2022

A New Generalization of the Gumbel Distribution With Climate Application

Hussein Eledum
Department of Statistics, Faculty of Science, University of Tabuk, KSA, heledum@yahoo.com

Mohammed Ahmmed Mosilhy
Department of Statistics, Faculty of Science, University of Tabuk, KSA\ Department of Mathematics, Faculty of Science, Cairo University, Egypt, heledum@yahoo.com

Follow this and additional works at: https://digitalcommons.aaru.edu.jo/isl

Recommended Citation
Eledum, Hussein and Ahmmed Mosilhy, Mohammed (2022) "A New Generalization of the Gumbel Distribution With Climate Application," Information Sciences Letters: Vol. 11 : Iss. 1 , PP -. Available at: https://digitalcommons.aaru.edu.jo/isl/vol11/iss1/32

This Article is brought to you for free and open access by Arab Journals Platform. It has been accepted for inclusion in Information Sciences Letters by an authorized editor. The journal is hosted on Digital Commons, an Elsevier platform. For more information, please contact rakan@aaru.edu.jo, marah@aaru.edu.jo, u.murad@aaru.edu.jo.
A New Generalization of the Gumbel Distribution With Climate Application

Hussein Eledum\textsuperscript{1,\ast} and Mohammed Ahmed Mostilly\textsuperscript{1,2}

\textsuperscript{1}Department of Statistics, Faculty of Science, University of Tabuk, KSA
\textsuperscript{2}Department of Mathematics, Faculty of Science, Cairo University, Egypt

Received: 28 Jan. 2021, Revised: 20 Oct. 2021, Accepted: 24 Oct. 2021
Published online: 1 Jan. 2022

Abstract: In this paper, we propose a new generalization of the Gumbel distribution named the Cubic Transmuted Gumbel distribution (CTGD) based on a cubic ranking transmutation map. Statistical properties of CTGD such as reliability function, hazard function, moments, moment generating function, quantile function, and simulation of the random sample are studied. The parameters of CTGD are estimated using the Maximum Likelihood method. Finally, an application of CTGD using two real data sets on climate change is conducted to illustrate and compare with the base Gumbel distribution (GD) and transmuted Gumbel distribution (TGD). CTGD was found to be a better fit than GD and TGD.

Keywords: Cubic transmutation, Maximum likelihood estimation, Extreme value distribution, Reliability function

1 Introduction

The Gumbel distribution (GD), named after Gumbel \cite{1} is also referred to as the Smallest Extreme Value (SEV) distribution or Type I Extreme Value distribution. The GD is a very popular distribution due to its extensive applicability in several areas, and its wide applications have been reported by Kotz and Nadarajah \cite{2}. Koutsoyiannis \cite{3} studied GD to observe the appropriateness of this distribution in modeling extreme rainfall. Aryal and Tsokos \cite{4} have given the necessary formulation of the GD to study the airline spill data. The applicability of GD in the field of flood frequency analysis, network, space, software reliability, structural, and wind engineering are reported by Cardeiro et al. \cite{5}. Generally, GD is used to analyze and model the behavior of random phenomena that occur in engineering, biology, environment among others. The cumulative distribution function (cdf) and the probability density function (pdf) of Gumbel random variable $X$ are defined, respectively, as

\[ G(x) = e^{-z} \quad ; \quad x \in \mathbb{R} \]  

where $z = e^{-(x-\mu)/\beta}$, while $\beta \in [0, \infty)$ and $\mu \in \mathbb{R}$ are a scale and location parameters, respectively.

Shaw and Buckley \cite{6} used the rank transmutation map to propose a new method for generating a family of distribution. According to \cite{6}, the cumulative distribution function of the ranking quadratic transformation map (QR TM) is:

\[ F(x) = \left(1 + \lambda\right)G(x) - \lambda G^2(x); \quad |\lambda| \leq 1 \] \hspace{1cm} (3)

where $G(x)$ is the cumulative distribution function (cdf) of the base distribution. Observe that, when $\lambda = 0$, the new distribution becomes the original one. Using OR TM of Eq.(3), Aryal and Tsokos \cite{7} developed transmuted Gumbel distribution (TGD).

Abed Al-Kadim \cite{8} proposed a generalized formula for transmuted distribution presented by \cite{6}, the cumulative distribution function of the Cubic Ranking transformation map (CRTM) is:

\[ F(x) = \left(1 + \lambda\right)G(x) - \lambda G^2(x) + \lambda G^3(x); \quad |\lambda| \leq 1 \] \hspace{1cm} (4)

This method used by Abed Al-Kadim and Mohammed \cite{9} to develop a cubic transmuted Weibull distribution. Another two classes of Cubic Transmuted distributions with two transmuted parameters have been developed, one by Granzatto et al. \cite{10}, the other by Rahman et al.

