Information Sciences Letters

Volume 11
Issue 1 *Jan. 2022*

Article 17

2022

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Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/isl/110113

A New Fractional Differential Approach: a Combination between Usual Derivative and a Fractional Differential Operator Without Singular Kernel

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Received: 2 Jul. 2021, Revised: 12 Sep. 2021, Accepted: 9 Oct. 2021
Published online: 1 Jan. 2022

Abstract: In this paper, authors introduce a new fractional differential order operator given as a combination between the usual derivative and a fractional differential operator without singular kernel. The new approach is defined through a fractional integral order and based on the Caputo viewpoint. Some properties are given to illustrate the results. Also calculus of integral of an interesting function is illustrated.

Keywords: Fractional calculus, Fractional derivative, Fractional integral.

1 Introduction

A fractional differential operator is an integral operator which generalizes the ordinary derivative, such that if the fractional derivative is represented by D^{α} then, when $\alpha = n$, it coincides with the usual differential operator D^n [1]. Its origin dates back to 1695 when L'Hopital raised by a letter to Leibniz the question of how the expression

$$D^n u(t) = \frac{d^n}{dt^n} u(t),$$

should be understood if n was a real number [1]. Since then, this new branch turned out to be very attractive to mathematicians such as Euler, Laplace, Fourier, Liouville, Riemann, Laurent, Weyl and Abel who first applied it in physics to solve the integral equation arising from the tautochron problem [2]. Since then, Fractional Calculus has become popular and useful due to its ability to describe some natural phenomena in numerous fields of engineering such as theory of viscoelasticity [3,4,5], study of the anomalous diffusion phenomenon [6,7,8], circuit theory [9,10,11] and image processing [12,13], among other applications. Various definition of fractional derivatives have been introduced [14]-[20]. In fact, the Grunwald-Letnikov fractional derivative, defined as a limit of a fractional order backward difference, is one of the first introduced fractional operators. Other definition which also plays a major role in Fractional Calculus is the Riemann-Liouville fractional derivative. The Caputo fractional derivative, which is useful for the formulation and solution of applied problems, has also been defined via a modified Riemann-Liouville fractional derivative. In 2015, Caputo and Fabrizio introduced a new fractional differential approach without singular kernel [20]. The interest for this new approach was born from the prospect that there is a class of non-local systems, which have the ability to describe the material heterogeneities and the fluctuations of different scales, which cannot be well described by classical local theories or by fractional models with singular kernel [20,21]. The propose of this paper is to suggest a new fractional differential operator which is given as a combination between the usual derivative and a fractional differential operator without singular kernel. We think that this way of defining a fractional derivative may be helpful in describing the real world problems which cannot be well described by traditional calculus theory or by models involving fractional operators with singular kernel only. The new approach is defined through a fractional integral order. The paper also contains some properties concerning the

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behavior of this new derivative. Finally, we have also calculated primitives of an interesting function.

2 Definitions

Here, we give some definitions which will be needed in our subsequent discussions.

Definition 2.01*A function $f:[a,b]\longrightarrow\mathbb{R}$ is said to be absolutely continuous, denoted by $f\in AC[a,b]$, on $[a,b]$ if, given $\varepsilon>0$ there exist some $\sigma>0$ such that*

$$\sum_{k=1}^{n}|f(y_k)-f(x_k)|<\varepsilon.$$

whenever $\{[x_k,y_k]:i=1,...,n\}$ is a finite collection of mutully disjoint subintervals of $[a,b]$ with

$$\sum_{k=1}^{n}|y_k-x_k|<\sigma.$$

Definition 2.02*Let $n\in\mathbb{N}:=1,2,3,\cdots$ and $k=1,2,\cdots,n-1$, the space $AC^n[a,b]$ is defined as*

$$AC^n[a,b] \\ =\{f:[a,b]\longrightarrow\mathbb{R}, f^k(t)\in C[a,b], f^{n-1}(t)\in AC[a,b]\}.$$

Definition 2.03*Let $u(t)\in L^1_{loc}(\mathbb{R})$ and $f(t)=0, t\in(0,+\infty)$. The Laplace transform of $f(t)$ is defined by*

$$\mathfrak{L}\{f(t)\}(s)=\lim_{A\longrightarrow\infty}\int_0^A e^{-st}f(t)dt,$$

for those numbers s for which this limit exists.

