{r-j-1} \binom{r-1}{j} \times \binom{r+g_1+g_2+1}{g_1+j+1} \times \int_x F_x^{g_1+j+1}(1-F_x)^{g_2+r-j}\ dx$$
$$= \frac{1}{(r+1)} \sum_{j=0}^{r-1} (-1)^{j} \binom{r-1}{r-j-1} \times \binom{r+g_1+g_2+1}{r+g_1-j} \times \int_x F_x^{r+g_1-j}(1-F_x)^{g_2+j+1}\ dx$$
$$= \frac{1}{(r+1)} \sum_{j=0}^{r-1} (-1)^{j} \binom{r-1}{r-j-1} \times \binom{r+g_1+g_2+1}{r+g_1-j} \frac{\chi_{r+g_1-j:r+g_1+g_2+1}}{\binom{r+g_1+g_2+1}{r+g_1-j}}$$
$$= \frac{1}{(r+1)} \sum_{j=0}^{r-1} (-1)^{j} \binom{r-1}{r-j-1} \times \chi_{r+g_1-j:r+g_1+g_2+1} \quad (19)$$

For $r \geq 1$. This presentation in terms of the U-statistics facilitates the interpretation and understanding of the GL-moments.

2.6 GL-Mean and GL-Variance

We define the **GL-mean** $G\mu^{(g_1,g_2)}$ and the **GL-variance** $GV^{(g_1,g_2)}$ which are the generalization of the standard population mean and population variance in terms of GL-moments.
The **GL-mean** and **GL-variance** can be defined as

$$G\mu^{(g_1,g_2)} = \Psi_1^{(g_1,g_2)} = \frac{\Gamma(g_1+g_2+2)}{\Gamma(g_1+1)\Gamma(g_2+1)} \times E\left[x(F)\ F^{g_1}(1-F)^{g_2}\right]. \quad (20)$$

And

$$GV^{(g_1,g_2)} = \sum_{r=1}^{\infty} \frac{(r+1)^2\Gamma(2r+g_1+g_2+1)}{r!(r+g_1+g_2+1)^2\Gamma(r+g_1+g_2+1)} \times \Gamma(r+g_1+1)\Gamma(r+g_2+1) \quad (21)$$

for$g_1,g_2 \geq 0$.

Proof:
When r=0 , we obtain the GT-mean from (9) as

$$\mu^{(g_1,g_2)} = \Psi_1^{(g_1,g_2)} = \frac{\Gamma(g_1+g_2+2)}{\Gamma(g_1+1)\Gamma(g_2+1)} \times E\left[x(F)\ F^{g_1}(1-F)^{g_2}\right].$$

This is the usual mean weighted by a power of the cumulative distribution function multiplied by a constant. In case the distribution is symmetric and $g_1 = g_2 = g$, $\mu^{(g)} = \mu = E(X)$ *is the population mean*

$$\mu = \mu^{(g)} = \Psi_1^{(g)} = \frac{\Gamma(2g+2)}{\Gamma^2(g+1)} E\left[x(F)\ F^{g}(1-F)^{g}\right].$$

Following Sillitto [24] *and Hosking* [2] *we find that*

$$0 = lim_{S\to\infty} \int_0^1 R^2(F)\,dF = \int_0^1 x^2(F)\ F^{g_1}(1-F)^{g_2}dF - \sum_{r=0}^{\infty} \frac{(r+1)^2(2r+g_1+g_2+1)}{r!(r+g_1+g_2+1)^2\Gamma(r+g_1+g_2+1)} \times \Gamma(r+g_1+1)\Gamma(r+g_2+1)\Psi_{r+1}^{2(g_1,g_2)} \quad (22)$$

This can be written as

$$0 = \int_0^1 x^2(F)\ F^{g_1}(1-F)^{g_2}dF - \frac{\Gamma(g_1+1)\Gamma(g_2+1)}{(g_1+g_2+1)\ \Gamma(g_1+g_2+1)}\Psi_1^{2(g_1,g_2)} - \sum_{r=1}^{\infty} \frac{(r+1)^2(2r+g_1+g_2+1)}{r!(r+g_1+g_2+1)^2\Gamma(r+g_1+g_2+1)} \times \Gamma(r+g_1+1)\Gamma(r+g_2+1)\Psi_{r+1}^{2(g_1,g_2)}$$

hence,

$$\int_0^1 x^2(F)\, F^{g_1}(1-F)^{g_2} dF - \frac{\Gamma(g_1+1)\Gamma(g_2+1)}{(g_1+g_2+1)\,\Gamma(g_1+g_2+1)} \Psi^{2\,(g_1,g_2)}_{1} = \sum_{r=1}^{\infty} \frac{(r+1)^2(2r+g_1+g_2+1)}{r!(r+g_1+g_2+1)^2\Gamma(r+g_1+g_2+1)} \times \Gamma(r+g_1+1)\Gamma(r+g_2+1)\Psi^{2\,(g_1,g_2)}_{r+1},$$

