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Dihedral Cohomology of an Infinity Algebras

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Abstract: In this research, the basic definitions of an operad, graded algebra, and $\mathscr{A}_\infty$-module are introduced. The $\mathscr{A}_\infty$-algebras and their (co)homology are studied to obtain the relations between the cyclic and dihedral (co)homology. The $\mathscr{L}_\infty$-algebras are discussed, and the relations of the isomorphism between primitive and indecomposable elements in the $\mathscr{L}_\infty$-algebras are presented. We demonstrate the relation between cyclic and dihedral (co)homology of $\mathscr{L}_\infty$-algebras. Finally, the Mayer-Vietoris sequence of $\mathscr{L}_\infty$-algebras is investigated.

Keywords: Operad, Module, Graded algebra, $\mathscr{A}_\infty$-algebras, $\mathscr{L}_\infty$-algebras, Mayer-Vietoris sequence.

1 Introduction

An operad is important algebraic tool used to study the operation on the elements of the given type of algebra. For example, the algebraic operads include commutative algebra, associative algebra, and Lie algebra. Recently, numerous algebras have received attention, including Poisson, Jordan, Gerstenhaber, Pre-Lie, Leibniz, Batalin-Vilkovisky, and dendriform algebras.
The operadic point is crucial because it applies the known results from classical algebras to other algebras, and the operadic language can express statements and proofs. Further, the operad theory can provide new results. Dmitry Tamarkin and Maxim Kontsevich applied the operadic theorem for the deformation-quatization of Poisson manifolds [1]. Operads play a fundamental role in many objects, including algebraic topology, $\mathscr{C}^*-$algebra, non-commutative geometry, differential geometry, symplectic geometry, quantum field theory, deformation theory, renormalization theory, category theory, string topology, combinational algebra, computer science and universal algebra.
In the mid- twentieth century, Michel Lazard started the operation composition of theoretical study [1].
In the sixties, F. Adam, J. Michael, A. Joyal, G. Kelly, P. May, S. Maclane, J. Stasheff, R. Vogt and numerous topologists used the operad theory in the algebraic topology as the efficient tool [2]. During the nineties, the deformation theory had envolved from algebra into topology by many scientists as E. Getzler, V. Ginzburg, V. Hinich, J. Jones, M. Kapranov, M. Kontsevich, Y. Manin, M. Markl, V. Schechtman, V. Smirnov and D. Tamarkin [3].
We encounter difficulties in the relationship between algebraic study and homotopy theory because the algebraic structure of a chain complex cannot be converted to a homotopy equivalent chain complex.
$\mathscr{A}_\infty-$algebras are the higher structure of associative algebra, which was discovered by Stasheff [1]. However, Kontsevich studied the higher structure of Lie algebras, called $\mathscr{L}_\infty-$algebras.
This problem was the main reason for defining the concept of an operad.
In the early sixties, Stusheff studied $\mathscr{A}_\infty-$algebras [5]. It was then developed by Kadeishvili, Smirnove, Huebschmann, Proute and others, who also applied it to topology [4].
Recently: in [6] and [7], A. H. Noreldeen introduced the Hochschild (co)homology of $\mathscr{A}_\infty-$algebras and $\mathscr{D}_\infty-$differential module. Then, in [8] he studied differential $\mathscr{A}_\infty-$algebras. In [9] and [10], Alaa thoroughly studied everything related to $\mathscr{E}_\infty-$algebras.
Here, the $\mathscr{A}_\infty-$algebras and $\mathscr{L}_\infty-$algebras were discussed. Further, $\mathscr{A}_\infty-$algebras was defined by examining it properties and providing a $\mathscr{A}_\infty-$module definition.
The second section provides a simplified definition of an operad and an explanation of its various types.

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In the third section: the definition of graded algebra in a simple form is presented with some examples.
In the fourth section: the $\mathscr{A}_\infty-$operad is presented as an introduction to the study of $\mathscr{A}_\infty-$algebras, the $\mathscr{A}_\infty-$algebra is investigated, the $\mathscr{A}_\infty-$algebra is defined, and some important definitions and properties are provided.
In the fifth section: the (co)homology theory of $\mathscr{A}_\infty-$algebras and some of it properties were studied as Hochschild, Cyclic and Dihedral (co)homology of $\mathscr{A}_\infty-$algebras are introduced.
In the sixth section: the Lie algebra and $\mathscr{L}_\infty-$algebras are investigated. Further, the Hochschild, Cyclic and Dihedral (co)homologies of $\mathscr{L}_\infty-$algebras are probed.
The seventh section is divided into two parts, the first of which introduces and establishes relations between both Hochschild and Cyclic homology of $\mathscr{A}_\infty-$algebras, and between Cyclic and Dihedral homologies of $\mathscr{A}_\infty-$algebras. In the second part, the isomorphism between primitive and indecomposable elements in the $\mathscr{L}_\infty-$algebras is introduced, and the relation between the Cyclic and Dihedral (co)homologies of $\mathscr{L}_\infty-$algebras iss obtained. Finally, the Mayer-Vietoris sequence of $\mathscr{L}_\infty-$algebras is studied and proved.

2 What is the Operad?

J. Micheal, J.Peter and Ranier M. pioneered the study of operads, a concept introduced in algebraic topology [1]. In the early nineties, Kontsevish, Ginzburg, and Kapranov developed Operads, which enabled them to clarify the duality phenomena in rational homotopy theory by utilizing the koszul duality of operands [11]. In mathematics, operad divided into two types:
(i) Non-Symmetric operad, which consists of:

1. The Sequence $(\mathscr{P}(n))_{n\in\mathbb{N}}$, which is the set of all n-ary operations elements,
2. The identity element $1\in\mathscr{P}(1)$,
3. The composition function

$$\circ:\mathscr{P}(n)\times\mathscr{P}(x_1)\times\cdots\times\mathscr{P}(x_n)\to\mathscr{P}(x_1+\ldots+x_n),$$
$$(\alpha,\alpha_1,\ldots,\alpha_n)\longmapsto\alpha\circ(\alpha_1,\ldots,\alpha_n),$$

for all $n,x_1,\ldots,x_n\in\mathbb{Z}$, satisfying that:

$$\alpha\circ(1,\ldots,1)=\alpha=1\circ\alpha,$$
$$\alpha\circ(\alpha_1\circ(\alpha_{1,1},\ldots,\alpha_{1,x_1}),\ldots,\alpha_n\circ(\alpha_{n,1},\ldots,\alpha_{n,x_n}))$$
$$=(\alpha\circ(\alpha_1,\ldots,\alpha_n))\circ(\alpha_{1,1},\ldots,\alpha_{1,x_1},\ldots,\alpha_{n,1},\ldots,\alpha_{n,x_n}) \tag{1}$$

(ii) Symmetric operad (called operad) is the same non-symmetric operad, but there is a symmetric group Σ_n action on $\mathscr{P}(n)$and satisfy (1) and:
$\forall \mathtt{t}\in\Sigma_{x_i},\ \ \mathtt{r}\in\Sigma_n;$

$$(\alpha*\nabla)\circ(\alpha_{\mathtt{r}_1},\ldots,\alpha_{\mathtt{r}_n})=(\alpha\circ(\alpha_1,\ldots,\alpha_n))*\mathtt{r},$$
$$\alpha\circ(\alpha_1*\mathtt{t}_1,\ldots,*\mathtt{t}_n=(\alpha\circ(\alpha_1,\ldots,\alpha_n))*(\mathtt{t}_1,\ldots,\mathtt{t}_n).$$

Hence, the morphisms of the operad can be defined. For two operads, $\mathscr{P},\mathscr{Q}$, there is a morphism $\mathtt{f}:\mathscr{P}\longrightarrow\mathscr{Q}$:

$$\mathtt{f}_n:\mathscr{P}(n)\longrightarrow\mathscr{Q}(n);\ \ n\in\mathbb{N},$$

since:

1. $\mathtt{f}(1)=1,$
2. $\mathtt{f}(\alpha\circ(\alpha_1,\ldots,\alpha_n))=\mathtt{f}(\alpha)\circ(\mathtt{f}(\alpha_1),\ldots,\mathtt{f}(\alpha_n)),$
3. $\mathtt{f}(x*\mathtt{t})=\mathtt{f}(x)*\mathtt{t}$

Then, there is a category of operads, denoted by the term oper [3].
Example (2-1)
By considering the field $\mathscr{K}$ and the vector space $\mathscr{V}$ with finite-dimensional over $\mathscr{K}$, the operad endomorphism of $\mathscr{V}$ can be defined, as $\mathscr{E}\mathtt{nd}_{\mathscr{V}}=\{\mathscr{E}\mathtt{nd}_{\mathscr{V}}(n)\}$, where $\mathscr{E}\mathtt{nd}_{\mathscr{V}}(n)$ is the set of all linear maps, satisfying the following:

1. $\mathscr{E}\mathtt{nd}_{\mathscr{V}}(n):\mathscr{V}^{\otimes n}\longrightarrow\mathscr{V},$
2. $\beta(\mathtt{f},\mathtt{g}_1,\ldots,\mathtt{g}_n)$: $V^{\otimes i_1}\bigotimes\ldots\bigotimes\mathscr{V}^{\otimes i_n}\xrightarrow{\mathtt{g}_1\otimes\ldots\otimes\mathtt{g}_n}\mathscr{V}^{\otimes n}\xrightarrow{\mathtt{f}}\mathscr{V},$
3. $\gamma:x\to\mathscr{E}\mathtt{nd}_{\mathscr{V}}(1),\ \ 1\longmapsto id_{\mathscr{V}},$
4. $(\beta(\mathtt{f},\mathtt{g}_1,\ldots,\mathtt{g}_n))*\eta=\mathtt{f}\circ\mathtt{g}_{\eta^{-1}(1)}\bigotimes\ldots\bigotimes\mathtt{g}_{\eta^{-1}(n)},\ \ \eta\in\Sigma_n.$

For another operad O, the operad algebra can be obtained as the morphism $O\longrightarrow\mathscr{E}\mathtt{nd}$.
Similarly, the operad algebra in algebraic topology can be obtained by substituting $\mathscr{V}$ and computing the tensor product with the topological spaces and the Cartesian product [1].
The following section discusses the graded algebras and introduces some examples for clarification.