3 A new fractional differential operator

To achieve the goal outlined above, we propose the following definitions.

Definition 3.01*Let $a\geq 0$ and $f\in AC[a,b]$. The new fractional integral of order $\alpha>0$ is given by*

$$I^{\alpha}_{at}f(t)=(1-\alpha)f(t)+\frac{1}{\Gamma(\alpha)}\int_a^t(t+\tau)^{\alpha-1}f(\tau)d\tau. \quad (1)$$

In the case of $f\in L^1(a,b)$, the operator $I^{\alpha}_{at}f(t)\in L^1(a,b)$. For $\alpha=0$, we define $I^0_{at}f(t)=f(t)$. This definition is motivated by the following reasoning. Suppose that $f\in C^1([a,b])$. Then, after integration by parts, from (1), we obtain

$$I^{\alpha}_{at}f(t)=(1-\alpha)f(t)+\frac{1}{\Gamma(\alpha+1)}\Big[(at)^{\alpha}f(t) \\ -(t+\alpha)^{\alpha}f(a)\Big]-\frac{1}{\Gamma(\alpha+1)}\int_a^t(t+\tau)^{\alpha}f'(\tau)d\tau.$$

So that

$$\lim_{\alpha\longrightarrow 0}I^{\alpha}_{at}f(t)=f(t)+f(t)-f(a)-f(t)+f(a)=f(t).$$

That is, when α is zero we recover the initial function and if also α is 1, we obtain the ordinary integral.

Definition 3.02*Let $a\geq 0$, $n-1<\alpha\leq n\in\mathbb{N}$ and $f\in AC^n[a,b]$. The new fractional derivative of order α is given by*

$$D^{\alpha,n}_{at}f(t)=I^{n-\alpha}_{at}\left(\frac{d^n}{dt^n}f(t)\right)=[(\alpha+1)-n]f^{(n)}(t) \\ +\frac{1}{\Gamma(n-\alpha)}\int_a^t(t+\tau)^{n-(\alpha+1)}f^{(n)}(\tau)d\tau. \quad (2)$$

Taking $\alpha\longrightarrow n$, we obtain

$$\lim_{\alpha\longrightarrow n}D^{n,\alpha}_{at}f(t)=I^0_{at}\left(\frac{d^n}{dt^n}f(t)\right)=f^{(n)}(t).$$

4 Some properties of the proposed operator

In this section, we study some properties related to the new proposed fractional differential operator.

Theorem 4.01*Let $a\geq 0$, $\beta,\mu\in\mathbb{R}$ such that $n-1<\alpha\leq n\in\mathbb{N}$. Then*

$$D^{\alpha,n}_{at}(\mu f(t)+\beta g(t))=\mu\cdot D^{\alpha,n}_{at}f(t)+\beta\cdot D^{\alpha,n}_{at}g(t).$$

Proof: From Definition 3.02, we obtain

$$D^{\alpha,n}_{at}(\mu f(t)+\beta g(t))= \\ =[(\alpha+1)-n]\cdot\left(\mu f^{(n)}(t)+\beta g^{(n)}(t)\right) \\ +\frac{1}{\Gamma(n-\alpha)}\int_a^t(t+\tau)^{n-(\alpha+1)}\left(\mu f^{(n)}(\tau)+\beta g^{(n)}(\tau)\right)d\tau \\ =[(\alpha+1)-n]\cdot\mu f^{(n)}(t) \\ +\frac{\mu}{\Gamma(n-\alpha)}\int_a^t(t+\tau)^{n-(\alpha+1)}f^{(n)}(\tau)d\tau \\ +[(\alpha+1)-n]\cdot\beta g^{(n)}(t) \\ +\frac{\beta}{\Gamma(n-\alpha)}\int_a^t(t+\tau)^{n-(\alpha+1)}g^{(n)}(\tau)d\tau \\ =\mu\cdot D^{\alpha,n}_{at}f(t)+\beta\cdot D^{\alpha,n}_{at}g(t)$$