* Corresponding author e-mail: heledum@yahoo.com
[11]. Rahman formula used to propose some new distributions, for examples, Pareto distribution [12], Weibull distribution [13] and Frechet distribution [14]. Based on the CRTM proposed by [11], the cdf and pdf respectively given as

\[ F(x) = (1 + \lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2G(x) \]  

and

\[ f(x) = g(x)[(1 + \lambda_1) + 2(\lambda_2 - \lambda_1)G(x) - 3\lambda_2G^2(x)] \]  

where \( \lambda_1, \lambda_2 \in [-1, 1], -2 \leq \lambda_1 + \lambda_2 \leq 1 \) and \( G(x), g(x) \) are the cdf and pdf of the base distribution respectively.

In this article, we use CRTM suggested by [11] to propose a new distribution which generalizes the Gumbel distribution. This new version of the Gumbel distribution is called Cubic Transmuted Gumbel (CTGD). Some statistical properties are studied, and the model parameters are estimated using the maximum likelihood method. Moreover, an application to two real data sets on change climate is illustrated and compared with the base GD and TGD.

The rest of this paper is structured as follows: The new proposed distribution Cubic Transmuted Gumbel (CTGD) is presented in Section 2. We have investigated some statistical properties such as reliability function, hazard function, moments and moment generating function for CTGD in Section 3. Section 4 provides the parameter estimation of CTGD. An application of the CTGD to two real data sets for the purpose of illustration is conducted in section 5. Finally, Section 6 gives some concluding remarks.

The new Gumbel model is motivated because it exhibits a maximum or minimum of a number of samples with data close to normal distribution or approximately positively skewed, as illustrated in Fig. 1. The justification for the new generalized extreme value (GEV) model’s practicality is based on its ability to model the wind velocity and the snow accumulation data sets, as illustrated in Section 5.

2 Cubic Transmuted Gumbel Distribution (CTGD)

In this section, the new proposed distribution CTGD is demonstrated. Including the cumulative distribution function (cdf), probability density function (pdf), survival and hazard function.

2.1 Cumulative and density functions for CTGD

**Theorem 1.** Let \( X \) be a random variable with the CTGD. The cdf and pdf are defined, respectively, as

\[
F(x) = e^{-\left[(1 + \lambda_1) + (\lambda_2 - \lambda_1)\right]e^{-z} - 2\lambda_2e^{-2z}} \quad (7)
\]

and

\[
f(x) = \frac{1}{\beta} z e^{-z} \left[(1 + \lambda_1) + 2(\lambda_2 - \lambda_1)e^{-z} - 3\lambda_2e^{-2z}\right] \quad (8)
\]

where \( z = e^{-(x-\mu)/\beta}, \beta \geq 0 \) and \( \mu \in R \) are a scale and location parameters respectively, \( \lambda_1, \lambda_2 \in [-1, 1] \) and \(-2 \leq \lambda_1 + \lambda_2 \leq 1.\)

**Proof.** The proof is straightforward. Eq.(7) is obtained by substituting Eq.(1) into Eq.(5) and Eq.(8) is gotten from substituting Eq.(2) into Eq.(6).

**Proposition 1.** The limit of CTGD density as \( x \to \infty \) and \( x \to -\infty \) is \( 0.\)

**Proof.** To show this proposition, we must prove that \( f(x) \to 0 \) and \( \int_{-\infty}^{\infty} f(x)dx = 1.\)

From Proposition 1 \( \lim_{x \to \infty} f(x) = 0 \) and \( \lim_{x \to -\infty} f(x) = 0.\) It follows that \( f(x) \geq 0.\)

**Proof of \( \int_{-\infty}^{\infty} f(x)dx = 1.\)**

\[
\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} e^{-z} \left[(1 + \lambda_1) + 2(\lambda_2 - \lambda_1)\right]e^{-z} - 3\lambda_2e^{-2z} \] 

let \( z = e^{-(x-\mu)/\beta} \Rightarrow \ln z = -(x-\mu)/\beta \) and \( x = \mu - \beta \ln z \) also if \( x \to -\infty \Rightarrow z \to -\infty \) and if \( x \to \infty \Rightarrow z \to 0 \) then

\[
\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{\infty} f(z)\beta z^{-\beta} \] 

\[
= \int_{0}^{\infty} \left(\frac{1}{\beta z}\right) e^{-z} \left[(1 + \lambda_1) + 2(\lambda_2 - \lambda_1)e^{-z} - 3\lambda_2e^{-2z}\right] \] 

\[
= (1 + \lambda_1) + (\lambda_2 - \lambda_1) - \lambda_2 = 1.
\]

Therefore, the proposition is proved.