3 Graded algebra

In mathematics, the degree of any element has a multiplicative property. For example, the polynomial ring with n-variables has a degree equal to the number of variables in the vector. The product of two polynomials is equal to the sum of their vectors, and its degree is equal to the sum of the vectors' degrees. After that, the graded algebras were defined as algebras with an additional property.
Along with defining graded algebra, we must first state that the polynomial of a single variable is unique and denoted by

$$\mathtt{C}[\mathtt{x}]\cong\mathtt{C}\bigoplus\mathtt{C}_{\mathtt{x}}\bigoplus\mathtt{C}_{\mathtt{x}^2}\bigoplus\ldots,$$

since $\mathtt{C}_{\mathtt{x}^n}$ is the polynomial on the form $\mathtt{a}_n\mathtt{x}^n$. For a semi-group $\mathscr{G}$, the $\mathscr{G}$-graded algebra is the algebra $\mathscr{A}$, which is defined by:

$$\mathscr{A}=\bigoplus_{\mathtt{g}\in\mathscr{G}}\mathscr{A}_{\mathtt{g}},$$

since $\odot : \mathscr{A}_\rangle \times \mathscr{A}_| \longrightarrow \mathscr{A}_{\rangle|}$. The elements in $\mathscr{A}_\rangle$ are called homogeneous elements, but the value of $\mathtt{i}$ is the degree [12]. Some examples of graded algebras would be demonstrated:

Example (3-1)

Consider the group $\mathscr{G}$, and $\mathscr{K}[\mathscr{G}]$ is the algebra of the $\mathscr{G}-$group; the direct sum is defined as;

$$\mathscr{K}[\mathscr{G}] = \bigoplus_{\mathtt{g}\in\mathscr{G}} \mathscr{L}_{\mathtt{g}},$$

given that $\mathscr{L}_\}$ is a free h-module with one-dimensional, the structure of graded algebra is as follows:

$$\ell_{\mathtt{i}} \cdot \ell_{\mathtt{j}} = \ell_{\mathtt{ij}}.$$

For more examples, consider the vector $\mathscr{V}$, where $n-$degree homogeneous elements of the tensor product $T^n\mathscr{V}$ are graded algebraic elements. The symmetric algebra $\mathscr{S}^n\mathscr{V}$ and exterior algebra $\Lambda^n\mathscr{V}$ are graded algebras. The $n^{th}-$cohomology ring in the cohomology theory is also graded [13].

Graded algebra applicable to various fields, including commutative algebra, homological algebra, algebraic topology and algebraic geometry.

The study aims to demonstrate some properties of $\mathscr{A}_\infty-$algebras as propositions.

4 $\mathbf{A}_\infty-$Operad

Before defining $\mathscr{A}_\infty-$algebras, some basic definitions would be introduced

We consider a field $\mathscr{F}$ and graded vector space as:

$$\mathscr{V} = \bigoplus \mathscr{V}^p, \quad p \in Z$$

then we could define the shift of $\mathscr{V}$ or the suspension $\mathscr{S}\mathscr{V}$ in terms, satisfying:

$$(\mathscr{S}\mathscr{V})^p = \mathscr{V}^{p+1} \qquad \forall p \in Z,$$

the tensor product of two homogenous maps $\mathtt{g} : \mathscr{V} \to \mathscr{V}$‘, $\mathtt{f} : \mathscr{W} \to \mathscr{W}$‘as

$$\mathtt{g} \bigotimes \mathtt{f} : \mathscr{V} \bigotimes \mathscr{W} \longrightarrow \mathscr{V}‘ \bigotimes \mathscr{W}‘,$$

and defined by

$$(\mathtt{g}\otimes\mathtt{f})(\mathscr{V}\otimes\mathscr{W}) = (-1)^{\mathtt{f}\mathscr{V}} \mathtt{g}(\mathscr{V}) \otimes \mathtt{f}(\mathscr{W}),$$
$$\mathscr{V} \in \mathscr{V}, \ \mathscr{W} \in \mathscr{W},$$

where $\mathscr{V}, \mathscr{V}‘, \mathscr{W}, \mathscr{W}$‘are graded spaces [7].

Example (4-1):

Consider the field $\mathtt{f}$ and the polynomial ring $\mathtt{f}[\mathtt{x},\mathtt{y}]$. For the homogeneous polynomials, if $\mathscr{P}_n = span\{\mathtt{x}^m\mathtt{y}^{n-m}; n \in \mathbb{N}\}$ with degree n, then $\mathtt{f}[\mathtt{x},\mathtt{y}] = \bigoplus \mathscr{P}_n, \quad n = 0,1,2,\ldots$ is an example for the graded vector space since the degree of every terms in $\mathscr{P}_n$ is n. Then $\mathtt{f}[\mathtt{x},\mathtt{y}] = \bigoplus \mathscr{P}_n, \quad n = 0,1,2,\ldots$ is also graded algebra since;

$$(\mathtt{x}^m\mathtt{y}^{n-m})(\mathtt{x}^k\mathtt{y}^{l-k}) = \mathtt{x}^{m+k}\mathtt{y}^{(n+l)-(m+k)} \in \mathtt{P}_{n+l}.$$

Consider $\mathscr{V}, \mathscr{V}$‘ as complexes with homogeneous first-degree differentials $\mathtt{d}$, and $\mathtt{d}^2 = 0$. If, $\mathtt{d}_S\mathscr{V} = -\mathtt{d}_\mathscr{V}$, for $\mathtt{g} : \mathscr{V} \longrightarrow \mathscr{V}$‘, we obtain the following:

$$\mathtt{d}(\mathtt{g}) = \mathtt{d}_{\mathscr{V}‘} \circ \mathtt{g} - (-1)^{\mathtt{g}} \mathtt{g} \circ \mathtt{d}_\mathscr{V}.$$

We stated that $\mathtt{g}$ is a morphism of complexes if and only if $\mathtt{d}(\mathtt{g}) = 0$. If there is morphism $\mathtt{h}$ between graded spaces, and $\mathtt{g}‘ = \mathtt{g} + \mathtt{d}(\mathtt{h})$, then $\mathtt{g}, \mathtt{g}‘$ are homotopic [7].

The $\mathscr{A}_\infty-$algebra can be defined as the graded space $\mathscr{A}$ defined by

$$\mathtt{b}_n : (\mathscr{S}\mathscr{A})^{\otimes p} \longrightarrow \mathscr{S}\mathscr{A}, \quad n \geq 1.$$

For $n = 1$, the map is homogenous but for $n > 1$, then we have:

$$\sum_{i+j+l=n} \mathtt{b}_{i+1+l} \circ \left(I^{\otimes i} \bigotimes \mathtt{b}_j \bigotimes I^{\otimes l}\right) = 0,$$

where I is the identity map of $\mathscr{S}\mathscr{A}$. For analyzing little value of n:

If $n = 1$, then $\mathtt{b}_1^2 = 0$. So, we get a complex $(\mathscr{S}\mathscr{A}, \mathtt{b}_1)$ and $m_1 = -\mathtt{b}_1$.

If $n = 2$, then we get

$$\mathtt{b}_1\mathtt{b}_2 + \mathtt{b}_2\left(\mathtt{b}_1 \bigotimes I + I \bigotimes \mathtt{b}_1\right) = 0.$$

Since the differential of $\mathscr{S}\mathscr{A} \otimes \mathscr{S}\mathscr{A}$ is $\mathtt{b}_1 \otimes I + I \otimes \mathtt{b}_1$, then we get $\mathtt{d}(\mathtt{b}_2) = 0$.

If $n = 3$, then we obtain the following:

$$\mathtt{b}_2(\mathtt{b}_2 \otimes I + I \otimes \mathtt{b}_2) + \mathtt{b}_1\mathtt{b}_3 +$$
$$\mathtt{b}_3(\mathtt{b}_1 \otimes I \otimes I + I \otimes \mathtt{b}_1 \otimes I + I \otimes I \otimes \mathtt{b}_1) = 0.$$

We define $m_2 : \mathscr{A} \otimes \mathscr{A} \longrightarrow \mathscr{A}$ as:

$$m_2(\mathtt{x},\mathtt{y}) = (-1)^{\mathtt{x}} \mathtt{b}_2(\mathtt{x},\mathtt{y}).$$

If $n > 3$, then $\mathscr{A}_\infty-$algebra can give the higher homotopies. If $\mathtt{b}_n$ vanished at $n > 1$, then the differential graded (dg) algebra $(\mathscr{A}, m_1, m_2)$ could be obtained [3].