Theorem 4.02*Let $a\geq 0$ and $n-1<\alpha\leq n\in\mathbb{N}$. Then*

$$D^{\alpha,n}_{at}f(t)= \\ =[(\alpha+1)-n]f^{(n)}(t) \\ +\sum_{k=0}^{n-1}\frac{1}{\Gamma(n-\alpha-k)}\Big[(-1)^k(2t)^{n-1-k-\alpha}f^{(n-1-k)}(t) \\ +(-1)^{k+1}(t+a)^{n-1-k-\alpha}f^{n-1-k}(a)\,\Big] \\ +\frac{(-1)^n}{\Gamma(n-\alpha)}\left[\Pi_{k=0}^{n-1}(k-\alpha)\right]\int_a^t(t+\tau)^{-(\alpha+1)}f(\tau)d\tau \quad (3)$$

Proof.Performing repeatedly the method of integration by parts n times, we obtain

$$\int_a^t (t+\tau)^{n-(\alpha+1)} f^{(n)}(\tau)d\tau =$$
$$= \sum_{k=0}^{n-1} \frac{\Gamma(n-\alpha)}{\Gamma(n-\alpha-k)} \left[(-1)^k (2t)^{n-1-k-\alpha} f^{(n-1-k)}(t)\right.$$
$$\left. + (-1)^{k+1}(t+a)^{n-1-k-\alpha} f^{n-1-k}(a)\,\right]$$
$$+ (-1)^n \left[\Pi_{k=0}^{n-1}(k-\alpha)\right] \int_a^t (t+\tau)^{-(\alpha+1)} f(\tau)d\tau. \quad (4)$$

Combining (2) with (4), we obtain (3).

Theorem 4.03*Let $a \geq 0$ and $n-1 < \alpha \leq n \in \mathbb{N}$. Then*

$$I_{at}^{\alpha}\left(D_{at}^{\alpha,n} f(t)\right) = (1-\alpha)D_{at}^{\alpha,n} f(t) + (\alpha+1-n) I_{at}^{\alpha} f^{(n)}(t)$$
$$-(1-\alpha)(\alpha+1-n) f^{(n)}(t). \quad (5)$$

Proof: We obtain

$$I_{at}^{\alpha}\left(D_{at}^{\alpha,n} f(t)\right) =$$
$$= (1-\alpha)D_{at}^{\alpha,n} f(t) + \frac{1}{\Gamma(\alpha)} \int_a^t (t+\tau)^{\alpha-1} D_{a\tau}^{\alpha,n} f(\tau)d\tau$$
$$= (1-\alpha)D_{at}^{\alpha,n} f(t) + \frac{\alpha+1-n}{\Gamma(\alpha)} \int_a^t (t+\tau)^{\alpha-1} f^{(n)}(\tau)d\tau$$
$$+ \frac{1}{\Gamma(\alpha)} \int_a^t (t+\tau)^{\alpha-1} \left\{ \int_a^\tau (\tau+s)^{\alpha-1} f^{(n)}(s)ds \right\} d\tau \quad (6)$$

Adding and resting terms, (6) can be written as

$$I_{at}^{\alpha}\left(D_{at}^{\alpha,n} f(t)\right) =$$
$$= (1-\alpha)D_{at}^{\alpha,n} f(t) + (1-\alpha)(\alpha+1-n) f^{(n)}(t)$$
$$+ \frac{\alpha+1-n}{\Gamma(\alpha)} \int_a^t (t+\tau)^{\alpha-1} f^{(n)}(\tau)d\tau$$
$$x - (1-\alpha)(\alpha+1-n) f^{(n)}(t)$$
$$+ \frac{1}{\Gamma(\alpha)} \int_a^t (t+\tau)^{\alpha-1} \left\{ (\alpha+1-n)\Gamma(n-\alpha) f^{(n)}(\tau) \right.$$
$$+ \frac{\Gamma(n-\alpha)}{\Gamma(n-\alpha)} \int_a^\tau (\tau+s)^{n-(\alpha+1)} f^{(n)}(s)ds$$
$$\left. -(\alpha+1-n)\Gamma(n-\alpha) f^{(n)}(\tau) \right\} d\tau \quad (7)$$