Fig. 1 and 2 illustrate respectively, some of possible shapes of the pdf and cdf of CTGD for selected values of parameters \( \lambda_1 \) and \( \lambda_2 \) where \( \mu = 2 \) and \( \beta = 2.\)

From plot of pdf of Fig. 1, we can observe that for the positive values of both transmuted parameters \( \lambda_1 \) and \( \lambda_2 \), the distribution is approximately mesokurtic symmetrical shape, while for the negative values of \( \lambda_1 \) and \( \lambda_2 \) the distribution is shifting right with platykurtic shape.

Fig. 1: The pdf of CTGD for different value of $\lambda_1$ and $\lambda_2$ where $\mu = 2$ and $\beta = 2$.

Fig. 2: The cdf of CTGD for different value of $\lambda_1$ and $\lambda_2$ where $\mu = 2$ and $\beta = 2$.

2.2 Survival and Hazard function

The survival function is defined as $S(x) = 1 - F(x)$ and for the CTGD is given as

$$s(x) = 1 - e^{-\gamma[(1 + \lambda_1) + (\lambda_2 - \lambda_1)e^{-\gamma} - \lambda_2e^{-2\gamma}]}$$

The hazard function is defined as $h(x) = \frac{f(x)}{S(x)}$ and for the CTGD is given as

$$h(x) = \frac{\gamma([1 + \lambda_1] + 2(\lambda_2 - \lambda_1)e^{-\gamma} - 3\lambda_2e^{-2\gamma}]}{\beta([1 + \lambda_1] + (\lambda_2 - \lambda_1)e^{-\gamma} - \lambda_2e^{-2\gamma})}$$

Fig. 3 and 4 show respectively some possible shapes of the survival and hazard functions for the CTGD using different combination of model parameters $\lambda_1$ and $\lambda_2$ where $\mu = 2$ and $\beta = 2$.

3 Statistical Properties

In this section, some statistical properties for the proposed distribution, CTGD is discussed. These properties involve moments, moment generating function, quantile function and simulation of the random sample.
3.1 The Moments

**Theorem 2.** Let \( X \) be a random variable having the CTGD, then the \( r \)th moment of \( X \) about the origin is

\[
E(X^r) = \sum_{i=0}^{r} \binom{r}{i} \mu^{(r-i)}(-\beta)^i \left[ (1 + \lambda_1)\Gamma^{(i)}(1) \right] \\
+ \sum_{k=0}^{i} \left( \frac{i}{k} \right) \Gamma^{(k)}(1) T_{i,k} : r = 0, 1, 2, \ldots
\]

where \( \Gamma^{(i)}(1) = \int_0^\infty [\ln(z)]^i e^{-z} dz \) is the \( r \)th derivative of gamma function and

\[
T_{i,k} = \left[ (\lambda_2 - \lambda_1)(\ln(\frac{1}{\beta}))^{(i-k)} - (\lambda_2)(\ln(\frac{1}{\beta}))^{(i-k)} \right].
\]

**Proof.** we know that

\[
E(X^r) = \int_{-\infty}^{\infty} x^r f(x) dx = \int_{0}^{\infty} [\mu - \beta \ln(z)]^r f(z) \left( \frac{-\beta}{z} \right) dz \quad \text{ (see Prop. 2)}
= \int_{0}^{\infty} [\mu - \beta \ln(z)]^r e^{-z} \left[ (1 + \lambda_1) + 2(\lambda_2 - \lambda_1) e^{-\beta z} - 3\lambda_2 e^{-2\beta z} \right] dz
\]

Therefore,

\[
E(X^r) = (1 + \lambda_1) I_1 + 2(\lambda_2 - \lambda_1) I_2 - 3\lambda_2 I_3
\]

where

\[
I_1 = \int_{0}^{\infty} [\mu - \beta \ln(z)]^r e^{-z} dz,
I_2 = \int_{0}^{\infty} [\mu - \beta \ln(z)]^r e^{-2z} dz \quad \text{and}
I_3 = \int_{0}^{\infty} [\mu - \beta \ln(z)]^r e^{-3z} dz
\]

Now, we calculate the value of \( I_1 \) using binomial

\[
I_1 = \int_{0}^{\infty} [\mu - \beta \ln(z)]^r e^{-z} dz
= \int_{0}^{\infty} \sum_{i=0}^{r} \binom{r}{i} \mu^{(r-i)}(-\beta)^i [\ln(z)]^i e^{-z} dz
= \sum_{i=0}^{r} \binom{r}{i} \mu^{(r-i)}(-\beta)^i \int_{0}^{\infty} [\ln(z)]^i e^{-z} dz
= \sum_{i=0}^{r} \binom{r}{i} \mu^{(r-i)}(-\beta)^i \Gamma^{(i)}(1)
\]