Definition (4-2):[14]

Assume that $\mathscr{A}$ is the graded space defined by the formula $\mathtt{b}_0 : \mathtt{F} \longrightarrow \mathscr{S}\mathscr{A}$, $\lfloor_n; n \geq 0$. Then, we can say that $\mathscr{A}$ is week $\mathscr{A}_\infty-$algebra in the sense that: $\sum_{i+j+l=n} \mathtt{b}_{i+1+l} \circ \left(I^{\otimes i} \otimes \mathtt{b}_j \otimes I^{\otimes l}\right) = 0; \quad n \geq 0.$ and we have:

$$\mathtt{b}_1^2 = -\lfloor_2(\mathtt{b}_0 \bigotimes I + I \bigotimes \mathtt{b}_0) \neq 0.$$

Definition (4-3): [24]

For any two $\mathscr{A}_\infty-$algebras $\mathscr{A}, \mathscr{B}$, the morphism $\mathtt{f} : \mathscr{A} \longrightarrow \mathscr{B}$ can be defined as:

$$\mathtt{f}_n : (\mathscr{S}\mathscr{A})^{\otimes n} \longrightarrow \mathscr{S}\mathscr{A}; \quad n \geq 1,$$

which is homogenous at $n=0$. For $n\geq 1$, we get

$$\sum_{i+j+l=n} \mathtt{f}_{i+1+l}\circ\left(I^{\otimes i}\otimes \mathtt{b}_j\otimes I^{\otimes l}\right) \\ =\sum_{i_1+\ldots+i_s} \mathtt{b}_s\circ\left(\mathtt{f}_{i_1}\otimes\ldots\otimes \mathtt{f}_{i_s}\right)$$

, then, we obtain their algebraic morphism $\mathtt{f}_1$ as $\mathscr{H}^*\mathscr{A}\longrightarrow\mathscr{H}^*\mathscr{B}$. If $\mathtt{f}_1$ is quasi-isomorphism, then $\mathtt{f}$ is $\mathscr{A}_\infty-$ quasi-isomorphism.
The composition of $\mathtt{f}$ and $\mathtt{g}$ can be defined as:

$$(\mathtt{f}\circ\mathtt{g})_n=\sum_{i_1+\ldots+i_s=n}\mathtt{f}_{i_s}\circ\left(\mathtt{g}_{i_1}\bigotimes\ldots\bigotimes\}_{i_s}\right).$$

Given that $\mathtt{f}_1=1$ and $\mathtt{f}_n=0\ \forall n\geq 2$, we can obtain the morphism of identical $\mathscr{S}\mathscr{A}$.
Proposition (4-4):[16]
If $U(\mathscr{A})$ is the dg-algebra and $\mathscr{A}$ is $\mathscr{A}_\infty-$algebra, then the $\mathscr{A}_\infty-$quasi-isomorphism $\varphi:\mathscr{A}\to U(\mathscr{A})$ is the universal $\mathscr{A}_\infty-$algebra morphism.
If there is another dg-algebra morphism $\psi: U(\mathscr{A})\to\mathscr{B}$, then we have $\mathscr{A}_\infty-$morphism $\mathtt{h}:\mathscr{A}\longrightarrow\mathscr{B}$ as $\mathtt{h}=\psi\circ\varphi$.
Proposition (4-5):
Consider an $\mathscr{A}_\infty-$algebra $\mathscr{A}$ and a complex $\mathscr{V}$, if $\mathtt{f}:\mathscr{A}\longrightarrow\mathscr{V}$ is quasi-isomorphism, then $\mathscr{V}$ is $\mathscr{A}_\infty-$algebra, and $\mathtt{f}$ can be extend to $\mathscr{A}_\infty-$quasi-isomorphism $\mathtt{h}:\mathscr{A}\to\mathscr{V}$.
Definition (4-6):
Consider the homological unital $\mathscr{A}_\infty-$algebra $\mathscr{A}$; then the $\mathscr{A}_\infty-$module is defined as $\mathscr{M}$, which is the graded space defined by

$$\mathtt{b}_n:\mathscr{S}\mathscr{M}\bigotimes(\mathscr{S}\mathscr{A})^{\otimes n-1}\longrightarrow(\mathscr{S}\mathscr{M}),\ \ n\geq 1,$$

which is homogeneous at $n=1$, satisfying the following:

$$\sum_{i+j+l=n}\mathtt{b}_{i+1+l}\circ\left(I^{\otimes i}\bigotimes\mathtt{b}_j\bigotimes I^{\otimes l}\right)=0.$$

and inducing the action:

$$\mathscr{H}^*\mathscr{M}\bigotimes\mathscr{H}^*\mathscr{A}\longrightarrow\mathscr{H}^*\mathscr{M},$$

which is unital.
The following section discusses the theory of (co)homology for $\mathscr{A}_\infty-$algebras and defines the Hochschild, Cyclic and dihedral homology, and cohomology of $\mathscr{A}_\infty-$algebras.

5 (Co)Homology theory of $A_\infty-$algebras

For the differential module $(\mathscr{C},\mathtt{d})$ such that $\mathtt{d}:\mathscr{C}_n\longrightarrow\mathscr{C}_{n-1}$ and if the simplicial faces are defined as $\partial_i:\mathscr{C}_n\longrightarrow\mathscr{C}_{n-1},\ 0\leq i\leq n$, since $\partial_i\partial_j=\partial_{j-1}\partial_i,\ i<j$, then we refer to this as ∂_i being the simplitial faces of $(\mathscr{C},\mathtt{d})$ (for more see [17], and [5]).
Given the arbitrary permutation σ in the symmetric group Σ_m of the m-elements of permutations, such that its components are $(\sigma(i_1),\ldots,\sigma(i_m))$ which acts on $(i_1,\ldots,i_m)$ such that $i_1<\ldots<i_m$and then we can define $\left(\widehat{\sigma(i_1)},\ldots,\widehat{\sigma(i_m)}\right)$ as:

$$\widehat{\sigma(i_s)}=\sigma(i_s)-\alpha(\sigma(i_s)),\ \ 1\leq s\leq m,$$

since $\alpha(\sigma(i_s))$ is the number of $(\sigma(i_1),\ldots,\sigma(i_s),\ldots,\sigma(i_m))$. Now, the $\mathtt{F}_\infty-$module $\left(\mathscr{C},\mathtt{d},\widetilde{\partial}\right)$ can be defined as the differential module $(\mathscr{C},\mathtt{d})$ with the family map;

$$\widetilde{\partial}=\left\{\partial_{(i_1,\ldots,i_m)}:\mathscr{C}_n\longrightarrow\mathscr{C}_{n-m}\right\}, \\ i\leq m\leq n,\ \ 0\leq i_1<\ldots<i_m\leq n,\ \ i_1,\ldots,i_m\in Z$$

and satisfy that

$$\mathtt{d}\left(\partial_{(i_1,\ldots,i_m)}\right)= \\ \sum_{\sigma\in\Sigma_m}\sum_{I_\sigma}(-1)^{1+sign(\sigma)}\partial_{\left(\widehat{\sigma(i_1)},\ldots,\widehat{\sigma(i_l)}\right)}\partial_{\left(\widehat{\sigma(i_{l+1})},\ldots,\widehat{\sigma(i_m)}\right)}.$$

Since I_σ denotes all permutations of $\left(\widehat{\sigma(i_1)},\ldots,\widehat{\sigma(i_m)}\right)$ such that $\widehat{\sigma(i_1)}<\ldots<\widehat{\sigma(i_l)}$, and $\widehat{\sigma(i_{l+1})}<\ldots<\widehat{\sigma(i_m)}$.
Then $\widetilde{\partial}=\left\{\partial_{(i_1,\ldots,i_m)}\right\}$ is the $\mathtt{F}_\infty-$differential of $\left(\mathscr{C},\mathtt{d},\widetilde{\partial}\right)$, whereas $\partial_{(i_1,\ldots,i_m)}$ is the $\infty-$ simplitial of faces of $\mathtt{F}_\infty$ module (see [2], and [11]).
Then, for$m=1$, we get:

$$\mathtt{d}\left(\partial_{(i_1)}\right)=0,\ \ i\geq 0$$

For $m=2$,

$$\mathtt{d}\left(\partial_{(i,j)}\right)=\partial_{(j-1)}\partial_{(i)}-\partial_{(i)}\partial_{(j)},\ \ i<j$$

For m=3,

$$\mathtt{d}\left(\partial_{(i,j,k)}\right)=-\partial_{(i)}\partial_{(j,k)}-\partial_{(i,j)}\partial_{(k)}-\partial_{(k-2)}\partial_{(i,j)} \\ -\partial_{(j-1,k-1)}\partial_{(i)}+\partial_{(j-1)}\partial_{(i,k)}+\partial_{(i,k-1)}\partial_{(j)},\ \ i<j<k$$

If we define the differential module $(\mathscr{C},\mathtt{d})$ with the map $\mathtt{t}=\{\mathtt{t}_n:\mathscr{C}_n\longrightarrow\mathscr{C}_n\}$ such that:

$$\forall n\geq 0;\ \ \mathtt{t}_n^{n+1}=I_{\mathscr{C}_n},\ \ \mathtt{d}\mathtt{t}_n=\mathtt{t}_n\mathtt{d}.$$

Then we get the cyclic differential module $(\mathscr{C},\mathtt{d},\mathtt{t})$, and if we also define the map $\mathtt{r}=\{\mathtt{r}_n:\mathscr{C}_n\longrightarrow\mathscr{C}_n\}$ such $\mathtt{r}_n^2=I_{\mathscr{C}_n}$, then we get the dihedral differential module $(\mathscr{C},\mathtt{d},\mathtt{t},\mathtt{r})$ and we get:

$$\mathtt{r}_n\mathtt{t}_n=\mathtt{t}_n^{-1}\mathtt{r}_n, \\ \mathtt{d}\mathtt{r}_n=\mathtt{r}_n\mathtt{d}.$$

For cyclic and dihedral module, we get;

$$\partial_i\mathtt{t}_n=\mathtt{t}_{n-1}\partial_{i-1},\ \ 0<i\leq n \\ \partial_0\mathtt{t}_n=\partial_n,\ \ \partial_i\mathtt{r}_n=\mathtt{r}_{n-1}\partial_{n-1},\ \ 0\leq i\leq n$$