thus, we obtain

$$\begin{aligned} GV^{(g_1,g_2)} &= \int_0^1 x^2(F)\, F^{g_1}(1-F)^{g_2} dF - \frac{\Gamma(g_1+1)\Gamma(g_2+1)}{(g_1+g_2+1)\,\Gamma(g_1+g_2+1)} \Psi^{2\,(g_1,g_2)}_{1} \\ &= \int_0^1 x^2(F)\, F^{g_1}(1-F)^{g_2} dF - \frac{\Gamma(g_1+g_2+2)}{\Gamma(g_1+1)\Gamma(g_2+1)}\left[\int_0^1 x(F)\, F^{g_1}(1-F)^{g_2} dF\right]^2 \\ &= \sum_{r=1}^{\infty} \frac{(r+1)^2(2r+g_1+g_2+1)}{r!(r+g_1+g_2+1)^2\Gamma(r+g_1+g_2+1)} \times \Gamma(r+g_1+1)\Gamma(r+g_2+1)\Psi^{2\,(g_1,g_2)}_{r+1}. \end{aligned}$$

When $g_1, g_2 = 0$ we obtain the population variance in terms of the L-moments; as

$$GV^{(0,0)} = \sigma^2 = \int_0^1 x^2(F)dF - \Psi_1^2 = \sum_{r=1}^{\infty} \Psi_{r+1}^2,$$

(see, [23] and [2]). It is clear from the representation of GL-variance in terms of GL-moments that $GV^{(g_1,g_2)} > 0$.

2.7 Sample GL-moments

In this section we introduce two methods of estimating the population GL-moments; the first method gives nearly unbiased estimators, and the estimators of the second method are biased.

(1) Nearly Unbiased Estimator **Definition 2.** Let $x_{1:n} \le x_{2:n} \le \cdots \le x_{n:n}$ denote the sample order statistics of a random sample $x_1, x_2, \ldots, x_n$ of size from the population. The sample GL-moments are given by

$$\begin{aligned} \psi^{(g_1,g_2)}_{r+1} &= (r+1)^{-1}\textstyle\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} \\ &\quad \times \frac{\Gamma(r+g_1+g_2+2)}{\Gamma(g_1+j+1)\Gamma(g_2+r-j+1)} \\ &\quad \times \textstyle\sum_{i=1}^{n} \frac{\Gamma(i)\Gamma(n-i+1)\Gamma(n-g_1-g_2-r)}{\Gamma(n+1)\Gamma(n-i-g_2-r+j+1)\Gamma(i-j-g_1)} x_{i:n} \\ &= (r+1)^{-1}\textstyle\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} \\ &\quad \times \textstyle\sum_{i=1}^{n} \frac{\binom{i-1}{g_1+j}\binom{n-i}{r+g_2-j}}{\binom{n}{r+g_1+g_2+1}} x_{i:n}\,, \\ &\quad r = 0,1,\ldots,n,\ \ g_1,g_2 \ge 0 \end{aligned} \tag{23}$$

Proof: It is known that

$$\begin{aligned} E(X_{k:r}) &= \frac{r!}{(k-1)!\,(r-k)!}\int_0^1 x(F)F^{k-1}(1-F)^{r-k}\,dF \\ &= \frac{\Gamma(r+1)}{\Gamma(k)\Gamma(r-k+1)} E\left[x(F)\ F^{k-1}(1-F)^{r-k}\right], \end{aligned}$$

(see;[24]. Thus

$$E\left[x(F)\ F^{k-1}(1-F)^{r-k}\right] = \frac{\Gamma(k)\Gamma(r-k+1)}{\Gamma(r+1)} E(X_{k:r}), \tag{24}$$

replacing r-j in place j in (11) and using binomial coefficient properties

$$\begin{aligned} \Psi^{(g_1,g_2)}_{r+1} &= (r+1)^{-1}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} \\ &\times \frac{\Gamma(r+g_1+g_2+2)}{\Gamma(g_1+j+1)\Gamma(r+g_2-j+1)} \\ &\times E\left[x(F)\ F^{g_1+j}(1-F)^{r+g_2-j}\right], \end{aligned}$$

then, the estimated GL-moments become

$$\begin{aligned} \psi^{(g_1,g_2)}_{r+1} &= (r+1)^{-1}\sum_{j=0}^{r}(-1)^{r-j}\binom{r}{j} \\ &\times \frac{\Gamma(r+g_1+g_2+2)}{\Gamma(g_1+j+1)\Gamma(r+g_2-j+1)} \\ &\times \hat{E}\left[x(F)\ F^{g_1+j}(1-F)^{r+g_2-j}\right]. \end{aligned} \tag{25}$$