Similarly, we can obtain the results

\[
I_2 = \frac{1}{2} \left\{ \sum_{i=0}^{r} \binom{r}{i} \mu^{(r-i)}(-\beta)^i \left[ \sum_{k=0}^{i} \binom{i}{k} \Gamma^{(k)}(1) \right] \right\} \times [-\ln(2)]^{(r-k)} \Gamma^{(l)}(1)
\]

and

\[
I_3 = \frac{1}{3} \left\{ \sum_{i=0}^{r} \binom{r}{i} \mu^{(r-i)}(-\beta)^i \left[ \sum_{k=0}^{i} \binom{i}{k} \Gamma^{(k)}(1) \right] \right\} \times [-\ln(3)]^{(r-k)} \Gamma^{(l)}(1)
\]

Substitute equations (11), (12), and (13) in Eq. (10) we get

\[
E(X^r) = \sum_{i=0}^{r} \binom{r}{i} \mu^{(r-i)}(-\beta)^i \left[ (1 + \lambda_1)\Gamma^{(i)}(1) \right] \\
+ \sum_{k=0}^{i} \left( \frac{i}{k} \right) \Gamma^{(k)}(1) T_{i,k}
\]

Therefore, the theorem is proved.

The mean and variance can be easily obtained by using \( r = 1, 2 \) in Eq(9) such that \( \Gamma^{(0)}(1) = 1, \Gamma^{(1)}(1) = -\gamma \) and \( \Gamma^{(2)}(1) = \gamma^2 + \frac{\pi^2}{6} \) where \( \gamma \approx 0.5772 \) is the Euler Mascheroni constant. We get

\[
E(X) = \theta(1 + \lambda_1) + (\lambda_2 - \lambda_1)(\theta + \beta \ln(2)) - \lambda_2(\theta + \beta \ln(3))
\]

\[
E(X^2) = (1 + \lambda_1) \left[ \theta^2 + \frac{(\beta \pi)^2}{6} \right] + (\lambda_2 - \lambda_1) \left[ \theta^2 + \theta \beta \ln(4) \right]
+ \beta^2 \left[ \frac{\pi^2}{6} + (\ln(2))^2 \right] - \lambda_2 \left[ \theta^2 + \theta \beta \ln(9) \right]
+ \beta^2 \left[ \frac{\pi^2}{6} + (\ln(3))^2 \right]
\]

Where \( \theta = \mu + \beta \gamma \)

\[
Var[X] = E(X^2) - [E(X)]^2
\]

The mean and variance of CTGD for various combinations of model parameters are given in Table 1 and Table 2 respectively. From Tables 1 and 2, it is observed that, holding the location and scale parameters \( \mu \) and \( \beta \) constants, as the transmuted parameters \( \lambda_1 \) and \( \lambda_2 \) increase the mean and variance of CTGD decrease. Whilst, holding \( \lambda_1 \) and \( \lambda_2 \) constants, as the scale parameter \( \beta \) increases the mean and variance also increase.
Table 1: Mean of the CTGD for various combinations of the parameters