Then we can define $\mathscr{DF}_\infty-$module $\left(\mathscr{C},\mathtt{r},\mathtt{t},\mathtt{d},\widetilde{\partial}\right)$, also known as a dihedral module, because it is with ∞-simplitial faces [11], since $(\mathscr{C},\mathtt{r},\mathtt{t},\mathtt{d})$ is the dihedral differential module and satisfies that:

$$\begin{aligned}\partial_{(i_1,\ldots,i_m)}\mathtt{t}_n &= \mathtt{t}_{n-m}\partial_{(i_1-1,\ldots,i_m-1)},\ i_1>0\\ &= (-1)^{m-1}\partial_{(i_2-1,\ldots,i_m-1,n)},\\ & i_1=0\end{aligned} \tag{2}$$

$$\partial_{(i_1,\ldots,i_m)}\mathtt{r}_n = (-1)^{\frac{m(m-1)}{2}}\mathtt{r}_{n-m}\partial_{(n-i_m,\ldots,n-i_1)} \tag{3}$$

Definition (5-1): [5] If we have a complex $\mathscr{A}=\{\mathscr{A}_n\}$, $\forall n\geq 0$, $n\in Z$, then we can define the $\mathscr{A}_\infty-$algebra $(\mathscr{A},\mathtt{d},\pi)$, as the differential module $(\mathscr{A},\mathtt{d})$, since $\mathtt{d}:\mathscr{A}_*\to\mathscr{A}_{*-1}$ and $\pi_n:\left(\mathscr{A}^{\otimes(n+2)}\right)_*\to\mathscr{A}_{*+n}$, which satisfy that:

$$\mathtt{d}\circ\pi_{n-1} = \mathtt{d}\pi_{n-1}+(-1)^n\pi_{n-1} = \sum_{m=1}^{n-1}\sum_{t=1}^{m+1} (-1)^{n+t(n-m)+1}\pi_{m-1}\left(I^{\otimes(t-1)}\otimes\pi_{n-m-1}\otimes I^{m-t+1}\right), \tag{4}$$

since $n=1:\ \mathtt{d}\circ\pi_0=0$.
For $n=2$:

$$\mathtt{d}\circ\pi_1 = \pi_0\left(\pi_0\bigotimes I\right)-\pi_0\left(1\bigotimes\pi_0\right).$$

For $n=3$:

$$\begin{aligned}\mathtt{d}\circ\pi_2 &= \pi_0\left(\pi_1\otimes I+I\otimes\pi_1\right)\\ &-\pi_1\left(\pi_0\otimes I^{\otimes 2}-1\otimes\pi_0\otimes I+I^{\otimes 2}\otimes\pi_0\right)\end{aligned}$$

Now, the involutive $\mathscr{A}_\infty-$algebra can be defined as the complex $(\mathscr{A},\mathtt{d},\pi_n,*)$, since $(\mathscr{A},\mathtt{d},\pi_n)$ is the $\mathscr{A}_\infty-$algebra and defined by automorphism $*:\mathscr{A}_n\to\mathscr{A}_n$ such that $\forall x\in\mathscr{A};\ *(x)=x^*$ and the following conditions are satisfied:

$$\begin{aligned}&(x^*)^*=x,\\ &\mathtt{d}(x^*)=\mathtt{d}(x)^*,\pi_n(x_0\otimes x_1\otimes\ldots\otimes x_n\otimes x_{n+1})^*\\ &=(-1)^\varepsilon\pi_n\left(x_{n+1}^*\otimes x_n^*\otimes\ldots\otimes x_1^*\otimes x_0^*\right).\end{aligned}$$

Since $\varepsilon=\frac{n(n+1)}{2}+\sum_{0\leq i\leq j\leq n}|x_i|\,|x_j|$, $n\geq 0$. Then the module of the dihedral differential is the complex $({}^{\alpha}\mathscr{M}(\mathscr{A}),\mathtt{t},\mathtt{r},\mathtt{d})$, since $\alpha=\pm1$ and:

$$\begin{aligned}&\mathtt{t}_n(x_0\otimes\ldots\otimes x_n)\\ &=(-1)^\beta x_n\otimes x_0\otimes x_1\otimes\ldots\otimes x_{n-1},\\ &\mathtt{r}_n(x_0\otimes\ldots\otimes x_n)\\ &=\alpha(-1)^\gamma x_0^*\otimes x_n^*\otimes x_{n-1}^*\otimes\ldots\otimes x_1^*,\\ &\mathtt{d}(x_0\otimes\ldots\otimes x_n)\\ &=\sum_{i=0}^n(-1)^\mu x_0\otimes\ldots\otimes x_{i-1}\otimes\mathtt{d}(x_i)\otimes x_{i+1}\otimes\ldots\otimes x_n.\end{aligned}$$

Where

$$\beta=|x_n|(|x_0|+\ldots+|x_n|),\quad \gamma=\sum_{0<i<j\leq n}|x_i|\,|x_j|,\quad \mu=|x_0|+\ldots+|x_{n-1}|.$$

Since $\forall n\geq 0;\ \mathtt{t}_n^{n+1}=1,\ \mathtt{r}_n^2=1,\ \mathtt{t}_n\mathtt{r}_n=\mathtt{r}_n\mathtt{t}_n^{-1}$.
Theorem (5-2): [15]

If $(\mathscr{A},\mathtt{d},\pi_n,*)$ is the involutive $\mathscr{A}_\infty-$algebra, then $\left({}^{\alpha}\mathscr{M}(\mathscr{A}),\mathtt{t},\mathtt{r},\mathtt{d},\widetilde{\partial}\right)$ is the dihedral module $(\mathscr{DF}_\infty-module)$. The notations $\Sigma\mathscr{K}$ and $\Sigma^{-1}\mathscr{K}$ denote the one-dimensional vector space with degree −1 and 1 with 0-differential, respectively, for the field $\mathscr{K}$ with characteristic zero. $\hat{T}\mathscr{X}$ is the free formal augmented differential of a graded associative algebra, which generated by $\mathscr{X}$ and given by

$$\hat{T}\mathscr{X}=\prod_{n=0}^{\infty}\mathscr{X}^{\otimes n}=\mathscr{K}\times\mathscr{X}\times\left(\mathscr{X}\bigotimes\mathscr{X}\right)\cdots$$

By $\hat{T}_{\geq i}\mathscr{X}$, we mean the sub-algebra with an order of element equal to or greater than i.
Definition (5-3): [18]
Consider the graded vector space $\mathscr{X}$, then the $\mathscr{A}_\infty-$algebra structure on $\mathscr{X}$, which is defined as the derivation $\mathtt{m}:\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*\longrightarrow\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*$ with the degree 1 and $\mathtt{m}^2=0$.
Definition (5-4):
If we consider two $\mathscr{A}_\infty-$algebras $(\mathscr{X},\mathtt{m})$ and $(\mathscr{Y},\mathtt{m}‘)$, we have $\mathscr{A}_\infty-$morphism of $(\mathscr{X},\mathtt{m})$ and $(\mathscr{Y},\mathtt{m}‘)$ as an associative algebras map $\varphi:\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{Y}^*\longrightarrow\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*$, where $\mathtt{m}\circ\varphi=\varphi\circ\mathtt{m}‘$.
Definition (5-5): [18]
For a space of derivations $Der(\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*)$ for the $\mathscr{A}_\infty-$algebra $(\mathscr{X},\mathtt{m})$, we can define the Hochschild cohomology complex of $\mathscr{X}$ as the differential graded vector space $\mathscr{X}$;

$$\mathscr{CH}^\bullet(\mathscr{X},\mathscr{X})=Der(\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*)$$

and we denote it by $\mathscr{HH}^\bullet(\mathscr{X},\mathscr{X})$.
Definition (5-6):
With the graded vector space $\mathscr{V}$ and the involution map $\mathscr{V}\longrightarrow\mathscr{V}^*$ as defined $(\mathscr{V}^*)^*=\mathscr{V}$, then we can define the differential graded associative algebra $\mathscr{X}$ with involution satisfying for $\mathtt{a},\mathtt{b}\in\mathtt{A};(\mathtt{ab})^*=(-1)^{|\mathtt{a}||\mathtt{b}|}\mathtt{b}^*\mathtt{a}^*$ and $d(\mathtt{a})^*=d(\mathtt{a}^*)$.
Definition (5-7):
If $\mathscr{X}$ is an involutive graded vector space, the structure of $\mathscr{A}_\infty-$algebra on $\mathscr{X}$ is the derivation $\mathtt{m}:\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*\longrightarrow\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*$, with the one-degree and $\mathtt{m}^2=0$.
Definition (5-8):
If $Der_+(\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*)$ is a subspace of the space of derivation $Der(\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*)$, the involutive Hochschild cohomology complex $\mathscr{H}_+^\bullet(\mathscr{X},\mathscr{X})$ can be defined as the differential graded vector space

$$\mathscr{CH}_+^\bullet(\mathscr{X},\mathscr{X})=\Sigma^{-1}Der_+(\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*)$$

noting that space $Der_+(\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*)$ is eigenspace corresponding to eigenvalue +1, while $Der_-(\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*)$ is the eigenspace corresponding to eigenvalue -1. Since we have that:

$$Der\left(\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*\right)=Der_+(\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*)\bigoplus Der_-(\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*).$$

Definition (5-9):

For the involutive $\mathscr{A}_\infty-$algebra $(\mathscr{X},\mathtt{m})$, the skew-involutive Hochschlid cohomology complex $\mathscr{H}\mathscr{H}^\bullet_-(\mathscr{X},\mathscr{X})$ of $(\mathscr{X},\mathtt{m})$ is defined as the differential graded vector space as

$$\mathscr{C}\mathscr{H}^\bullet_-(\mathscr{X},\mathscr{X}) = \Sigma^{-1}Der\left(\hat{T}_{\geq 1}\Sigma^{-1}\mathscr{X}^*\right).$$

Theorem (5-10): If $(\mathscr{X},\mathtt{m})$ is the involutive $\mathscr{A}_\infty-$algebra, then we have

$$\mathscr{H}\mathscr{H}^\bullet(\mathscr{X},\mathscr{X}) = \mathscr{H}\mathscr{H}^\bullet_+(\mathscr{X},\mathscr{X})\bigoplus\mathscr{H}\mathscr{H}^\bullet_-(\mathscr{X},\mathscr{X}).$$

Definition (5-11): [19]

For an involutive $\mathscr{A}_\infty-$algebra $(\mathscr{X},\mathtt{m})$, the cyclic cohomology $\mathscr{H}\mathscr{C}^\bullet(\mathscr{X})$ of $\mathscr{X}$ can be defined as the differential graded vector space $\mathscr{C}\mathscr{C}^\bullet(\mathscr{X})$ scince:

$$\mathscr{C}\mathscr{C}^\bullet(\mathscr{X}) = \sum\prod_{i=1}^{\infty}\left[\left(\Sigma^{-1}\mathscr{X}^*\right)^{\otimes i}\right]_{Z_i},$$

for the cyclic group Z_i of the order i.

