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The Lifetime Performance Index of Power Lomax Distribution Based on Progressive First-Failure Censoring Scheme

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Abstract: Evaluating the lifetime performance is a vital topic in manufacturing process. This paper is devoted to evaluate the lifetime performance index C_L for the three-parameter power Lomax distribution(POLO) under progressive first-failure type II right censoring sample with respect to a lower specification limit (L). The statistical inference concerning C_L is conducted via obtaining the maximum likelihood of C_L on the base of progressive first-failure censoring. The asymptotic normal distribution of the MLE of C_L and the confidence interval are proposed. Moreover, the hypothesis testing of C_L for evaluating the lifetime performance of POLO data is conducted. Providers can practice the innovative hypothesis testing to improve the process capability. Finally, two examples are given, one of them considering a real life data of the number of revolutions before failure of a ball bearing in endurance lifetime test and the other is a simulated example to illustrate the usage of the proposed procedure.

Keywords: Inference; Performance; Censoring; First-Failure Progressive; Power Lomax Distribution.

1 Introduction

Dimensions of the product quality have different descriptions and evaluated via several issues for example; performance, reliability, conformance to the standards. Process capability indices (PCIs) have been used to measure the attainment of the product quality level. One of PCIs indices measures are the target-the better type. The other measures are the larger-the better type quality features and the smaller-the better type. The lifetime performance index C_L is one of the recommended PCIs indices, which exhibits the larger-the-better quality measurement. [1] recommended the usage of C_L for evaluating the performance of the products lifetime. The analysis of process capability is mainly depended on the normality assumption of the population. However, the normality is very problematic in manufactures, engineering and business processing. The lifetime model for many products may follow non-normal distributions. It may include; exponential, gamma, Rayleigh, Weibull, Burr, Lomax , power Lomax and others.

Censored data are recommended to solve many problems in life testing experiments, saving time and money, working the test under restrictions in materials or any difficulties in planning the experimental test. There are many censoring schemes in survival analysis see [2]. Type-II right censoring is the most common. The construction of the type II right censoring is as follows; suppose that out of n items put on life test, put m items $X_{1:n} \leq X_{2:n}, \ldots . \leq X_{m:n}$ only under observation. The rest $n-m$ components remain unobserved or missing. One of the generalization of the type II right censoring is the progressive type II censoring which allows for units to be removed from the test at a different time of termination the test. The description of this type is as follows; n items are put on a test and the termination of the test is determined when the $m-^{th}$ fails occurred. As well as the i^{th} item fails $(i = 1,2,\ldots m-1\}$, randomly R_i of the surviving items are removed. At the end of the test all $R_m = n-m-\sum_{i=1}^{m} R_i$ are removed. To have shorter expected test times than the progressive type II censoring scheme, the progressive first-failure right type II censoring scheme is performed to generalize all the above schemes, see for example; [3]. Its construction is as follows; N items are divided in to n disjoint groups, each has k items $(N = n \times k)$, put on a test. The life test is terminated at the m-th fail happened. When the i th item fails $(i = 1,2,\ldots m-1\}$, select randomly R_i groups and remove the group which has the i th failure. When the $m-^{th}$ failure

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occurred, remove all the remaining groups from the test. Actually, the progressive first failure type II right censoring has generalized the above other schemes.

For analyzing the performance index, there are many publications in the literature on different censoring schemes for some lifetime distributions, for example; [3] estimating the life performance index with Weibull distribution under first failure progressive censoring. The procedure of the performance index under Pareto distribution with right type II censored is studied in [4]. Implementing of performance index of Burr XII distribution are given in [5] and [6] under progressive censoring. Furthermore, many publications are conducted to study the progressive censoring and evaluating the performance index under exponential distribution for different censoring schemes for instance; [7], [8] and [9]. Moreover, inferences of the lifetime performance index for Lomax distribution based on progressively type II censored data is introduced in [10].