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\beta$</th>
<th>$\lambda_2 / \lambda_1$</th>
<th>$\lambda_1 = -1$</th>
<th>$\lambda_1 = -0.5$</th>
<th>$\lambda_1 = 1$</th>
<th>$\lambda_1 = 0.5$</th>
<th>$\lambda_1 = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -1$</td>
<td>-0.3242</td>
<td>-0.6707</td>
<td>-1.7105</td>
<td>-1.3639</td>
<td>-1.0173</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -0.5$</td>
<td>-0.5269</td>
<td>-0.8735</td>
<td>-1.9132</td>
<td>-1.5666</td>
<td>-1.2201</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0$</td>
<td>-0.7296</td>
<td>-1.0762</td>
<td>-2.1159</td>
<td>-1.7694</td>
<td>-1.4228</td>
</tr>
<tr>
<td></td>
<td>$\beta = 1$</td>
<td>$\lambda_2 = 0.5$</td>
<td>-0.9324</td>
<td>-1.2789</td>
<td>-2.0233</td>
<td>-1.8582</td>
<td>-1.6255</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>-1.1351</td>
<td>-1.4817</td>
<td>-2.2264</td>
<td>-2.0023</td>
<td>-1.8282</td>
</tr>
<tr>
<td>$\mu = -2$</td>
<td>$\beta = 1$</td>
<td>$\lambda_2 = -1$</td>
<td>3.0275</td>
<td>1.9878</td>
<td>-1.1314</td>
<td>-0.0917</td>
<td>0.948</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -0.5$</td>
<td>2.4193</td>
<td>1.3796</td>
<td>-1.7396</td>
<td>-0.6999</td>
<td>0.3398</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0$</td>
<td>1.8111</td>
<td>0.7714</td>
<td>-2.3478</td>
<td>-1.3081</td>
<td>-0.5684</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0.5$</td>
<td>1.2029</td>
<td>0.1632</td>
<td>-1.9163</td>
<td>-0.8766</td>
<td>-0.447</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>0.5947</td>
<td>-0.445</td>
<td>-1.4847</td>
<td>-0.7334</td>
<td>-1.1847</td>
</tr>
<tr>
<td>$\mu = 0$</td>
<td>$\beta = 1$</td>
<td>$\lambda_2 = -1$</td>
<td>1.6758</td>
<td>1.3293</td>
<td>0.2895</td>
<td>0.6361</td>
<td>0.9827</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -0.5$</td>
<td>1.4731</td>
<td>1.1265</td>
<td>0.0868</td>
<td>0.4334</td>
<td>0.7799</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0$</td>
<td>1.2704</td>
<td>0.9238</td>
<td>-0.1159</td>
<td>0.2306</td>
<td>0.5772</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0.5$</td>
<td>1.0676</td>
<td>0.7211</td>
<td>-0.0279</td>
<td>0.3748</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>0.8649</td>
<td>0.5183</td>
<td>-</td>
<td>-</td>
<td>0.1718</td>
</tr>
<tr>
<td>$\beta = 3$</td>
<td>$\mu = 0$</td>
<td>$\lambda_2 = -1$</td>
<td>-1</td>
<td>5.0275</td>
<td>3.9878</td>
<td>0.8686</td>
<td>1.9083</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -0.5$</td>
<td>4.4193</td>
<td>3.3796</td>
<td>0.2604</td>
<td>1.3001</td>
<td>2.3398</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0$</td>
<td>3.8111</td>
<td>2.7714</td>
<td>-0.3478</td>
<td>0.6919</td>
<td>1.7316</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0.5$</td>
<td>3.2029</td>
<td>2.1632</td>
<td>-</td>
<td>0.0837</td>
<td>1.1234</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>2.5947</td>
<td>1.5550</td>
<td>-</td>
<td>-</td>
<td>0.5153</td>
</tr>
<tr>
<td>$\mu = -2$</td>
<td>$\beta = 1$</td>
<td>$\lambda_2 = -1$</td>
<td>3.6758</td>
<td>3.3293</td>
<td>2.2895</td>
<td>2.6361</td>
<td>2.9827</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -0.5$</td>
<td>3.4731</td>
<td>3.1265</td>
<td>2.0868</td>
<td>2.4334</td>
<td>2.7799</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0$</td>
<td>3.2704</td>
<td>2.9238</td>
<td>1.8841</td>
<td>2.2306</td>
<td>2.5772</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0.5$</td>
<td>3.0676</td>
<td>2.7211</td>
<td>-</td>
<td>2.0279</td>
<td>2.3745</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>2.8649</td>
<td>2.5183</td>
<td>-</td>
<td>-</td>
<td>2.1718</td>
</tr>
<tr>
<td>$\beta = 2$</td>
<td>$\mu = 2$</td>
<td>$\lambda_2 = -1$</td>
<td>7.0275</td>
<td>5.9878</td>
<td>2.8686</td>
<td>3.9083</td>
<td>4.948</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -0.5$</td>
<td>6.4193</td>
<td>5.3796</td>
<td>2.2604</td>
<td>3.3001</td>
<td>4.3398</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0$</td>
<td>5.8111</td>
<td>4.7714</td>
<td>1.6522</td>
<td>2.6919</td>
<td>3.7316</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0.5$</td>
<td>5.2029</td>
<td>4.1632</td>
<td>-</td>
<td>2.0837</td>
<td>3.1234</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>4.5947</td>
<td>3.555</td>
<td>-</td>
<td>-</td>
<td>2.5153</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$\beta$</td>
<td>$\lambda_2 / \lambda_1$</td>
<td>$\lambda_1 = -1$</td>