From ([24]) we get

$$\begin{aligned} E\left[x(F)\ F^{g_1+j}(1-F)^{r+g_2-j}\right] &= \frac{\Gamma(g_1+j+1)}{\Gamma(r+g_1+g_2+2)} \\ &\times \frac{\Gamma(r+g_2-j+1)}{\Gamma(r+g_1+g_2+2)} \times E\left(X_{g_1+j+1:r+g_1+g_2+1}\right). \end{aligned} \tag{26}$$

But, by using Downton's estimator as

$$E\left(X_{k+1:k+l+1}\right) = \frac{1}{\binom{n}{k+l+1}}\sum_{i=1}^{n}\binom{i-1}{k}\binom{n-i}{l} X_{i:n},$$

then

$$E\left(X_{g_1+j+1:r+g_1+g_2+1}\right)=\frac{1}{\binom{n}{r+g_1+g_2+1}}\times\sum_{i=1}^{n}\binom{i-1}{g_1+j}\binom{n-i}{r+g_2-j}X_{i:n},\tag{27}$$

substituting (27) in (26); the estimator of $E\left[x(F)\ F^{g_1+j}(1-F)^{r+g_2-j}\right]$ is

$$\hat{E}\left[x(F)\ F^{g_1+j}(1-F)^{r+g_2-j}\right]=\sum_{i=1}^{n}\frac{\Gamma(i)\Gamma(n-i+1)\Gamma(n-g_1-g_2-r)}{\Gamma(n+1)\Gamma(n-i-g_2-r+j+1)\Gamma(i-j-g_1)}x_{i:n}.\tag{28}$$

Substituting (28)in (26); we obtain the sample GL-moments that shown in (25).
Under the condition

$$\sum_{i=1}^{n}\binom{n-g_1-g_2-r-1}{i-1-j-g_1}\times F^{i-1-j-g_1}(1-F)^{n-i-g_2-r+j}=1,$$

the sample GL-moments, in (24), are said to be "*nearly unbiased*" estimators of the corresponding population GL-moments, in (7). This can be shown as follows

$$E\left[\sum_{i=1}^{n}\frac{\Gamma(i)\Gamma(n-i+1)\Gamma(n-g_1-g_2-r)}{\Gamma(n+1)\Gamma(n-i-g_2-r+j+1)\Gamma(i-j-g_1)}x_{i:n}.\right]$$
$$=\sum_{i=1}^{n}\frac{\Gamma(n-g_1-g_2-r)}{\Gamma(n-i-g_2-r+j+1)\Gamma(i-j-g_1)}\times\int_0^1 x(F)F^{i-1}(1-F)^{n-i}\,dF$$
$$=\sum_{i=1}^{n}\frac{\Gamma(n-g_1-g_2-r)}{\Gamma(n-i-g_2-r+j+1)\Gamma(i-j-g_1)}\times\int_0^1 x(F)F^{g_1+j}(1-F)^{r+g_2-j}F^{i-1-j-g_1}\times(1-F)^{n-i-g_2-r+j}\,dF$$
$$=\int_0^1 x(F)F^{g_1+j}(1-F)^{r+g_2-j}\times\sum_{i=0}^{n}\binom{n-g_1-g_2-r-1}{i-1-j-g_1}\times F^{i-1-j-g_1}(1-F)^{n-i-g_2-r+j}\,dF$$
$$=E\left[\int_0^1 x(F)F^{g_1+j}(1-F)^{r+g_2-j}df\right]$$

Note, It possible to write the equation(25) as following

$$\psi_{r+1}^{(g_1,g_2)}=\sum_{i=1}^{n}w_i(r,g_1,g_2)\,x_{i:n},$$

where

$$w_i(r,g_1,g_2)=\sum_{j=0}^{r}\frac{(-1)^{r-j}\binom{r}{j}\binom{i-1}{g_1+j}\binom{n-i}{r+g_2-j}}{(r+1)\binom{n}{r+g_1+g_2+1}},$$
$$r=0,1,\ldots,\ i=1,2,\ldots,n,\ g_1,g_2\geq 0$$

is the weights function.

Some values of these weights are given in table (1). From this table we see that the trimmed L-moments assign zero weights for the extreme observation, but sample GL- moments assigns small weight for the extreme observations which capture more information about the tail of the distribution. $\boldsymbol{w_1}$

Table 1: The weights of the sample GL-moments $\psi_1^{(g_1,g_2)}$, $\psi_2^{(g_1,g_2)}$, and $\psi_3^{(g_1,g_2)}$ for sample size $n=10$ and different choices of g_1 andg_1.