Large sample is the cornerstone of statistical inference for quality performance and capability evaluation model. The limiting distribution of a statistic provides an approximate distributional result that are often direct determined, even in complicate quality performance evaluation procedure [6]. This work is proposed to evaluate the quality of the lifetime performance index and study the statistical inference of the product under power Lomax distribution (POLO) with large sample and first-failure progressively type II right censoring sample.

The organization of the paper is as follows. Section 2 involved the lifetime performance index of POLO distribution. The conforming rate is studied in Section 3. Section 4 includes the Maximum Likelihood Estimator (MLE) of lifetime performance index. The testing process for the lifetime performance index is given in Section 5. Finally, two numerical examples of real lifetime data and simulated data are given in Section 6 to apply the theoretical obtained results.

2 The Lifetime Performance Index

A longer lifetime implies a better product quality. Hence, the lifetime is a larger-the better-type quality characteristic. [1] has developed a process capability performance index C_L to measure the above characteristic. Then, C_L is defined by

$$C_L = \frac{\mu - L}{\sigma} \tag{1}$$

where μ, σ are the mean and the standard deviation of the process and L is the lower specification limit where the lifetime is required to exceed L unit times to be both money-wise profitable and satisfying customers.

To evaluate the lifetime performance of products, C_L can be defined as the *lifetime performance index*. Throughout this paper, consider that the random variable X follows the Power Lomax distribution $POLO(\alpha,\beta,\lambda)$ [11] with the probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) which are

$$f_X\left(x;\alpha,\beta,\lambda\right) = \alpha\beta\lambda^{\alpha}x^{\beta-1}\left(\lambda+x^{\beta}\right)^{-\alpha-1} \quad , \ x>0,\alpha,\beta,\lambda>0 \tag{2}$$

and

$$F_X\left(x;\alpha,\beta,\lambda\right) = 1-\lambda^{\alpha}\left(x^{\beta}+\lambda\right)^{-\alpha} \qquad , \ x>0,\alpha,\beta,\lambda>0 \tag{3}$$

with mean μ and standard deviation σ as follows:

$$\mu = \frac{\lambda^{\frac{1}{\beta}}\Gamma\left[\alpha-\frac{1}{\beta}\right]\Gamma\left[\frac{1}{\beta}\right]}{\beta\Gamma[\alpha]} \quad , \ \alpha\,\beta>1, \quad \alpha,\beta,\lambda>0 \tag{4}$$

$$\sigma = \frac{\lambda^{\frac{1}{\beta}}\sqrt{\left(\Gamma[\alpha]\Gamma\left[\alpha-\frac{2}{\beta}\right]\ \Gamma\left[\frac{2+\beta}{\beta}\right]-\Gamma\left[\alpha-\frac{1}{\beta}\right]^{2}\Gamma\left[1+\frac{1}{\beta}\right]^{2}\right)}}{\Gamma[\alpha]}, \alpha\,\beta>2\ ,\ \alpha,\beta,\lambda>0 \tag{5}$$

Then the lifetime performance index C_L is derived as

$$C_L = \frac{\Gamma\left[\alpha-\frac{1}{\beta}\right]\Gamma\left[\frac{1}{\beta}\right]-\beta\lambda^{\beta}L\ \Gamma[\alpha]}{\beta\sqrt{\Gamma[\alpha]\Gamma[\alpha-\frac{2}{\beta}]\Gamma[\frac{2+\beta}{\beta}]-\Gamma[\alpha-\frac{1}{\beta}]^{2}\Gamma[1+\frac{1}{\beta}]^{2}}} \tag{6}$$

where $-\infty < C_L < \frac{\Gamma\left[\alpha-\frac{1}{\beta}\right]\Gamma\left[\frac{1}{\beta}\right]}{\beta\sqrt{\Gamma[\alpha]\Gamma[\alpha-\frac{2}{\beta}]\Gamma[\frac{2+\beta}{\beta}]-\Gamma[\alpha-\frac{1}{\beta}]^{2}\Gamma[1+\frac{1}{\beta}]^{2}}}$, $\alpha\,\beta>2$, $\alpha,\beta,\lambda>0$

The failure rate $h(x)$ is

$$h(x) = \frac{x^{\beta-1}\alpha\beta}{x^{\beta}+\lambda}, \quad x>0,\alpha,\beta,\lambda>0 \tag{7}$$

For different values of α, β and λ, the failure rate function takes various shapes where, it decreases at $\alpha>0$, $0<\beta\le 1, \lambda>0$, and if $\alpha>0, \beta>1, \lambda>0$, it is unimodal with critical point $x=(\lambda\beta-\lambda)^{\frac{1}{\beta}}$.