Definition (5-12):

For two cyclic $\mathscr{A}_\infty-$algebras $(\mathscr{X},\mathtt{m})$ and $(\mathscr{Y},\mathtt{m}`)$ with d, the cyclic $\mathscr{A}_\infty-$morphism can be defined as the map $\varphi$ such that $\varphi(\dashv`) = \mathtt{a}$ for all $\mathtt{a}\in\Sigma^{-1}\mathscr{X}^*\bigotimes\Sigma^{-1}\mathscr{X}^*$ and $\mathtt{a}`\in\Sigma^{-1}\mathscr{Y}^*\bigotimes\Sigma^{-1}\mathscr{Y}^*$ with $d+2$ degree.

Definition (5-13):

Consider the graded vector space $\mathscr{X}$ with involution and the dihedral group of order $2n$ is denoted by D_n; since $D_n = \langle r,s;r^n = s^2 = 1, srs^{-1} = r^{-1}\rangle$, then we define the dihedral action as $\forall \mathtt{x}_i\in\mathscr{X}$;

$$1.\mathtt{r}(\mathtt{x}_1\otimes\mathtt{x}_2\otimes\ldots\otimes\mathtt{x}_n) = (-1)^{|\mathtt{x}_n|\sum_{i=1}^{n-1}|\mathtt{x}_i|}\mathtt{x}_n\otimes\mathtt{x}_1\otimes\ldots\otimes\mathtt{x}_{n-1},$$
$$2.\mathtt{s}(\mathtt{x}_1\otimes\mathtt{x}_2\otimes\ldots\otimes\mathtt{x}_n) = (\mathtt{x}_1\otimes\mathtt{x}_2\otimes\ldots\otimes\mathtt{x}_n)^*.$$

And the skew-dihedral action as

$$1.\mathtt{r}(\mathtt{x}_1\otimes\mathtt{x}_2\otimes\ldots\otimes\mathtt{x}_n) = (-1)^{|\mathtt{x}_n|\sum_{i=1}^{n-1}|\mathtt{x}_i|}\mathtt{x}_n\otimes\mathtt{x}_1\otimes\ldots\otimes\mathtt{x}_{n-1},$$
$$2.\mathtt{s}(\mathtt{x}_1\otimes\mathtt{x}_2\otimes\ldots\otimes\mathtt{x}_n) = -(\mathtt{x}_1\otimes\mathtt{x}_2\otimes\ldots\otimes\mathtt{x}_n)^*.$$

Proposition (5-14): [18]

If we consider involutive $\mathscr{A}_\infty-$algebra $(\mathscr{X},\mathtt{m})$, we obtain the graded vector spaces $\mathscr{C}\mathscr{D}^\bullet_+(\mathscr{X})$ and $\mathscr{C}\mathscr{D}^\bullet_-(\mathscr{X})$ as follows:

$$1.\mathscr{C}\mathscr{D}^\bullet_+(\mathscr{X}) = \sum\prod_{i=1}^{\infty}\left[\left(\Sigma^{-1}\mathscr{X}^*\right)^{\otimes i}\right]_{D_i},$$
$$2.\mathscr{C}\mathscr{D}^\bullet_-(\mathscr{X}) = \sum\prod_{i=1}^{\infty}\left[\left(\Sigma^{-1}\mathscr{X}^*\right)^{\otimes i}\right]_{D_i}.$$

Since D_i is the dihedral action in (i) but a skew-dihedral action in (ii).

Definition (5-15):

For the involutive $\mathscr{A}_\infty-$algebra $(\mathscr{X},\mathtt{m})$, then the dihedral cohomology $\mathscr{H}\mathscr{D}^\bullet_+(\mathscr{X})$ is space $\mathscr{C}\mathscr{D}^\bullet_+(\mathscr{X})$ which differential with $\mathtt{m}$ and the skew-dihedral cohomology $\mathscr{H}\mathscr{D}^\bullet_-(\mathscr{X})$ is space $\mathscr{C}\mathscr{D}^\bullet_-(\mathscr{X})$ which differential with $\mathtt{m}$.

Theorem (5-16): [18]

If $(\mathscr{X},\mathtt{m})$ is $\mathscr{A}_\infty-$algebra with involution, then we have

$$\mathscr{H}\mathscr{C}^\bullet(\mathscr{X}) \cong \mathscr{H}\mathscr{D}^\bullet_-(\mathscr{X})\oplus\mathscr{H}\mathscr{D}^\bullet_+(\mathscr{X}).$$

Proof: We know that both of$\mathscr{C}\mathscr{D}^\bullet_+(\mathscr{X})$ and $\mathscr{C}\mathscr{D}^\bullet_-(\mathscr{X})$ are quotients of $\mathscr{C}\mathscr{C}^\bullet(\mathscr{X})$by two distinct actions of $\mathbb{Z}_2$ with involution on $\mathscr{X}$. By the relation

$$\mathscr{H}\mathscr{H}^\bullet(\mathscr{X},\mathscr{X} = \mathscr{H}\mathscr{H}^\bullet_+(\mathscr{X},\mathscr{X})\oplus\mathscr{H}\mathscr{H}^\bullet_-(\mathscr{X},\mathscr{X}).$$

And from the isomorphism with the involution of $\mathscr{C}\mathscr{C}^\bullet(\mathscr{X})$, we get the result.

Definition (5-17): [15]

Let $(\mathscr{A},\mathtt{d},\pi_n,*)$ be the involutive $\mathscr{A}_\infty-$algebra and $\left({}^{\alpha}\mathscr{M}(\mathscr{A}),\mathtt{t},\mathtt{r},\mathtt{d},\widetilde{\partial}\right)$ is $\mathscr{D}\mathrm{F}_\infty-$module, then we can define the dihedral homology ${}^{\alpha}\mathscr{H}\mathscr{D}(\mathscr{A})$ as the dihedral homology $\mathscr{H}\mathscr{D}({}^{\alpha}\mathscr{M}(\mathscr{A}))$ of $\left({}^{\alpha}\mathscr{M}(\mathscr{A}),\mathtt{t},\mathtt{r},\mathtt{d},\widetilde{\partial}\right)$ which is also defined as the homology of a chain complex $\left(Tot\left(\mathscr{D}\left(\overline{{}^{\alpha}\mathscr{M}(\mathscr{A})}\right)\right),\hat{\delta}\right)$.

Corollary (5-18): [15]

For any involutive $\mathscr{A}_\infty-$algebra $(\mathscr{A},\mathtt{d},\pi_n,*)$ and chain complex $\left(\mathscr{M}\left(\overline{{}^{\alpha}\mathscr{M}(\mathscr{A})}\right),\mathtt{b}\right)$, then the dihedral homology ${}^{\alpha}\mathscr{H}\mathscr{D}(\mathscr{A})$ is isomorphic to the homology of $\left(\mathscr{M}\left(\overline{{}^{\alpha}\mathscr{M}(\mathscr{A})}\right),\mathtt{b}\right)$.

Definition (5-19):

If $\mathscr{K},\mathscr{L},\mathscr{M}$ are examples of $\mathscr{A}_\infty-$algebra which related in the short exact sequence as;

$$0\to\mathscr{K}_\bullet\to\mathscr{L}_\bullet\to\mathscr{M}_\bullet\to 0.$$

With connected morphism $\partial_n:\mathscr{H}_n(\mathscr{M}_\bullet)\to\mathscr{H}_{n-1}\mathscr{K}_\bullet$. Then the long exact sequence is in the form;

$$\ldots\to\mathscr{H}_{n+1}(\mathscr{M}_\bullet)\underset{\to}{\partial}\mathscr{H}_n(\mathscr{K}_\bullet)\underset{\to}{\mathrm{g}}\mathscr{H}_n(\mathscr{L}_\bullet)\underset{\to}{\mathrm{h}}\mathscr{H}_{n+1}(\mathscr{M}_\bullet)\underset{\to}{\partial}\mathscr{H}_{n-1}(\mathscr{K}_\bullet).$$

Now, we study the dihedral cohomology of $\mathscr{L}_\infty-$algebra as a form of graded algebra.

6 Graded Lie algebra

The $\mathscr{L}_\infty-$algebra is a strong homotopy Lie algebra. M. Gerstenhaber introduced the bracket structure on cochain spaces of an associative algebra, demonstrating that the bracket can be the co-derivations bracket [13].

In this section, the Lie algebra ([14], [21], [22], [23], and [24]) and $\mathscr{L}_\infty-$algebra with some examples ([2], [19], [25], and [26]).