Est^*	w_1	w_2	w_3	w_4	w_5
$\psi_1^{(0,0)}$	0.100	0.100	0.100	0.100	0.100
$\psi_1^{(1,1)}$	0	0.067	0.117	0.150	0.167
$\psi_1^{(.99,.99)}$	0.001	0.067	0.117	0.149	0.166
$\psi_2^{(0,0)}$	-.100	-.077	-.055	-.033	-.011
$\psi_2^{(1,1)}$	0	-.066	-.083	-.064	-.024
$\psi_3^{(.99,.99)}$	-.001	-.067	-.083	-.064	-.023
	w_6	w_7	w_8	w_9	w_{10}
$\psi_1^{(0,0)}$	0.100	0.100	0.100	0.100	0.100
$\psi_1^{(1,1)}$	0.167	0.150	0.117	0.067	0
$\psi_1^{(.99,.99)}$	0.166	0.149	0.117	0.067	0.001
$\psi_2^{(0,0)}$	0.011	0.033	0.055	0.078	0.100
$\psi_2^{(1,1)}$	0.024	0.064	0.083	0.067	0
$\psi_3^{(.99,.99)}$	0.023	0.064	0.083	0.067	0.001

Est^*: The sample GL-moments.

2.8 The First Two Sample GL-moments

noindent We can obtain the first two sample GL-moments, from (25), as follows.

$$\psi_1^{(g_1,g_2)}=\frac{1}{\binom{n}{g_1+g_2+2}}\sum_{i=1}^{n}\left[\binom{i-1}{g_1}\binom{n-i}{g_2}\right]x_{i:n},\tag{29}$$

and

$$\psi_2^{(g_1,g_2)}=\frac{1}{2\binom{n}{g_1+g_2+2}}\times\sum_{i=1}^{n}\left[\binom{i-1}{g_1+1}\binom{n-i}{g_2}-\binom{i-1}{g_1}\binom{n-i}{g_2+1}\right]x_{i:n}.\tag{30}$$

the first two sample GL-moments for the first two sample GL-moments, in the symmetric case,

$$\psi_1^{(g)} = \frac{1}{\binom{n}{2g+2}} \sum_{i=1}^{n} \left[\binom{i-1}{g} \binom{n-i}{g} \right] x_{i:n}, \quad (31)$$

and

$$\psi_2^{(g)} = \frac{1}{2\binom{n}{2g+2}}$$
$$\times \sum_{i=1}^{n} \left[\binom{i-1}{g+1}\binom{n-i}{g} - \binom{i-1}{g}\binom{n-i}{g+1} \right] x_{i:n}. \quad (32)$$

3 Examples of GL-moments for Symmetric Distributions

In this section we give the first two GL-moments, in the symmetric case,$g_1 = g_2 = g$, and estimate the parameters using GL-moments of Symmetric Generalized Lambda Distribution and normal distribution.

3.1 The Symmetric Generalized Lambda Distribution (GLD) with Three Parameters m,a and b

If X is distributed according to the generalized lambda distribution (GLD), then the quantile function is

$$x(F) = m + \frac{F^b - (1-F)^d}{a}, \qquad 0 < F < 1,$$

where m is a location parameter, a is a scale parameter and b and d are a shape parameters (see, [26,27]).
In the symmetric case, where $b = d$, the quantile function is given by

$$x(F) = m + \frac{F^b - (1-F)^b}{a}, \quad 0 < F < 1, \quad (33)$$

by substituting(29) in (12), we get the rth population GL-moment for the symmetric lambda distribution as

$$\begin{aligned} \Psi_{r+1}^{(g)} &= (r+1)^{-1} \sum_{j=0}^{r} (-1)^j \binom{r}{j} \\ &\times \frac{\Gamma(r+2g+2)}{\Gamma(r+g-j+1)\Gamma(g+j+1)} \\ &\times m\beta(r+g-j+1,\, g+j+1) \\ &+ a^{-1}\beta(b+r+g-j+1,\, g+j+1) \\ &- \beta(r+g-j+1,\, b+g+j+1), \end{aligned} \quad (34)$$

Where $\beta(a,\, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$, $g \geq 0$ and $r = 0,1,2,\ldots$

Thus, the first two GL-moments for symmetric lambda distribution are

$$\Psi_1^{(g)} = m,$$

and

$$\begin{aligned} \Psi_2^{(g)} &= \frac{a^{-1}\Gamma(2g+3)}{\Gamma(b+2g+3)} \left[\frac{\Gamma(b+g+2)}{\Gamma(g+2)} - \frac{\Gamma(b+g+1)}{\Gamma(g+1)} \right] \\ &= \frac{a^{-1}b\,\Gamma(2g+3)\Gamma(b+g+1)}{\Gamma(g+2)\Gamma(b+2g+3)}. \end{aligned}$$