When the process mean $\frac{\lambda^{\frac{1}{\beta}}\Gamma\left[\alpha-\frac{1}{\beta}\right]\Gamma\left[\frac{1}{\beta}\right]}{\beta\Gamma[\alpha]} > L$, then the lifetime performance index $C_L>0$ where $\alpha\,\beta>2$, $\alpha,\beta,\lambda>0$. For $\beta>1$, $\alpha\,\beta>2$, $x>(\lambda\beta-\lambda)^{\frac{1}{\beta}}$, if x is large, and α is small then the lifetime performance C_L is relatively large and the failure rate is relatively small. *Consequently, the lifetime performance index* C_L *is reasonably and definitely describes the lifetime performance of products.*

3 The Conforming Rate

The product is defined as a conforming product, if its lifetime exceeds the lower specification limit L. The ratio of conforming product is known as the conforming rate and can be defined for $X \sim POLO(\alpha,\ \beta,\ \lambda\ )$ as follows

$$\begin{aligned} P_r &= P(X \ge L) \\ &= \lambda^{\alpha}\left(\left(\left(\frac{\lambda^{\frac{1}{\beta}}\Gamma[\alpha-\frac{1}{\beta}]\Gamma[\frac{1}{\beta}]}{\beta\Gamma[\alpha]} - \frac{C_L\,\lambda^{\frac{1}{\beta}}}{\Gamma[\alpha]}\sqrt{\Gamma[\alpha]\Gamma[\alpha-\frac{2}{\beta}]\Gamma[\frac{2+\beta}{\beta}-\Gamma[\alpha-\frac{1}{\beta}]^{2}\Gamma[1+\frac{1}{\beta}]^{2}]}\right)^{\beta}+\lambda\right)^{-\alpha}\right) \end{aligned} \tag{8}$$

where $-\infty < C_L < \frac{\Gamma\left[\alpha-\frac{1}{\beta}\right]\Gamma\left[\frac{1}{\beta}\right]}{\beta\sqrt{\Gamma[\alpha]\Gamma[\alpha-\frac{2}{\beta}]\Gamma[\frac{2+\beta}{\beta}]-\Gamma[\alpha-\frac{1}{\beta}]^{2}\Gamma[1+\frac{1}{\beta}]^{2}}}$, $\alpha\,\beta>2$, $\alpha,\beta,\lambda>0$

Table 1: The lifetime performance index C_L v.s. the conforming rate P_r for POLO distribution with $\left(\widehat{\alpha},\widehat{\beta},\widehat{\lambda}\right) = (1.00706,\ 4.92447,\ 0.250471)$

C_L	P_r	C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.00000000	-4.00	0.00623618	0.2	0.519161	0.8	0.827807
-11.00	0.00016704	-3.00	0.0141697	0.25	0.546198	0.85	0.847822
-10.00	0.000244677	-2.00	0.0374334	0.3	0.573561	0.9	0.866413
-9.00	0.000369982	-1.00	0.119076	0.4	0.628616	0.95	0.883544
-8.00	0.000580882	-0.50	0.224834	0.5	0.682948	1.00	0.899202
-7.00	0.000954028	-0.25	0.308518	0.6	0.73512	1.5	0.98511
-6.00	0.0016554	0.000	0.416883	0.7	0.783781	2.00	0.999512
-5.00	0.00307586	0.10	0.466619	0.75	0.806431	2.4	1

Note that: $C_L \to \frac{\Gamma\left[\alpha-\frac{1}{\beta}\right]\Gamma\left[\frac{1}{\beta}\right]}{\beta\sqrt{\Gamma[\alpha]\Gamma[\alpha-\frac{2}{\beta}]\Gamma[\frac{2+\beta}{\beta}]-\Gamma[\alpha-\frac{1}{\beta}]^{2}\Gamma[1+\frac{1}{\beta}]^{2}}} \approx 2.49515 \Rightarrow P_r \to 1.0.$

Table 2: The lifetime performance index C_L v.s. the conforming rate P_r for POLO distribution with $\left(\widehat{\alpha},\widehat{\beta},\widehat{\lambda}\right) = (24.656, 14.1082,\ 0.92693)$.