For the field $\mathscr{K}$, let $\mathscr{V}$ be Lie algebra over $\mathscr{K}$ with $[-,-]$. The linear map $\Lambda^n\ \mathscr{V}\longrightarrow\mathscr{V}$ gives the bracket antisymmetry. If $\mathscr{M}$ is $\mathscr{V}-$ module, then the antisymmtry functions space is $\mathscr{C}^n(\mathscr{V},\mathscr{M}) = Hom(\Lambda^n\mathscr{V},\mathscr{M})$ with n-degree. We can define the coboundary operator d of $\mathscr{V}$ since $d:\mathscr{C}^n(\mathscr{V},\mathscr{M}) \longrightarrow \mathscr{C}^{n+1}(\mathscr{V},\mathscr{M})$ as

$$d\varphi(a_1,\ldots,a_{n+1}) = \sum_{1\le i<j\le n+1}(-1)^{i+j-1}\varphi([a_i,a_j],a_1,\ldots,\hat{a}_i,\ldots,\hat{a}_j,\ldots,a_{n+1}) + \sum_{1\le i\le n+1}(-1)^i.\varphi(a_1,\ldots,\hat{a}_i,\ldots,a_{n+1}).$$

Then the cohomology of Lie algebra $\mathscr{V}$ is

$$\mathscr{H}^n(\mathscr{V},\mathscr{M}) = \frac{\ker\left(d:\mathscr{C}^n(\mathscr{V},\mathscr{M}) \longrightarrow \mathscr{C}^{n+1}(\mathscr{V},\mathscr{M})\right)}{Im\left(d:\mathscr{C}^{n-1}(\mathscr{V},\mathscr{M}) \longrightarrow \mathscr{C}^{n}(\mathscr{V},\mathscr{M})\right)}.$$

If $\mathscr{M}=\mathscr{V}$, then $\mathscr{C}^n(\mathscr{V},\mathscr{V}) = \mathscr{C}^n(\mathscr{V})$ and $\mathscr{H}^n(\mathscr{V},\mathscr{V}) = \mathscr{H}^n(\mathscr{V})$ [13].
If we relate $\mathscr{H}^n(\mathscr{V})$ to the deformations of $\mathscr{V}$, this relation is given by $\mathscr{H}^2(\mathscr{V})$. If we consider ℓ as the bracket in $\mathscr{V}$ and ℓ_t as the infinitesimally deformed product since $\ell_t = \ell + t\varphi$ such that $t^2=0$, then we find that $\varphi:\Lambda^n\mathscr{V} \longrightarrow \mathscr{V}$ is cocycle and thus we have:

$$\ell_t(a_1,\ell_t(a_2,a_3)) = \ell_t(\ell_t(a_1,a_2),a_3) + \ell_t(a_2,\ell_t(a_1,a_3)).$$

Then

$$[a_1,\varphi(a_2,a_3)] + \varphi(a_1,[a_2,a_3]) = [\varphi(a_1,a_2),a_3] + \varphi(a_2,[a_1,a_3])$$

since $d\varphi = 0$.
Consider the inner product of $\mathscr{V}$; $\langle -,-\rangle : \mathscr{V}\otimes\mathscr{V} \longrightarrow \mathscr{K}$ such satisfy that

$$\langle [a_1,a_2],a_3\rangle = \langle a_1[a_2,a_3]\rangle$$

and the tensor $\widetilde{\ell}$ is given by

$$\widetilde{\ell}(a_1,a_2,a_3) = \langle [a_1,a_2],a_3\rangle,$$

since $\widetilde{\ell} \in Hom\left(\Lambda^3\mathscr{V},\mathscr{K}\right)$ and we have

$$\widetilde{\ell}(a_1,a_2,a_3) = \widetilde{\ell}(a_3,a_1,a_2).$$

Hence, we find that $\widetilde{\ell}$ is invariant under cyclic permutations. The cohomology with trivial coefficients is denoted by $\mathscr{H}^3(\mathscr{V},\mathscr{K})$ ([14], and [21]).
If $\varphi \in \mathscr{C}^n(\mathscr{V})$ is the cyclic element of the following formula:

$$\langle\varphi(a_1,...,a_n),a_{n+1}\rangle = (-1)^n\langle a_1,\varphi(a_2,...,a_n)\rangle$$

then φ is cyclic. If and only of $\widetilde{\varphi}:\mathscr{V}^n \longrightarrow \mathscr{V}$ is anti-symmetric such that

$$\widetilde{\varphi}(a_1,...,a_{n+1}) = \langle\varphi(a_1,...,a_n),a_{n+1}\rangle,$$

and we have also, $\widetilde{\varphi} \in Hom\left(\Lambda^{n+1}\mathscr{V},\mathscr{K}\right)$, then $\widetilde{\varphi}$ is cyclic, satisfying that

$$\widetilde{\varphi}(a_1,\ldots,a_{n+1}) = (-1)^n\,\widetilde{\varphi}(a_{n+1},a_1,\ldots,a_n).$$

If the map $\varphi \longmapsto \widetilde{\varphi}$ is the isomorphism between $\mathscr{C}\mathscr{C}^n(\mathscr{V})$ and $\mathscr{C}^{n+1}(\mathscr{V},\mathscr{K})$, then $d\varphi$ is cyclic if φ is cyclic, and thus the cyclic cohomology of $\mathscr{V}$ is as follows:

$$\mathscr{H}\mathscr{C}^n(\mathscr{V}) = \frac{\ker\left(d:\mathscr{C}\mathscr{C}^n(\mathscr{V}) \longrightarrow \mathscr{C}\mathscr{C}^{n+1}(\mathscr{V})\right)}{Im\left(d:\mathscr{C}\mathscr{C}^{n-1}(\mathscr{V}) \longrightarrow \mathscr{C}\mathscr{C}^{n}(\mathscr{V})\right)}.$$

We notice that the isomorphism between $\mathscr{C}^{n+1}(\mathscr{V},\mathscr{K})$ and $\mathscr{C}\mathscr{C}^n(\mathscr{V})$ is commutative with respect to the coboundary operator [26], and thus that

$$\mathscr{H}\mathscr{C}^n(\mathscr{V}) \cong \mathscr{H}^{n+1}(\mathscr{V},\mathscr{K}).$$

Definition (6-1):[26]
Assuming $\mathscr{L}$ is the graded vector space, the graded Lie algebra can be defined as the vector space $\mathscr{L}$ with the map $[-,-]:\mathscr{L}\bigotimes\mathscr{L} \longrightarrow \mathscr{L}$ since this map is said to be Lie bracket and satisfy the following formula for all $\dashv,\lfloor,\rfloor \in \mathscr{L}$:

$$[a,[b,c]] = [[a,b],c] + (-1)^{|a||b|}[b,[a,c]] = 0.$$

The graded Lie algebra is denoted by $(\mathscr{L},[-,-])$.
For an example, if $\mathscr{X}$ is the graded associative algebra and defined with the graded commutator $[-,-]:\mathscr{X}\bigotimes\mathscr{X} \longrightarrow \mathscr{X}$ as;

$$[x_1,x_2] = x_1x_2 - (-1)^{|x_1||x_2|}x_2x_1 \qquad \forall x_1,x_2 \in X,$$

then $\mathscr{X}$ is graded Lie algebra.
Definition (6-2): [11]
DGLA is an abbreviation of the differential graded Lie algebra which is defined as DG algebra, since the algebra is graded Lie algebra.
If $\mathscr{L}$ is graded Lie algebra and $a \in \mathscr{L}$ since $\frac{1}{2}[a,a] = 0$, then $d = [a,-]; d^2 = 0$ and $(\mathscr{L},[-,-],d)$ is *DGLA*.
Definition (6-3):
If $(\mathscr{L},[-,-],d)$ is DGLA, then the Maurer-Cartan element is $a \in \mathscr{L}$ satisfying the Maurer-Cartan equation:

$$d(a) + \frac{1}{2}[a,a] = 0.$$

For example, if $(L,[-,-],d)$ is DGLA and $b \in \mathscr{L}$, then

$$\frac{1}{2}[a+b,a+b] = 0,$$

only if b satisfies the Maurer-Cartan equation.
Definition (6-4): [22]
For the graded vector space $\mathscr{L}$, the $\mathscr{L}_\infty-$algebra is $\mathscr{L}$ with the maps $l_k : \mathscr{L}^{\bigotimes k} \longrightarrow \mathscr{L}$, which are anti-symmetric linear and $|l_k| = 2-k$ since $1 \le k \le \infty$, satisfying the following:

$$\sum_{i+j=n+1}\sum_{\sigma\in sh^{-1}_{i,n-i}}(-1)^{i(j-1)}\chi(\sigma)\bullet l_j(l_j(a_1,\ldots,a_i),a_{i+1},\ldots,a_n) = 0,$$

where $\mathtt{a}_1,\ldots,\mathtt{a}_n \in \mathscr{L}$, $n \geq 1$. The set $\{l_k : 1 \leq k \leq \infty\}$ is the $\mathscr{L}_\infty-$structure of $\mathscr{L}$.