<td>$\lambda_1 = -0.5$</td>
<td>$\lambda_1 = 1$</td>
<td>$\lambda_1 = 0.5$</td>
<td>$\lambda_1 = 0$</td>
</tr>
<tr>
<td>-------</td>
<td>---------</td>
<td>-------------------------</td>
<td>------------------</td>
<td>------------------</td>
<td>----------------</td>
<td>----------------</td>
<td>----------------</td>
</tr>
<tr>
<td></td>
<td>$=1$</td>
<td>$\lambda_2 = 1$</td>
<td>1.6449</td>
<td>2.0461</td>
<td>1.8082</td>
<td>2.1277</td>
<td>2.2078</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -0.5$</td>
<td>1.686</td>
<td>1.9467</td>
<td>1.2872</td>
<td>1.7473</td>
<td>1.9671</td>
</tr>
<tr>
<td>$=2$</td>
<td></td>
<td>$\lambda_2 = 0$</td>
<td>1.6449</td>
<td>1.765</td>
<td>0.684</td>
<td>1.2846</td>
<td>1.6451</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0.5$</td>
<td>1.5216</td>
<td>1.5012</td>
<td></td>
<td>0.7397</td>
<td>1.2406</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>1.3161</td>
<td>1.1552</td>
<td></td>
<td></td>
<td>0.754</td>
</tr>
<tr>
<td>$=3$</td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>14.8044</td>
<td>18.4148</td>
<td>16.2739</td>
<td>19.1496</td>
<td>19.8633</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -0.5$</td>
<td>15.1743</td>
<td>17.52</td>
<td>11.585</td>
<td>15.7254</td>
<td>17.7037</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0$</td>
<td>14.8044</td>
<td>15.8854</td>
<td>6.1563</td>
<td>11.5613</td>
<td>14.8044</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0.5$</td>
<td>13.6947</td>
<td>13.511</td>
<td></td>
<td>6.6575</td>
<td>11.1653</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>11.8452</td>
<td>10.3968</td>
<td></td>
<td></td>
<td>6.7863</td>
</tr>
<tr>
<td></td>
<td>$=1$</td>
<td>$\lambda_2 = 1$</td>
<td>1.6449</td>
<td>2.0461</td>
<td>1.8082</td>
<td>2.1277</td>
<td>2.2078</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -0.5$</td>
<td>1.686</td>
<td>1.9467</td>
<td>1.2872</td>
<td>1.7473</td>
<td>1.9671</td>
</tr>
<tr>
<td>$=2$</td>
<td></td>
<td>$\lambda_2 = 0$</td>
<td>1.6449</td>
<td>1.765</td>
<td>0.684</td>
<td>1.2846</td>
<td>1.6449</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0.5$</td>
<td>1.5216</td>
<td>1.5012</td>
<td></td>
<td>0.7397</td>
<td>1.2406</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>1.3161</td>
<td>1.1552</td>
<td></td>
<td></td>
<td>0.754</td>
</tr>
<tr>
<td>$=3$</td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>14.8044</td>
<td>18.4148</td>
<td>16.2739</td>
<td>19.1496</td>
<td>19.8633</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -0.5$</td>
<td>15.1743</td>
<td>17.52</td>
<td>11.585</td>
<td>15.7254</td>
<td>17.7037</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0$</td>
<td>14.8044</td>
<td>15.8854</td>
<td>6.1563</td>
<td>11.5613</td>
<td>14.8044</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0.5$</td>
<td>13.6947</td>
<td>13.511</td>
<td></td>
<td>6.6575</td>
<td>11.1653</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>11.8452</td>
<td>10.3968</td>
<td></td>
<td></td>
<td>6.7863</td>
</tr>
<tr>
<td></td>
<td>$=1$</td>
<td>$\lambda_2 = 1$</td>
<td>1.6449</td>
<td>2.0461</td>
<td>1.8082</td>
<td>2.1277</td>
<td>2.2078</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -0.5$</td>
<td>1.686</td>
<td>1.9467</td>
<td>1.2872</td>
<td>1.7473</td>
<td>1.9671</td>
</tr>
<tr>
<td>$=2$</td>
<td></td>
<td>$\lambda_2 = 0$</td>
<td>1.64488</td>
<td>1.765</td>
<td>0.684</td>
<td>1.2846</td>
<td>1.64488</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0.5$</td>
<td>1.5216</td>
<td>1.5012</td>
<td></td>
<td>0.7397</td>
<td>1.2406</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>1.3161</td>
<td>1.1552</td>
<td></td>
<td></td>
<td>0.754</td>
</tr>
<tr>
<td>$=3$</td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>14.8044</td>
<td>18.4148</td>
<td>16.2739</td>
<td>19.1496</td>
<td>19.8633</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = -0.5$</td>
<td>15.1743</td>
<td>17.52</td>
<td>11.585</td>
<td>15.7254</td>
<td>17.7037</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0$</td>
<td>14.8044</td>
<td>15.8854</td>
<td>6.1563</td>
<td>11.5613</td>
<td>14.8044</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 0.5$</td>
<td>13.6947</td>
<td>13.511</td>
<td></td>
<td>6.6575</td>
<td>11.1653</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lambda_2 = 1$</td>
<td>11.8452</td>
<td>10.3968</td>
<td></td>
<td></td>
<td>6.7863</td>
</tr>
</tbody>
</table>
3.2 The Moment Generating Function of CTGD