By any method of moments, the parameters are estimated by equating the population rth moment with the corresponding sample moment, we obtain as many equations as the number of unknown parameters. The estimators are simply the solution of the system of equations obtained. Thus, the GL-moments estimators of the parameters $\boldsymbol{m}, \boldsymbol{b}$ and $\boldsymbol{a}$ denoted $\widehat{\boldsymbol{m}}, \widehat{\boldsymbol{b}}$ and $\widehat{\boldsymbol{a}}$, respectively, are given by

$$\hat{m} = \psi_1^{(g)},$$

$$\hat{b} = \frac{\left(-B \pm \sqrt{B^2 - 4AC}\right)}{2A},$$

where

$$\begin{aligned} A &= \left[\psi_4^{(g)}(g+3) - \psi_2^{(g)}(2g+3) \right], \\ B &= \left[\psi_4^{(g)}(g+3)(4g+7) - 3\psi_2^{(g)}(2g+3) \right], \\ C &= \left[\psi_4^{(g)}(g+3)\left(4g^2+14g+12\right) - 2\psi_2^{(g)}(2g+3) \right]. \end{aligned}$$

The close value for $\hat{b}$ must satisfy the condition on b.

$$\hat{a} = \frac{\hat{b}\,\Gamma(2g+3)\Gamma(\hat{b}+g+1)}{\psi_2^{(g)}\Gamma(g+2)\Gamma(\hat{b}+2g+3)}.$$

The four-parameter GLD is useful for modeling probability distributions and representing data when the underlying distribution is known (see,[28]) and provides good approximations to other well-known distributions. For example, the distribution with $m = 0$, $a = 0.1975$, $b = d = 0.1349$ results in an approximation to the standard normal distribution (see,[29]).

3.2 The Normal Distribution

If X is distributed according to the normal distribution, then

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp \left[\frac{-1}{2} \left(\frac{x-\mu}{\sigma} \right)^2 \right],$$

$$-\infty < x < \infty,\ -\infty < \mu < \infty,\ \sigma > 0$$

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} exp\left[\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right],$$

$x(F)$has no explicit analytical form. For the normal distribution, using the Generalized Lambda Distribution (GLD), approximations introduced by [29] can be used to obtain the first two GL-moments, when$g = 0, 0.5, 1, 1.5$, along with estimating the parameters.
Thus, by the approximations of GLD when $m = 0$, $a = 0.1975$ and $b = d = 0.1349$ and $g = 0, 0.5, 1, 1.5$ GL-moments estimators of μ, σ for the normal distribution (the details are given in appendix) are

$$\widehat{\mu} = \psi_1^{(0)} = \psi_1^{(0.5)} = \psi_1^{(1)} = \psi_1^{(1.5)}$$

$$\widehat{\sigma} = \frac{\psi_2^{(0)}}{0.5638} = \frac{\psi_2^{0.5}}{0.3885} = \frac{\psi_2^{1}}{0.2962} = \frac{\psi_2^{1.5}}{0.2393}. \quad (35)$$

4 A literature review of control charts

4.1 Statistical process control charts

A control chart is a valuable method in controlling the development process to detect process changes and recognize unexpected situations. The usage of attribute control charts occurs when measuring objects against a criterion and classifying them as either reaching or dropping short of the standard. Walter Shewhart designed the first control chart and proposed the following general model for control charts. Let τ be a sample statistic that measures some quality characteristic of interest, and suppose that the mean of τ is μ_τ and the standard deviation of τ is σ_τ. Then the center line (CL), the upper control limit (UCL) and the lower control limit (LCL) in the relevant control chart are defined as follows:

$$\begin{cases} LCL = \mu_\tau - k\ \sigma_\tau, \\ CL = \mu_\tau, \\ UCL = \mu_\tau + k\ \sigma_\tau, \end{cases} \quad (36)$$

Usually, the value k equals 3 is used due to the Normal distribution approximation, corresponding a control region $= 0.9973$ and $ARL_0 = 370$. To set control limits we must estimate μ_τ and σ_τ based on m samples of size n obtained from the process when it is under control. In line with the presumption that all measured data are independent and come from the same normal distribution while the process is under control. We can presume that the process mean and variance are unchanged.

4.2 Control chart for monitoring fractional, rates and proportions data based on Binomial distribution

For simplification, we can get a constant indefinitely broad manufacturing mechanism consistent, it will consist of generated products of varying characteristics, and the performance will be either conforming (0) or non-conforming (1). Considering a random sample of n items, as n independent Bernoulli trials, we obtain n stochastically independent random variables X_i (with $i = 1, \ldots, n$). Each X_i follows a zero-one distribution with $P(X_i = 0) = 1-p$ and $P(X_i = 1) = p$, where p [0, 1] holds. The number of non-conforming items in the sample, $X = \sum_{i=1}^{n} X_i$ is then defined as Binomial-distributed random variable X with $\mathrm{E}[X] = np$, $\mathrm{Var}(X) = np(1-p)$ as well as the fraction of nonconforming substance, which can be calculated by the provided data, by the sample fraction nonconforming:

$$\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n} \quad (37)$$

Since the estimator (37) can be obtained from the Binomial distribution, its mean is given by $\mathrm{E}[\hat{p}] = p$ and its variance by $\mathrm{Var}(\hat{p}) = p(1-p)/n$. Selecting m random samples each of size
n, where $X_1, \ldots, X_m$ (with X_j as the number of non-conforming items in the j-th sample, $j = 1, \ldots, m$) are stochastically independent, We measure the average of the actual sample fractions that were substantially non-conforming as

$$\overline{p} = \frac{\sum_{j=1}^{m} \hat{p}_j}{m} \quad (38)$$

the Binomial p-chart is defined as follows (compare for further details Montgomery,[30]):

$$UCL = min\left[1,\ \overline{p} + k\sqrt{\overline{p}(1-\overline{p})/n}\right]$$

$$CL = \overline{p}$$

$$LCL = max\left[0,\ \overline{p} - k\sqrt{\overline{p}(1-\overline{p})/n}\right]$$

4.3 Existing Literature Overview and Study Holes

Control charts for fraction nonconforming as a classical method of statistical process control are commonly applied in numerous branches such as industrial production, transactional market, service industry, or recently in the health care sector (e.g., the volume of virus C or B in blood). There are some guidelines that deal with the suppositions of symmetry and normal distribution approximation. Schader and Schmid [15] counseled that a normal approximation to the binomial distribution is satisfactory if two rules are satisfied: (i) $np\ (1-\ p) \geq 9$ and (ii) $np \geq 5$ when $0\ < p\ \leq\ 0.5\ \leq\ (1-p)$. further, [31]described an approximation as satisfactory, when the size of p is in the range $(0.3\ \leq\ p\ \leq\ 0.7)$ and n is extremely large for $np \geq 5$ and $n(1-p) \geq n$, the variance $p\ (1-p)$ remains constant, while[30] indicated that a Normal approximation to the Binomial distribution is satisfactory when $np \geq 10$ and p is inside the range $(0.1\ \leq\ p\ \leq\ 0.9)$. In lots of research the p-Charts used are in conditions where the parameter p is considered small $(i.e.\ p\ =\ 0.001;\ 0.01;\ 0.05;\ 0.1;\ etc.)$. In these cases, a Binomial distribution is quite skewed and the approximation by a Normal distribution is not satisfactory, as it allows values negative or greater than one. Quesenberry [32] introduced a Binomial Q Chart to monitor nonconforming proportion using a nonlinear transformation for the control limits and demonstrated that it approximates the Normal distribution closer to the Binomial. Heimann [33] proposed a modification of the p-Chart control limits for large sample sizes $(n\ >\ 10,000)$, noting that in this case the control limits are narrow, thus the false alarm rate increases. Ryan and Schwertman[34] introduced a adaptation of the np-Chart control limits to fit on the Normal approximation when $p\ <\ 0.03$, while Chen[17] introduced an adjustment to the p-Chart control limits and compared them with the traditional p-Chart and the Binomial Q Chart using the false alarm rate. Sim and Lim [35] adapted the attribute control charts to monitor zero-inflated data and used the Blyth-Still interval with 3-sigma to calculate control limits if this data follows a Binomial and Poisson distribution. Bourke [36] compared the performance of four control charts by monitoring shifts of the nonconforming proportion in industrial processes. He noted similarities in the performance of the Synthetic control chart and np-Chart over a long-time period of in-control process. With similar purposes, Aebtarm and Bouguila [37] compared the performance with eleven control charts for monitoring defects with Poisson distribution. Anna and Caten [38] introdced control charts for proportion non-conforming, where the control limits are based on the Beta distribution. Mohammed et al.[39]discuss p-charts in the light of very large sample sizes and present solutions for the problem of narrow control limits. There are also many works regarding np-charts, which are equivalent to p-charts when the sample size n is constant (see, [40,41,42]). The addressed efforts strengthen the capability of the Binomial p-chart commonly, though not enough when tracking proportions non-conforming in high-yield processes (see[43]). Moreover, this process control area is governed by so-called cumulative count statistics dependent on the Geometric distribution. (see, for example, [44,45]) or, more generally, the negative Binomial distribution (see, for example, [46,47,48]). Moreover, there are also p-charts for monitoring high-yield processes, such as a p chart based on an adjusted confidence interval (Wang, [43]) or an improved p-chart using the Cornish–Fisher quantile correction [49]. Further related works are given by [50,51,52].

In addition, We will like to comment on the most current works regarding control charts relating to tracking production processes. Haridy, et al.[53] discuss a Binomial EWMA chart based on a curtailment method to monitor the fraction of non-conforming items. Lima-Filho and Bayer[19] present control charts for fraction non-conforming, where the control limits are based on the Kumaraswamy distribution. Argoti and Carrion-Garcia [54] present a heuristic method to obtain quasi ARL-unbiased p-charts. In [55], the Beta distribution is used for the construction of a Beta regression control chart for monitoring fractions non-conforming. Lee Ho, et al. [56] are concerned with control charts for fractions non-conforming when the monitored proportions are not results of Bernoulli experiments. Lima-Filho et al.[19] present control charts for fraction non-conforming, where the control limits are based on the inflated beta distribution. Aslam et al. [57] suggested an optimum mixed attribute variable control chart for the Weibull distribution in an accelerated hybrid censoring scheme keeping the advantages of both attribute and variable control charts. Prior studies are based on developing elements of the binomial chart or the alternatives to it. Nevertheless, the chart's most significant point is the specification control limit in phase I analysis. It is mostly neglected or solved using approximations. To avoid an upper control limit exceeding one in p-chart, we propose a control chart based on generalized linear moments for monitoring fractional, rates, and proportions data.