C_L	P_r	C_L	P_r	C_L	P_r	C_L	P_r
$-\infty$	0.00000000	0.3	0.661717	0.7	0.781501	4.00	0.99866
-2.00	0.00476918	0.35	0.678701	0.75	0.793892	4.50	0.999502
-1.00	0.146998	0.4	0.69512	0.8	0.805727	5.00	0.999828
-0.5	0.336318	0.45	0.710964	0.85	0.817019	5.5	0.999946
-0.25	0.444122	0.5	0.72623	0.9	0.827779	6.00	0.999985
0	0.548916	0.55	0.740915	0.95	0.838021	7.00	0.999995
0.1	0.588435	0.6	0.755019	2.00	0.961161	7.5	0.999999
0.25	0.644181	0.65	0.768546	2.5	0.98188	11.3	1

Note that: $C_L \to \dfrac{\Gamma\left[\alpha-\frac{1}{\beta}\right]\Gamma\left[\frac{1}{\beta}\right]}{\beta\sqrt{\Gamma[\alpha]\Gamma[\alpha-\frac{2}{\beta}]\Gamma[\frac{2+\beta}{\beta}]-\Gamma[\alpha-\frac{1}{\beta}]^2\Gamma[1+\frac{1}{\beta}]^2}} \approx 11.3724 \Rightarrow P_r \to 1.0$

By observing the above values of C_L and P_r, there is a strictly increasing relationship between them for given $\alpha,\ \beta,\ and\ \lambda$. The construction of Tables (1,2) is depended on the obtained values of the parameter estimates and it helps to get the corresponding values of C_L which satisfy the required conforming rate for the Examples given in Section (6).

4 Maximum Likelihood Estimator of Lifetime Performance Index

Let $X_{1:m:n:k},\ X_{2:m:n:k},\ldots\ldots,X_{m:m:n:k}$ be the progressive first-failure type II right censored sample from a continuous population with p.d.f and c.d.f $f_X\left(.;\ \theta\right)$ and $F_X\left(.;\ \theta\right)$ respectively, where θ is a vector of parameters. Following [12], the associated likelihood function of the observed data $X = \left(x_{1:m:n:k},\ x_{2:m:n:k},\ldots\ldots,x_{m:m:n:k}\right)$ is given by

$$L\left(\theta,X\right) = Ck^m\prod_{i=1}^{m} f_X\left(x_{i:m:n:k};\ \theta\right)\left(1-F_X\left(x_{i:m:n:k};\ \theta\right)\right)^{k(R_i+1)-1} \tag{9}$$

where $0 < x_{1:m:n:k} <\ x_{2:m:n:k} < \ldots\ldots < x_{m:m:n:k} < \infty$ and $C = n\left(n-R_1-1\right)\left(n-R_2-1\right)\ldots\ldots\ldots\left(n-\sum_{i=1}^{m-1}R_i-m+1\right)$.