The moment generating function of CTGD is present in this theorem.

Theorem 3. Let $X$ be a random variable having the CTGD, then the moment generating function (mgf) of $X$ is

$$M_X(t) = e^{t\Gamma(1 - \beta t)[(1 + \lambda_1) + 2\beta(\lambda_2 - \lambda_1) - 3\beta^2\lambda_2]}$$

where $\Gamma(1 - \beta t)$ is the Gamma function.

Proof. We know that

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_{-\infty}^{\infty} e^{t[x + \beta \ln(z)]} f(z) e^{-\frac{z}{\beta}} dz$$

$$= \int_{-\infty}^{\infty} e^{t[x + \beta \ln(z)] - \frac{z}{\beta}} [(1 + \lambda_1) + 2\beta(\lambda_2 - \lambda_1) - 3\beta^2\lambda_2]$$

where $I_1 = \int_{\infty}^{0} z^{-\beta} e^{-z} \, dz$, $I_2 = \int_{\infty}^{0} z^{\beta} e^{-z} \, dz$ and $I_3 = \int_{\infty}^{0} z^{\beta} e^{-3z} \, dz$

Now, we calculate the value of $I_1, I_2, I_3$ using the relation $\int_{0}^{\infty} t^{n} e^{-at} \, dt = \frac{n!}{a^n}$, we obtain

$I_1 = \int_{0}^{\infty} z^{-\beta} e^{-z} \, dz = \Gamma(1 - \beta t)$,

$I_2 = \int_{0}^{\infty} z^{\beta} e^{-z} \, dz = 2^{\beta - 1} \Gamma(1 - \beta t)$ and

$I_3 = \int_{0}^{\infty} z^{\beta} e^{-3z} \, dz = 3^{\beta - 1} \Gamma(1 - \beta t)$.

By Substituting in Eq. (15), we get

$M_X(t) = e^{t\Gamma(1 - \beta t)[(1 + \lambda_1) + 2\beta(\lambda_2 - \lambda_1) - 3\beta^2\lambda_2]}$

So the theorem is proved.

3.3 Quantile function of CTGD

The quantile function for CTGD is derived by finding the value of $Q$ for which $F(x) = p$:

$$Q(p, \mu, \beta) = \mu - \beta \ln[-\ln(p)]$$

where

$$y = \left[\frac{\sqrt{\theta_2 + \sqrt{\theta_2^2 + 4\theta_1}}}{2a\sqrt{3}}\right] - \theta_1 \frac{\sqrt{\theta_3}}{3a\sqrt{\theta_2 + \sqrt{\theta_2^2 + 4\theta_1}}} - \left(\frac{b}{3a}\right)$$

such that

$$\theta_1 = 3ac - b^2, \theta_2 = -2b^3 + 9abc - 27da^2$$

and

$$\theta_3 = \theta_2^3, \theta_4 = -1 + \lambda_1 + \lambda_2, c = -1 + \lambda_1 + \lambda_2$$

are the coefficients of the cubic equation of $y$

The three quartiles $Q_1, Q_2$ and $Q_3$ can be obtained by using $p = 0.25, 0.50$ and $0.75$ in Eq. (16), respectively.

3.4 Simulating the Random Sample of CTGD

Random numbers from the CTGD can be obtained by equating cdf of the distribution in Eq. (7) with a uniform random number and inverting the expression, that is the random number from CTGD is obtained by solving $F(x) = u$ for $x$. The random sample from CTGD can be further expressed as

$$x = \mu - \beta \ln[-\ln(y)]$$

where $y$ is given in Eq. (17) with $d = a$ and $a$ is an arbitrary continuous uniform point over $(0, 1)$.

4 Parameters Estimation

This section pertains to discuss the maximum likelihood estimation (MLE) for parameters of CTGD.