5 Determination of control limits

When constructing p-chart based on process data, m regularly spaced samples, each of size n, must be taken to estimate the process Centre and spread. To set control limits we must estimate μ_τ and σ_τ based on m samples of size n obtained from the process when it is under control. In line with the presumption that while the mechanism is under operation, all measured data are independent and come from the same normal distribution, we can presume that the process mean and variance are unchanged.

To avoid an upper control limit exceeding one, here, the control limits of the p-chart will be modified by replacing $\overline{p}_j$ with trimmed value. The trimmed value is defined as (29). Thus, the procedure involves trimming g of the order observations from each tail and computing the mean of the remaining observations.

The steps required to compute the proposed control limits are summarized as follows:

Step 1. Select the values of $g_1 and\ g_2$.

Step 2. Compute $\hat{p}_j$ value, rank it and eliminate the r smallest and the r largest values.

Step 3. Compute the overall GL-mean as follows:

$$\overline{P}(g_1,g_2) = \psi_1^{(g_1,g_2)} = \frac{1}{\left(\binom{n}{g_1+g_2+2}\right)} \times \sum_{i=1}^{n}\left[\left(\binom{i-1}{g_1}\right)\left(\binom{n-i}{g_2}\right)\right]x_{i:n},$$

Step 4. Compute the control limits for the process as follows:

$$UCL = \overline{P}(g_1,g_2) + k\sqrt{\overline{P}(g_1,g_2)(1-\overline{P}(g_1,g_2))/(n-g_1-g_2)},$$

$$CL = \overline{P}(g_1,g_2),$$

$$LCL = \overline{P}(g_1,g_2) - k\sqrt{\overline{P}(g_1,g_2)(1-\overline{P}(g_1,g_2))/(n-g_1-g_2)}.$$

Where the value k equals 3 is used due to the Normal distribution approximation.

6 Average run length of proposed control charts

Control limits for the proposed chart are based upon the assumption that the process outcomes are normal approximation to the binomial distribution and independently distributed. In truth, the observation can be made without any likelihood distribution. Processes from distributions with heavy tails than a standard distribution appear to have more points beyond the control limits for example, average run length (ARL). Disturbing variations from normalcy are not uncommon. We examine defects resulting from implementing the proposed method for nonnormal data.
We consider observations generated from Kumaraswamy, Beta and uniform distributions. Table 2 contains results for independent observations generated from Kumaraswamy, Beta and uniform distributions. The control limits for the non-trimmed and trimmed control charts in Table 2 was computed by averaging the results of 150 replications of $m = 20$ random samples of size $n =$ 5 observations from each distribution. The trimmed charts limits were based on trimming 3 out of 20 sample values of $\hat{p}_j$. An additional 1000 samples of size 5 were generated from each distribution, and the number of points falling outside the control limits an estimate of the ARL was noted.
Table (2) reveals that the non-trimmed and trimmed control charts' simulated control limits are identical when the process generates uniform data. On the other hand, the Kumaraswamy and Beta distributions results differ: the trimmed charts are far tighter than the corresponding limits for the non-trimmed charts. The effect of these tighter limits is that more points fall outside, signaling the need for appropriate corrective measures when extreme departures from normality are encountered.

Table 2: Average Run Length (out of 1000) and UCL and LCL for samples of size 5 from some distributions

distribution	% trimming	UCL	LCL	ARL
Kumaraswamy	non-trimmed	0.851	0.935	17
	15% trimmed	0.871	0.923	20
Beta	non-trimmed	0.731	0.832	14
	15% trimmed	0.744	0.819	16
Uniform	non-trimmed	0.892	0.963	8
	15% trimmed	0.895	0.957	9