From (7), yields Formula (6) can be rewritten as

$$I_{at}^{\alpha}\left(D_{at}^{\alpha,n} f(t)\right) =$$
$$= (1-\alpha)D_{at}^{\alpha,n} f(t) + (\alpha+1-n) I_{at}^{\alpha} f^{(n)}(t)$$
$$-(1-\alpha)(\alpha+1-n) f^{(n)}(t)$$
$$+ \frac{\Gamma(n-\alpha)}{\Gamma(\alpha)} \int_a^t (t+\tau)^{\alpha-1} D_{a\tau}^{\alpha,n} f(\tau)d\tau$$
$$- \frac{(\alpha+1-n)\Gamma(n-\alpha)}{\Gamma(\alpha)} \int_a^\tau (t+\tau)^{\alpha-1} f^{(n)}(\tau)d\tau.$$

By adding and resting again, we get

$$I_{at}^{\alpha}\left(D_{at}^{\alpha,n} f(t)\right) =$$
$$= (1-\alpha)D_{at}^{\alpha,n} f(t) + (\alpha+1-n) I_{at}^{\alpha} f^{(n)}(t)$$
$$-(1-\alpha)(\alpha+1-n) f^{(n)}(t)$$
$$+ \Gamma(n-\alpha)\left\{ (1-\alpha)D_{at}^{\alpha,n} f(t) + \frac{1}{\Gamma(\alpha)} \cdot \right.$$
$$\left. \cdot \int_a^t (t+\tau)^{\alpha-1} D_{a\tau}^{\alpha,n} f(\tau)d\tau - (1-\alpha)D_{at}^{\alpha,n} f(t) \right\}$$
$$-(\alpha+1-n)\Gamma(n-\alpha)\left\{ (1-\alpha) f^{(n)}(t) + \frac{1}{\Gamma(\alpha)} \cdot \right.$$
$$\left. \cdot \int_a^t (t+\tau)^{\alpha-1} f^{(n)}(\tau)d\tau - (1-\alpha) f^{(n)}(t) \right\}. \quad (8)$$

From (8), we obtain

$$I_{at}^{\alpha}\left(D_{at}^{\alpha,n} f(t)\right) =$$
$$= (1-\alpha)(1-\Gamma(n-\alpha))D_{at}^{\alpha,n} f(t)$$
$$+ \Gamma(n-\alpha) I_{at}^{\alpha}\left(D_{at}^{\alpha,n} f(t)\right)$$
$$+ (\alpha+1-n)(1-\Gamma(n-\alpha)) I_{at}^{\alpha} f(t)$$
$$-(1-\alpha)(\alpha+1-n)(1-\Gamma(n-\alpha)) f^{(n)}(t). \quad (9)$$

Equality (5) follows from (9).

Theorem 4.04*Let $a \geq 0$ and $n-1 < \alpha \leq n \in \mathbb{N}$. Then*

$$D_{at}^{\alpha,n}\left(I_{at}^{\alpha} f(t)\right)$$
$$= (1-\alpha)D_{at}^{\alpha,n} f(t) + \left[\alpha+1\right.$$
$$\left. -n - \frac{(1-\alpha)\Gamma(\alpha)}{\Gamma(n-\alpha)}\right] \frac{d^n}{dt^n}\left(I_{at}^{\alpha} f(t)\right)$$
$$+ \left[\frac{(1-\alpha)^2\Gamma(\alpha)}{\Gamma(n-\alpha)} - (\alpha\right.$$
$$\left. +1-n)(1-\alpha)\right] f^{(n)}(t)$$
$$- \frac{(1-\alpha)\Gamma(\alpha)}{\Gamma(n-\alpha)} I_{at}^{\alpha} f^{(n)}(t)$$
$$+ \frac{\Gamma(\alpha)}{\Gamma(n-\alpha)} I_{at}^{\alpha}\left(\frac{d^n}{dt^n}\left(I_{at}^{\alpha} f(t)\right)\right) \quad (10)$$