Let $X_1, X_2, \ldots, X_n$ be a random sample of size $n$ from CTGD, then the likelihood function is given by

$$L = \prod_{i=1}^{n} f(x_i, \mu, \beta, \lambda_1, \lambda_2)$$

$$= \prod_{i=1}^{n} \left\{ \left(\frac{1}{\beta}\right) z_i e^{-\frac{z_i}{\beta}} \left[ (1 + \lambda_1) + 2(\lambda_2 - \lambda_1)e^{-\frac{z_i}{\beta}} - 3\lambda_2 e^{-2\frac{z_i}{\beta}} \right]\right\}$$

where $z_i = e^{-\frac{\lambda_1 x_i}{\beta}}; i = 1, 2, \ldots, n.$

so, the log-likelihood function is

$$l = -n \ln(\beta) + \sum_{i=1}^{n} \left\{ \ln z_i - z_i + \ln \left[ (1 + \lambda_1) + 2(\lambda_2 - \lambda_1)e^{-\frac{z_i}{\beta}} - 3\lambda_2 e^{-2\frac{z_i}{\beta}} \right]\right\}$$

(19)

Therefore, the maximum likelihood estimates of $\mu, \beta, \lambda_1$ and $\lambda_2$ which maximize Eq. (19), must satisfy the four equations (20), (21), (22) and (23).
The maximum likelihood estimates \( \hat{\theta} = (\hat{\mu}, \hat{\beta}, \hat{\lambda}_1, \hat{\lambda}_2) \) of \( \theta = (\mu, \beta, \lambda_1, \lambda_2) \) is obtained by solving the non-linear system of equations (20), (21), (22) and (23).

5 Application of CTGD

In this section, the CTGD is applied to two real data sets. The first data in Table 3 corresponds to a wind velocity (WVD) involving 246 observations of the maximum of monthly wind speed (mph) in Palm Beach, Florida (USA) for the months January 1984 to December, 2005. Data is available for download from NOAA website. The second data in Table 4 is related to the snow accumulation (SAD) in inches in the Raleigh-Durham airport, North Carolina, from 1948 to 2000. The data set contains 63 observations. Summary statistics of the two data sets are demonstrated in Table 5. The maximum likelihood estimates, the log-likelihood value (-Log(L)), the Kolmogorov-Smirnov (k-s) test statistic and the p-value for the k-s statistic for the fitted distributions are demonstrated in Tables 6 and 7 respectively. Latterly, Yolanda et al. [15] used the data in Tables 3 and 4 to investigate model fitting for Gumbel (GD), Slash (SD) and Slashed Gumbel (SGD) distributions.
Table 7: Parameters estimates, -log (L), k-s test value and p-value for Gumbel, Transmuted Gumbel, and Cubic Transmuted Gumbel for SAD data

<table>
<thead>
<tr>
<th>Model</th>
<th>Parameters estimates</th>
<th>-log(L)</th>
<th>k-s</th>
<th>P-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>GD</td>
<td>$\beta = 0.0802$  $\mu = 36.466$</td>
<td>898.880</td>
<td>0.0671</td>
<td>0.1848</td>
</tr>
<tr>
<td>TGD</td>
<td>$\beta = 7.260$  $\mu = 38.837$  $\lambda = 0.6233$</td>
<td>896.752</td>
<td>0.0612</td>
<td>0.2759</td>
</tr>
<tr>
<td>CTGD</td>
<td>$\beta = 7.5512$  $\mu = 39.7094$  $\lambda_1 = 0.99$  $\lambda_2 = -0.360$</td>
<td>895.894</td>
<td>0.0629</td>
<td>0.2456</td>
</tr>
</tbody>
</table>

By comparing the goodness of fit statistics in Table 6 among the three distributions, it is clear that all distributions are competitors and fit the wind velocity data well but the proposed distribution CTGD leads to a better fit than the other two distributions (see Fig. 5). Moreover, basing on -2log (L) criteria (the smaller the better), CTGD performs better than GD and TGD. Regarding the snow accumulation data, from Table 7 it is clear that the three distributions fit the data well but the TGD is the best (see Fig. 6), whilst, basing on -2log (L) criteria, the CTGD performs better than GD and TGD.

6 Conclusions

In this paper, a new generalized version of the Gumbel distribution which called cubic transmuted Gumbel distribution (CTGD) is introduced by using the generalization formula for transmuted distribution proposed by [11]. Some statistical properties of CTGD are derived. The model parameters are estimated by the maximum likelihood method. Finally, an application of CTGD to two real data sets and compared with some distributions based on Gumbel distribution is explained. We conclude that the applications suggest that the proposed distribution CTGD fits the two data sets very well. Generally, CTGD performs better than the other distributions. We recommend the CTGD for modelling extreme data sets in the field of flood frequency analysis, network, space, software reliability, structural and wind engineering and hope that it would receive significant applications in the future.

Acknowledgement

The authors are grateful to the editor and the anonymous reviewers for their valuable comments and suggestions, which have substantially improved this paper.

Conflicts of Interests

The authors declare that they have no conflicts of interests.

References