Definition (6-5): [22]

For two $\mathscr{L}_\infty-$algebras, $\mathscr{L}$ with the $\mathscr{L}_\infty-$structure $\{l_k\}_{k\in\mathbb{N}}$ and $\mathscr{L}$‘ with $\{l_k^{'}\}_{k\in\mathbb{N}}$, the strict homomorphism of $\mathscr{L}_\infty-$algebra is the degree to which the linear map $\mathtt{f} : \mathscr{L} \longrightarrow \mathscr{L}$‘ preserved while satisfying that:

$$\mathtt{f} \circ l_k = l_k^{'} \circ \mathtt{f}^{\otimes k} \qquad \forall 1 \leq k \leq \infty.$$

Definition (6-6): [26]

If $(\mathfrak{g},[-,-])$ is the graded Lie algebra and $\mathscr{V}$ is the vector space, then the representation of $\mathfrak{g}$ on $\mathscr{V}$ is the homomorphism of $(\mathfrak{g},[-,-])$ as $\rho : \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathscr{V})$. Thus, the Chevally-Eilenberg cohomology of the complex $\left(\bigoplus_{n\geq 0} Hom(\Lambda^n \mathfrak{g},\mathscr{V}),\delta\right)$ denotes the cohomology of $(\mathfrak{g},[-,-])$ where $\omega : \mathfrak{g}^{\otimes n} \longrightarrow \mathscr{V}$, and satisfying the following

$$\delta\omega(\dashv_1,\ldots,\mathtt{a}_{n+1}) = \sum_{i=1}^{n+1}(-1)^{i+1}\rho(\mathtt{a}_i) \\ (\omega(\mathtt{a}_1,\ldots,\hat{\mathtt{a}}_i,\ldots,\mathtt{a}_{n+1})) + \sum_{1\leq j<k\leq n+1}(-1)^{j+k} \times \\ \omega([\mathtt{a}_j,\mathtt{a}_k],\mathtt{a}_1\ldots,\hat{\mathtt{a}}_j,\ldots,\hat{\mathtt{a}}_n,\ldots,\mathtt{a}_{n+1}).$$

All $\mathtt{a}_1,\ldots,\mathtt{a}_{n+1} \in \mathfrak{g}$ and the element $\hat{\mathtt{a}}_i$ are deleted.

Definition (6-7):

If B is the polynomial algebra of the form: $B = B^1 \supset B^2 = B.B \supset \ldots \supset B^n \supset \ldots$, since $\hat{B} = \lim \frac{B}{B^n}$, then $\hat{B}$ is also polynomial algebra. If natural mapping: $B \to \hat{B}$ exists, then B is said to be perfect algebra.

Definition (6-8):

For $\mathscr{L}_\infty-$algebra $\mathscr{L}$ and the Massy sequence $(x^2,\ldots,x^n)$, where $x^i \in S\mathscr{L}^{\otimes i}$ satisfies the following conditions:

$$\left(\pi(2)\bigotimes\ldots\bigotimes 1 + 1\bigotimes\ldots\bigotimes\pi(2)\right)(x^n) = 0,$$

$$(\pi(3)\otimes\ldots\otimes 1 + \ldots + 1\otimes\ldots\otimes\pi(3))(x^n) \\ +\pi(2)\otimes\ldots\otimes 1 + \ldots + 1\otimes\ldots\otimes\pi(2))(x^{n-1}) = 0,$$

....

$$(\pi(n-1)\otimes 1 + 1\otimes\pi(n-1))(x^n) \\ +(\pi(n-2)\otimes 1 + 1\otimes\pi(n-2))(x^{n-1}) + \\ \ldots + \ldots(\pi(2)\otimes 1 + 1\otimes\pi(2))(x^3) = 0,$$

the Massy product is defined as

$$\mu(x^2,\ldots,x^n) = \pi(2)(x^2) + \ldots + \pi(2)(x^n).$$

The decomposable element $x \in \mathscr{L}$ is similar to a massy product, except $J\mathscr{L}$ that is considered indecomposable elements.

Definition(6-9):

If $\mathscr{K}$ is $\mathscr{L}_\infty-$algebra and $\hat{F}\mathscr{K}$ is co-B-construction from the short exact sequence:

$$0 \to \hat{F}^1\mathscr{K} \xrightarrow{\mathtt{i}} \hat{F}\mathscr{K} \xrightarrow{\mathtt{p}} \mathscr{K} \to 0,$$

by using the connecting homomorphism we get the long exact homology sequence:

$$\ldots \to \mathscr{H}_*(\hat{F}\mathscr{K}) \xrightarrow{\mathtt{p}} \mathscr{K} \xrightarrow{\mathtt{v}} \mathscr{H}_*(\hat{F}^1\mathscr{K}) \to \ldots.$$

If $x \in Ker \;\; \mathtt{v}_* : K \to \mathscr{H}_*(\hat{F}^1\mathscr{K})$ or $x \in Im \sqrt{}^* : \mathscr{H}_*(\hat{F}\mathscr{K}) \to \mathscr{K}$ and $P\mathscr{K}$ is the module of the primitive element of $\mathscr{L}_\infty$co-algebra $\mathscr{K}$, then the element $x \in \mathscr{K}$ is called primitive.

Whereas in this research we have thoroughly studied two important topics in the infinity algebras, which are the $\mathscr{A}_\infty-$algebras and $\mathscr{L}_\infty-$algebras, so the next section will review the results we obtained, which are explained as we will explain now.

7 Main result

In the first and second parts of this section, we present the of our study of $\mathscr{A}_\infty-$algebras and $\mathscr{L}_\infty-$algebras, respectively.

7.1 Main result of $\mathbf{A}_\infty-$*algebras*

After a thorough examination of $\mathbf{A}_\infty-$algebras, we present the relation between the cyclic and dihedral homology, which is one of the most important relationships in the homology theory, and we prove several of them in the following theorems.

Theorem (7-1):

Consider the involutive $\mathbf{A}_\infty-$algebra $(\mathscr{A},\mathtt{d},\pi_n,*)$, where ${}^\alpha\mathscr{H}\mathscr{D}(\mathscr{A})$ is its dihedral homology and $\mathscr{H}\mathscr{C}(\mathscr{A})$ is the cyclic homology. The relation between them is as follows:

$$\cdots \to {}^-\mathscr{H}\mathscr{D}_{n+1}(\mathscr{A}) \xrightarrow{\partial} {}^+\mathscr{H}\mathscr{D}_n(\mathscr{A}) \xrightarrow{j} \mathscr{H}\mathscr{C}_n(\mathscr{A}) \xrightarrow{i} \\ {}^-\mathscr{H}\mathscr{D}_n(\mathscr{A}) \xrightarrow{\partial} {}^+\mathscr{H}\mathscr{D}_{n-1}(\mathscr{A}) \to \ldots$$

where

$$i_n : \mathscr{H}\mathscr{C}_n(\mathscr{A}) \to {}^-\mathscr{H}\mathscr{D}_n(\mathscr{A}), \\ j_n : {}^+\mathscr{H}\mathscr{D}_n(\mathscr{A}) \to \mathscr{H}\mathscr{C}_n(\mathscr{A}), \\ \partial : {}^-\mathscr{H}\mathscr{D}_n(\mathscr{A}) \to {}^+\mathscr{H}\mathscr{D}_{n-1}(\mathscr{A})$$

Proof:

By the short exact sequence;

$$0 \to Tot\,{}^-\mathscr{D}(\mathscr{A}) \to Tot\,{}^+\mathscr{D}(\mathscr{A}) \to Tot\,\mathscr{C}(\mathscr{A}) \to 0,$$

then we get the required.

Theorem (7-2):

For the involutive $\mathscr{A}_\infty-$algebra $(\mathscr{A},\mathtt{d},\pi_n,*)$, we have the following:

$$\mathscr{H}\mathscr{C}_n(\mathscr{A}) \cong \bigoplus {}^\alpha\mathscr{H}\mathscr{D}_n(\mathscr{A}), \;\; \alpha = \pm 1.$$

Proof: If we define the morphisms:

$$\mu : x \to x + \mathscr{R}_n.x,\;\; \gamma : x \to x - \mathscr{R}_n.x,\;\; \mathscr{R}_n = (-1)^{\frac{n(n+1)}{2}}\, \mathfrak{r}_n,$$

then we get the required from the diagram:

$$\begin{array}{cccc}
\cdots \to {}^{+}\mathcal{HD}_n(\mathcal{A}) \overset{\partial_n}{\to} & {}^{+}\mathcal{HD}_{n-1}(\mathcal{A}) \to \cdots \to & {}^{+}\mathcal{HD}_1(\mathcal{A}) \overset{\partial_1}{\to} & {}^{+}\mathcal{HD}_0(\mathcal{A}) \to 0 \\
i \upuparrows \mu & i \upuparrows \mu & i \upuparrows \mu & i \upuparrows \mu \\
\cdots \to \mathcal{HC}_n(\mathcal{A}) \overset{\partial_n}{\to} & \mathcal{HC}_{n-1}(\mathcal{A}) \to \cdots \to & \mathcal{HC}_1(\mathcal{A}) \overset{\partial_1}{\to} & \mathcal{HC}_0(\mathcal{A}) \to 0 \\
i \downdownarrows \gamma & i \downdownarrows \gamma & i \downdownarrows \gamma & i \downdownarrows \gamma \\
\cdots \to {}^{-}\mathcal{HD}_n(\mathcal{A}) \overset{\partial_n}{\to} & {}^{-}\mathcal{HD}_{n-1}(\mathcal{A}) \to \cdots \to & {}^{-}\mathcal{HD}_1(\mathcal{A}) \overset{\partial_1}{\to} & {}^{-}\mathcal{HD}_0(\mathcal{A}) \to 0
\end{array}$$

It has been reported that the results for cyclic and Hochschild cohomology of $\mathscr{A}_\infty-$algebra can also be applied to $\mathscr{L}_\infty-$algebra [22]. Therefore, the following section applies some results about the homology of $\mathscr{L}_\infty-$algebra and the dihedral cohomology of $\mathscr{A}_\infty-$algebra to the dihedral cohomology of $\mathscr{L}_\infty-$algebra.

7.2 Main result of $\mathscr{L}_\infty-$algebra

After a thorough examination of $\mathscr{A}_\infty-$algebras, we present the relation between the cyclic and dihedral homology, which is one of the most important relationships in the homology theory, and we prove several of them in the following theorems.