Proof: We have

$$D_{at}^{\alpha,n}\left(I_{at}^{\alpha}f(t)\right)$$
$$= D_{at}^{\alpha,n}\left[(1-\alpha)f(t)+\frac{1}{\Gamma(\alpha)}\cdot\right.$$
$$\left.\cdot\int_a^t (t+\tau)^{\alpha-1}f(\tau)d\tau\right]$$
$$=(1-\alpha)D_{at}^{\alpha,n}f(t)+\frac{1}{\Gamma(\alpha)}\cdot$$
$$\cdot D_{at}^{\alpha,n}\left[\int_a^t (t+\tau)^{\alpha-1}f(\tau)d\tau\right]$$
$$+(1-\alpha)D_{at}^{\alpha,n}f(t)+\frac{\alpha+1-n}{\Gamma(\alpha)}\cdot\frac{d^n}{dt^n}\int_a^t (t+\tau)^{\alpha-1}f(\tau)d\tau$$
$$+\frac{1}{\Gamma(\alpha)\Gamma(n-\alpha)}\int_a^t (t+\tau)^{\alpha-1}\cdot$$
$$\cdot\left\{\frac{d^n}{d\tau^n}\int_a^\tau (\tau+\xi)^{\alpha-1}f(\xi)d\xi\right\}d\tau. \quad (11)$$

Equality (11) can be rewritten as

$$D_{at}^{\alpha,n}\left(I_{at}^{\alpha}f(t)\right)$$
$$=(1-\alpha)D_{at}^{\alpha,n}f(t)+(\alpha+1-n)\frac{d^n}{dt^n}\left(I_{at}^{\alpha}f(t)\right)$$
$$-(1-\alpha)(\alpha+1-n)f^{(n)}(t)$$
$$-\frac{1-\alpha}{\Gamma(n-\alpha)}\int_a^t (t+\tau)^{\alpha-1}f^{(n)}(\tau)d\tau$$
$$+\frac{1}{\Gamma(n-\alpha)}\int_a^t (t+\tau)^{\alpha-1}\left(\frac{d^n}{d\tau^n}I_{at}^{\alpha}f(t)\right)d\tau. \quad (12)$$

From (12), we obtain

$$D_{at}^{\alpha,n}\left(I_{at}^{\alpha}f(t)\right)$$
$$=(1-\alpha)D_{at}^{\alpha,n}f(t)+(\alpha+1-n)\frac{d^n}{dt^n}\left(I_{at}^{\alpha}f(t)\right)$$
$$-(1-\alpha)(\alpha+1-n)f^{(n)}(t)$$
$$+\frac{\Gamma(\alpha)}{\Gamma(n-\alpha)}I_{at}^{\alpha}\left(\frac{d^n}{dt^n}I_{at}^{\alpha}f(t)\right)$$
$$-\frac{(1-\alpha)\Gamma(\alpha)}{\Gamma(n-\alpha)}\left(\frac{d^n}{dt^n}I_{at}^{\alpha}f(t)\right)$$
$$-\frac{(1-\alpha)\Gamma(\alpha)}{\Gamma(n-\alpha)}I_{at}^{\alpha}f^{(n)}(t)+\frac{(1-\alpha)^2\Gamma(\alpha)}{\Gamma(n-\alpha)}f^{(n)}(t). \quad (13)$$

The result (10) follows from (13).

5 Integral of an interesting function

This section contains a set of statements in the form of theorem and lemmas according to the standard calculus for undergraduate courses. The main gaol is to prove the following theorem:

Theorem 5.01*Let $b, C\in\mathbb{R}$, $n\in\mathbb{N}$, $n>1$, $i=\sqrt{-1}$, θ given by (23) and*

$$\alpha=-\sqrt{\frac{n+1}{n}}i-\frac{n+1}{n}-b.$$

Then

$$\int x\sqrt[n]{x+b}e^x dx$$
$$=\frac{n}{2\alpha n+(\theta+b)n+2n+1}\left[(x-\theta)\sqrt[n]{(x+b)^{n+1}}\right.$$
$$-\left(x-\theta-\alpha+\sqrt{\frac{n+1}{n}}i-1\right)\cdot\left(x-\theta-\alpha-\sqrt{\frac{n+1}{n}}i\right.$$
$$\left.\left.-\frac{n+1}{n}\right)\sqrt[n]{x-\theta-\alpha-\sqrt{\frac{n+1}{n}}i-\frac{n+1}{n}}\right]e^x+C. \quad (14)$$