7 Example

The chart control limits were calculated for the probability of false alarms (a = 0.0027) in the monitoring process, based on the Normal distribution after Shewhart, Ryan[58], Chen [17], Sant'Anna and Caten[18](Beta), and Lima-Fiho et al.[19] (Kumaraswamy) charts and based proposed chart.
The data set consists of a data set of the study of contaminated peanut by toxic substances in 34 batches of 120 pounds. reported in Draper and Smith[59], p.101.
0.971,0.979,0.982,0.971,0.957,0.961,0.956,0.972,0.889,
0.961,0.982,0.975,0.942,0.932,0.908,0.970,0.985,0.933,
0.858,0.987,0.958,0.909,0.859,0.863,0.811,0.877,0.798,
0.855,0.788,0.821,0.830,0.718,0.642,0.658
Table.3 shows that the Shewhart, Ryan, Chen, Beta and Kumaraswamy charts extrapolate the upper limit for monitoring variable, which is restricted to the *[0, 1]*-interval, while the proposed chart 12% trimmed $(g_1 = 0, g_2 = 4)$ does not extrapolate this region, considering in-control process.
Finally, the control charts with the approximation to the Normal distribution and Beta, Kumaraswamy charts showed above one values. In contrast, the proposed Chart with control limits based on the 12% trimmed $(g_1 = 0, g_2 = 4)$ presented satisfactory estimates within the [0, 1]-interval. At that point, the proposed Chart is more robust for monitoring fraction data.

8 Conclusions

In this paper we have introduced the method of GL-moments as a generalization to the TL-moments and the L-moments methods. We have defined population GL-moments in terms of the quantile function and shifted Jacobi polynomials. We have also derived, depending on Jacobi Polynomials, some relations between

Table 3: Control Charts limits for the data

	LCL	UCL
Shewhart	0.686	**1.095**
Ryan	0.747	**1.15**
Chen	0.6369	**1.0453**
Beta	0.6692	**1.001**
Kumaraswamy	0.5804	**1.015**
Proposed $(g_1 = 0, g_2 = 4)$12% trimmed	0.51	0.991

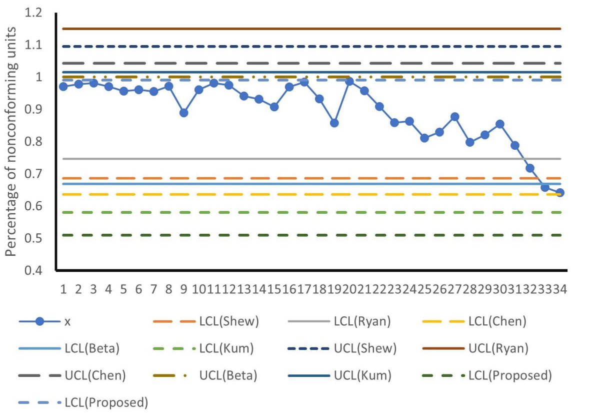

Fig. 1: Control Chart based on Shewhart, Ryan, Chen, Beta and Kumaraswamy and proposed Charts for the data.

GL-moments. Furthermore, we have introduced interpretation of the first two population GL-moments by U-statistics. We have also obtained two expressions to estimate the population GL-moments: the first expression, which depends on Downton's estimator, is nearly unbiased estimator. By re-writing this expression in terms of weights function, we have shown, in table 1, that GL-moments capture more information about the tail of the distribution than TL-moments. Finally, to avoid an upper control limit exceeding one, we proposed a control chart based on generalized linear moments for monitoring fractional, rates and proportions data. This control chart assumes that the fraction data can be approximated by a normal distribution and proposes new control limits based on GL-moments.

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Appendix

The First two GL-moments for Normal Distributions with Parameters (μ,σ), by the Approximations of GLD
Let F_x and F_y denote the distribution functions of X and $Y = g(x)$. Similarly, let $x_x(F)$ and $y_y(F)$ denote their quantile functions, then $y_Y(F) = g[x_X(F)]$.
The important consequence of this
If $Y = \mu + \sigma X$, then $y_Y(F) = \mu + \sigma x_X(F)$, See [60].
since all normal distributions can be obtained by location and scale adjustment to $N(0,1)$, Karian and Dudewicz [28] considered a GLD (m,a,b) fit to $N(0,1)$, *i.e* GLD$(0,0.1975,0.1349)$
the approximation to the Normal distribution with$(\mu = 0,\sigma = 1)$;
The quantile function for the generalized lambda distribution (GLD), which given as

$$x(F) = m + \frac{F^b - (1-F)^b}{a}, \quad 0 < F < 1.$$

If $z = \left(\frac{x-\mu}{\sigma}\right) \approx N(0,1)$,
then $z(F) = \frac{F^b-(1-F)^b}{a}$, since $m = 0$ *in*

approximation the normal distribution.
Thus,
if $x = \mu + \sigma z$,then

$x(F) = \mu + \sigma\, z(F) = \mu + \sigma \frac{F^b-(1-F)^b}{a}$ (A.1)
The first two GL-moments for the normal distributions with (μ,σ), by the approximations of GLD, by substituting (A.1) in (12), are

$$\Psi_1^{(g)} = \mu,$$

$$\Psi_2^{(g)} = \sigma \frac{a^{-1} b\, \Gamma(2g+3)\Gamma(b+g+1)}{\Gamma(g+2)\,\Gamma(b+2g+3)}, \qquad b > -(g+1).$$