Theorem (7-3):

Consider the perfect algebra $\mathscr{L}$ and the B-construction $B\mathscr{L}$. If $\mathscr{H}_*(B\mathscr{L})$ is the homology of $B\mathscr{L}$, then for the primitive element space $P\mathscr{H}_*(B\mathscr{L})$, we find that:

$$P\mathscr{H}_*(B\mathscr{L}) \cong J\mathscr{L}.$$

Proof: We obtain the following from the definitions (6-9):

$$P\mathscr{H}_*(B\mathscr{L}) = Im\left(\mathscr{H}_*\left(\hat{F}\mathscr{H}_*(B\mathscr{L})\right) \to \mathscr{H}_*(B\mathscr{L})\right),$$

given that $\mathscr{H}_*\left(\hat{F}B\mathscr{L}\right) \cong \mathscr{L}$ exists, we obtain

$$P\mathscr{H}_*(B\mathscr{L}) \cong Im\left(\mathscr{L} \to \mathscr{H}_*(B\mathscr{L})\right) \cong J\mathscr{L}.$$

Theorem (7-4):

If $\mathscr{K}$ is the perfect co-algebra,$\mathscr{F}\mathscr{K}$ is the co-B-construction and $\mathscr{H}_*(FK)$ is the homology of $\mathscr{F}\mathscr{K}$, then we obtain the following for the indecomposable elements space

$$J\mathscr{H}_*(\mathscr{F}\mathscr{K}) : J\mathscr{H}_*(\mathscr{F}\mathscr{K}) \cong \mathscr{P}\mathscr{K},$$

where $\mathscr{P}\mathscr{K}$ is the primitive element space.

Proof: We obtain the following from definitions (6-8):

$$J\mathscr{H}_*(\mathscr{F}\mathscr{K}) = I\left(\mathscr{H}_*(\mathscr{F}\mathscr{K}) \to \mathscr{H}_*(B\mathscr{H}_*(\mathscr{F}\mathscr{K}))\right),$$

since $\mathscr{H}_*(B\mathscr{H}_*(\mathscr{F}\mathscr{K})) = \mathscr{K}$, then

$$J\mathscr{H}_*(\mathscr{F}\mathscr{K}) \cong Im\left(\mathscr{H}_*(\mathscr{F}\mathscr{K}) \to \mathscr{K}\right) \cong \mathscr{P}\mathscr{K}.$$

The following theorem explains the relationship between cyclic and dihedral cohomology of $\mathscr{L}_\infty-$algebra.

Theorem (7-5):

For the $\mathscr{L}_\infty-$algebra $\mathscr{L}$ with involutive, we obtain the following:

$$\mathscr{H}\mathscr{C}^\bullet(\mathscr{L}) \cong \mathscr{H}\mathscr{D}^\bullet_-(\mathscr{L}) \oplus \mathscr{H}\mathscr{D}^\bullet_+(\mathscr{L}).$$

Proof: As we have that in $\mathscr{A}_\infty-$algebra's dihedral cohomology:

$$\mathscr{H}\mathscr{C}^\bullet(\mathscr{X}) \cong \mathscr{H}\mathscr{D}^\bullet_-(\mathscr{X}) \oplus \mathscr{H}\mathscr{D}^\bullet_+(\mathscr{X}).$$

Then, similarly, we obtain in the dihedral cohomology of $\mathscr{L}_\infty-$algebra:

$$\mathscr{H}\mathscr{C}^\bullet(\mathscr{L}) \cong \mathscr{H}\mathscr{D}^\bullet_-(\mathscr{L}) \oplus \mathscr{H}\mathscr{D}^\bullet_+(\mathscr{L}).$$

The Mayer-Vietoris sequence relates between any complex $\mathscr{C}^n(\mathscr{L})$ and its sub-complexes $\mathscr{M}^n(\mathscr{L})$ and $\mathscr{N}^n(\mathscr{L})$ such that $\mathscr{C} = \mathscr{M} \cup \mathscr{N}$.

Theorem (7-6):

For the $\mathscr{L}_\infty-$algebra $\mathscr{L}$, we denote $\mathscr{C}^n(\mathscr{L})$ for the chain complex of $\mathscr{L}_\infty-$algebra and $\mathscr{M}^n(\mathscr{L})$, $\mathscr{N}^n(\mathscr{L})$ for the sub-complexes of $\mathscr{C}^n(\mathscr{L})$, since their sequences are:

$$\mathscr{C}^n : \cdots \longrightarrow \mathscr{C}^{n-1}(\mathscr{L}) \xrightarrow{\mathrm{d}_{n-1}} \mathscr{C}^n(\mathscr{L}) \xrightarrow{\mathrm{d}_n} \mathscr{C}^{n+1}(\mathscr{L}) \to \cdots,$$

$$\mathscr{M}^n : \cdots \longrightarrow \mathscr{M}^{n-1}(\mathscr{L}) \xrightarrow{d_{n-1}} \mathscr{M}^n(\mathscr{L}) \xrightarrow{\mathrm{d}_n} \mathscr{M}^{n+1}(\mathscr{L}) \to \cdots,$$

and

$$\mathscr{N}^n : \cdots \longrightarrow \mathscr{N}^{n-1}(\mathscr{L}) \xrightarrow{d_{n-1}} \mathscr{N}^n(\mathscr{L}) \xrightarrow{\mathrm{d}_n} \mathscr{N}^{n+1}(\mathscr{L}) \to \cdots,$$

Then, the Mayer-Vietoris sequence is:

$$\cdots \to \mathscr{H}\mathscr{D}^n(\mathscr{L}) \longrightarrow k,l\,\mathscr{H}\mathscr{D}^n(\mathscr{M}) \bigoplus \mathscr{H}\mathscr{D}^n(\mathscr{N}) \xrightarrow{i,j} \mathscr{H}\mathscr{D}^n(\mathscr{M} \cap \mathscr{N}) \xrightarrow{\partial} \mathscr{H}\mathscr{D}^{n-1}(\mathscr{L}) \to \cdots,$$

Since $i : \mathscr{M} \hookrightarrow \mathscr{M} \cap \mathscr{N},\;\; j : \mathscr{N} \hookrightarrow \mathscr{M} \cap \mathscr{N},\;\; k : \mathscr{L} \hookrightarrow \mathscr{M},$ *and* $l : \mathscr{L} \hookrightarrow \mathscr{N}.$

Proof: If we relate the dihedral cohomology of three sequences to the two sub-complexes $\mathscr{M}$ and $\mathscr{N}$ of $\mathscr{C}$, then $\mathscr{M} \cup \mathscr{N} = \mathscr{C}$, as shown in the following:

$$\mathscr{C}^n : \cdots \longrightarrow \mathscr{C}^{n-1}(\mathscr{L}) \xrightarrow{\mathrm{d}_{n-1}} \mathscr{C}^n(\mathscr{L}) \xrightarrow{\mathrm{d}_n} \mathscr{C}^{n+1}(\mathscr{L}) \to \cdots,$$

$$\mathscr{M}^n : \cdots \longrightarrow \mathscr{M}^{n-1}(\mathscr{L}) \xrightarrow{d_{n-1}} \mathscr{M}^n(\mathscr{L}) \xrightarrow{\mathrm{d}_n} \mathscr{M}^{n+1}(\mathscr{L}) \to \cdots,$$

and

$$\mathscr{N}^n : \cdots \longrightarrow \mathscr{N}^{n-1}(\mathscr{L}) \xrightarrow{d_{n-1}} \mathscr{N}^n(\mathscr{L}) \xrightarrow{\mathrm{d}_n} \mathscr{N}^{n+1}(\mathscr{L}) \to \cdots,$$

8 Conclusion

In the current study, the $\mathscr{A}_\infty-$algebras and $\mathscr{L}_\infty-$algebras were investigated. The relations between Hochchild and Cyclic homology of $\mathscr{A}_\infty-$algebras and between Cyclic and Dihedral homology of $\mathscr{A}_\infty-$algebras were found, as shown in the following:

$$\cdots \to {}^{-}\mathscr{H}\mathscr{D}_{n+1}(\mathscr{A}) \xrightarrow{\partial} {}^{+}\mathscr{H}\mathscr{D}_n(\mathscr{A}) \xrightarrow{j} \mathscr{H}\mathscr{C}_n(\mathscr{A}) \xrightarrow{i} {}^{-}\mathscr{H}\mathscr{D}_n(\mathscr{A}) \xrightarrow{\partial} {}^{+}\mathscr{H}\mathscr{D}_{n-1}(\mathscr{A}) \to \cdots.$$

The isomorphism between primitive and indecomposable elements in the $\mathscr{L}_\infty-$algebras was introduced. It has been proved that

$$P\mathscr{H}_*(B\mathscr{L}) \cong J\mathscr{L} and J\mathscr{H}_*(\mathscr{F}\mathscr{K}) \cong \mathscr{P}\mathscr{K}$$

The relation between the Cyclic and Dihedral cohomology of $\mathscr{L}_\infty-$algebras was obtained as

$$\mathscr{H}\mathscr{C}^\bullet(\mathscr{L}) \cong \mathscr{H}\mathscr{D}^\bullet_-(\mathscr{L}) \oplus \mathscr{H}\mathscr{D}^\bullet_-(\mathscr{L}).$$

Finally the Mayer-Vietoris sequence of $\mathscr{L}_\infty-$algebras was studied in the form of

$$\cdots \to \mathscr{H}\mathscr{D}^n(\mathscr{L}) \xrightarrow{k,l} \mathscr{H}\mathscr{D}^n(\mathscr{M}) \bigoplus \mathscr{H}\mathscr{D}^n(\mathscr{N}) \xrightarrow{i,j} \mathscr{H}\mathscr{D}^n(\mathscr{M}\cap\mathscr{N}) \xrightarrow{\partial} \mathscr{H}\mathscr{D}^{n-1}(\mathscr{L}) \to \cdots.$$

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