To prove theorem 5.01, we need the following lemmas:

Lemma 5.01*Let $n\in\mathbb{N}$, $n>1$, $i=\sqrt{-1}$ and $C\in\mathbb{R}$. Then*

$$\int\left(x+\sqrt{\frac{n+1}{n}}i\right)^2\sqrt[n]{x-\frac{n+1}{n}}e^x dx$$
$$=\left(x+2\sqrt{\frac{n+1}{n}}i-1\right)\cdot\left(x-\frac{n+1}{n}\right)\sqrt[n]{x-\frac{n+1}{n}}e^x+C. \quad (15)$$

Proof: We obtain

$$\int\left(x+2\sqrt{\frac{n+1}{n}}ix-\frac{n+1}{n}\right)\sqrt[n]{x-\frac{n+1}{n}}e^x dx$$
$$=\left(x+2\sqrt{\frac{n+1}{n}}i-1\right)\cdot\left(x-\frac{n+1}{n}\right)\sqrt[n]{x-\frac{n+1}{n}}e^x, \quad (16)$$

by using the method of integration by parts. Since

$$x^2+2\sqrt{\frac{n+1}{n}}ix-\frac{n+1}{n}=\left(x+\sqrt{\frac{n+1}{n}}i\right)^2,$$

then, the formula (15) follows from (16).

Lemma 5.02*Let $n\in\mathbb{N}(n>1)$, $i=\sqrt{-1}$ and $C\in\mathbb{R}$. Then*

$$\int x^2\sqrt[n]{x-\sqrt{\frac{n+1}{n}}i-\frac{n+1}{n}}\cdot e^x dx$$
$$=\left(x+\sqrt{\frac{n+1}{n}}i-1\right)\cdot\left(x-\sqrt{\frac{n+1}{n}}i\right.$$
$$\left.-\frac{n+1}{n}\right)\sqrt[n]{x-\sqrt{\frac{n+1}{n}}i-\frac{n+1}{n}}e^x+C. \quad (17)$$

Proof: From the change of variable

$$x = t + \sqrt{\frac{n+1}{n}} i,$$

we obtain

$$\int x^2 \sqrt[n]{x - \sqrt{\frac{n+1}{n}} i - \frac{n+1}{n}} \cdot e^x dx$$
$$= e^{\sqrt{\frac{n+1}{n}} i} \int \left(t + \sqrt{\frac{n+1}{n}} i\right)^2 \sqrt[n]{t - \frac{n+1}{n}} e^t dt. \qquad (18)$$

Formula (17) follows from combining the lemma 5.01 with the equality (18).

Lemma 5.03*Let $n \in \mathbb{N}$, $n > 1$, $i = \sqrt{-1}$ and $b, C \in \mathbb{R}$ and*

$$\alpha = -\sqrt{\frac{n+1}{n}} i - \frac{n+1}{n} - b.$$

Then

$$\int (x-\alpha)^2 \sqrt[n]{x+b} e^x dx$$
$$= \left(x - \alpha + \sqrt{\frac{n+1}{n}} i - 1\right) \cdot \left(x - \alpha - \sqrt{\frac{n+1}{n}} i - \frac{n+1}{n}\right) \sqrt[n]{x - \alpha - \sqrt{\frac{n+1}{n}} i - \frac{n+1}{n}} e^x + C. \qquad (19)$$

Proof: Using the change of variable $x = \alpha + t$, we obtain

$$\int (x-\alpha)^2 \sqrt[n]{x+b} e^x dx$$
$$= e^{\alpha} \int t^2 \sqrt[n]{t - \sqrt{\frac{n+1}{n}} i - \frac{n+1}{n}} e^t dt. \qquad (20)$$

Formula (19) is obtained by combining (20) with the lemma 5.02.

Lemma 5.04*Let b, $C \in \mathbb{R}$, $n \in \mathbb{N}$, $n > 1$, $i = \sqrt{-1}$ and*

$$\alpha = -\sqrt{\frac{n+1}{n}} i - \frac{n+1}{n} - b$$

then

$$\int \left[x + \frac{(b-\alpha^2)n}{2\alpha n + bn + 2n + 1}\right] \sqrt[n]{x+b} e^x dx$$
$$= \frac{n}{2\alpha n + bn + 2n + 1} \Bigg[x \sqrt[n]{(x+b)^{n+1}} e^x - \left(x - \alpha + \sqrt{\frac{n+1}{n}} i - 1\right) \cdot \left(x - \alpha - \sqrt{\frac{n+1}{n}} i - \frac{n+1}{n}\right) \sqrt[n]{x - \alpha - \sqrt{\frac{n+1}{n}} i - \frac{n+1}{n}} e^x\Bigg] + C. \qquad (21)$$

Proof: Integrating by parts, we obtain

$$\int \left[x + \frac{(b-\alpha^2)n}{2\alpha n + bn + 2n + 1}\right] \sqrt[n]{x+b} e^x dx$$
$$= \frac{n}{2\alpha n + bn + 2n + 1} \left[x \sqrt[n]{(x+b)^{n+1}} e^x - \int (x-\alpha)^2 \sqrt[n]{x+b} e^x dx\right] \qquad (22)$$

Combining (22) with Lemma 5.03, we obtain (21).

Lemma 5.05*Let b, $C \in \mathbb{R}$, $n \in \mathbb{N}$, $n > 1$, $i = \sqrt{-1}$ and*

$$\alpha = -\sqrt{\frac{n+1}{n}} i - \frac{n+1}{n} - b,$$
$$\Delta = (1 + b^2 + 4\alpha + 4b\alpha + 6b)n^2 + (4\alpha + 2 + 2b)n + 1.$$

Then the two following equalities are equivalent:

$$\theta = \frac{(\theta + b - \alpha^2)n}{2\alpha n + (\theta + b)n + 2n + 1}$$
$$\theta = \frac{-(bn + n + 1 + 2\alpha n) \pm \sqrt{\Delta}}{2n}. \qquad (23)$$

Proof: Formula

$$\theta = \frac{(\theta + b - \alpha^2)n}{2\alpha n + (\theta + \beta)n + 2n + 1},$$

is equivalent to equality

$$n\theta^2 + (bn + n + 1 + 2\alpha n)\theta - (b - \alpha^2)n = 0, \qquad (24)$$

which again can be rewritten as

$$\theta = \frac{-(bn + n + 1 + 2\alpha n) \pm \sqrt{\Delta}}{2n},$$

where

$$\Delta = (1 + b^2 + 4\alpha + 4b\alpha + 6b)n^2 + (4\alpha + 2 + 2b)n + 1.$$

Now we are in conditions to prove Theorem 5.01.

Proof: Let θ given by (23). Using the change of variable $x = t + \theta$, we obtain

$$\int x \sqrt[n]{x+b} e^x dx$$
$$= \int (t+\theta) \sqrt[n]{t + \theta + b} e^{t+\theta} dt.$$

Combining Lemma 5.05 with the previous equality, we get

$$\int x \sqrt[n]{x+b} e^x dx$$
$$= e^{\theta} \int \left(t + \frac{(\theta + b - \alpha^2)n}{2\alpha n + (\theta + b)n + 2n + 1}\right) \sqrt[n]{t + (\theta + b)} e^t dt. \qquad (25)$$

Formula (14) is obtained by combining (25) with Lemma 5.04.

6 Conclusion

The aim of this paper was to suggest a new fractional differential operator by combining the usual derivative with an integral operator without singular kernel. The fractional differential operator term is based upon the Caputo viewpoint. Some properties have been obtained to illustrate the results. Also primitives of an interesting function have been calculated.

Conflict of Interest

The authors declare that they have no conflict of interest.

Acknowledgement

This work was supported by Universidad Nacional de Guinea Ecuatorial (UNGE), Havana University and Instituto de Cibernética, Matemática y Física (ICIMAF).

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