Consider that the progressive first- failure type II right censoring sample from a life test of n products whose lifetimes follow $POLO\left(\alpha,\ \beta,\ \lambda\ \right)$ distribution. From (2) and (3), the likelihood function is as follows

$$L\left(\alpha,\beta,\lambda;X\right) = C\ k^m\ \prod_{i=1}^{m}\alpha\ \beta\ \lambda^\alpha x_{i:m:n:k}^{\ \beta-1}\left(\lambda+x_{i:m:n:k}^{\ \beta}\right)^{-(\alpha+1)}\left[\lambda^\alpha\left(\lambda+x_{i:m:n:k}^{\ \beta}\right)^{-\alpha}\right]^{k(R_i+1)-1} \tag{10}$$

The natural Logarithm of $L\left(\alpha,\beta,\lambda;X\right)$ is obtained as

$$\begin{aligned}\ln\left(L\left(\alpha,\beta,\lambda;X\right)\right) &= \ln\left(C\right)+m\ \ln\left(k\right)+m\ln\left(\alpha\right)\ +m\ln\left(\beta\right)+\left(\beta-1\right)\sum_{i=1}^{m}\ln\left(x_{i:m:n:k}\right)\\ &-\sum_{i=1}^{m}\ln\left(\left(\lambda+x_{i:m:n:k}^{\ \beta}\right)\right)\ \left[1+\alpha\ k\ \left(R_i+1\right)\right]+\alpha\ k\ \ln(\lambda)\sum_{i=1}^{m}\left(R_i+1\right)\end{aligned} \tag{11}$$

The MLE $\widehat{\theta} = \left(\widehat{\alpha},\ \widehat{\beta},\ \widehat{\lambda}\right)$ can be obtained by equating the first partial derivative of (11) with respect to $\alpha,\ \beta,\ \ and\ \lambda$.The likelihood Equations for the parameters $\alpha,\ \beta\ and\ \lambda$ are obtained as follows

$$\begin{aligned}\frac{\partial\ ln\left(L\left(\alpha,\beta,\lambda;X\right)\right)}{\partial\alpha} &= \frac{m}{\alpha}-\sum_{i=1}^{m}k\left(R_i+1\right)\left[ln(\ \lambda+x_{i:m:n:k}^{\ \beta})-ln(\lambda)\right]\\ \frac{\partial\ ln\left(L\left(\alpha,\beta,\lambda;X\right)\right)}{\partial\beta} &= \frac{m}{\beta}+\sum_{i=1}^{m}ln(x_{i:m:n:k})-\sum_{i=1}^{m}\left[1+\alpha\ k\ \left(R_i+1\right)\right]\frac{x_{i:m:n:k}^{\ \beta}ln(x_{i:m:n:k})}{\lambda+x_{i:m:n:k}^{\ \beta}}\\ \frac{\partial\ ln\left(L\left(\alpha,\beta,\lambda;X\right)\right)}{\partial\lambda} &= -\sum_{i=1}^{m}\left[1+\alpha\ k\ \left(R_i+1\right)\right]\frac{1}{\lambda+x_{i:m:n:k}^{\ \beta}}+\frac{\alpha\ k}{\lambda}\sum_{i=1}^{m}\left(R_i+1\right)\end{aligned} \tag{12}$$

Hence;

The MLE of α, β, *and* λ can be obtained by

$$\widehat{\alpha} = \frac{m}{\sum_{i=1}^{m} k\,(R_i+1)\,[ln(\,\widehat{\lambda}+x_{\,i:m:n:k}^{\,\widehat{\beta}}) - \ln(\widehat{\lambda})]} \tag{13}$$

$$\widehat{\beta} = \frac{m}{\sum_{i=1}^{m} ln\,(x_{i:m:n:k})\;[[1+\,\widehat{\alpha}\;k\;(R_i+1)]\,\frac{x_{\,i:m:n:k}^{\,\widehat{\beta}}}{\widehat{\lambda}+x_{\,i:m:n:k}^{\,\widehat{\beta}}} - 1]} \tag{14}$$

$$\widehat{\lambda} = \frac{\widehat{\alpha}\;k\;\sum_{i=1}^{m}(R_i+1)}{\sum_{i=1}^{m}[1+\,\widehat{\alpha}\;k\;(R_i+1)]\,\frac{1}{\widehat{\lambda}+x_{\,i:m:n:k}^{\,\widehat{\beta}}}} \tag{15}$$

The closed form of the above Equations are very hard to analytically solved, hence, these non- linear Equations will be solved numerically.

Following [13], the invariance property of the MLE satisfies, then the MLE of C_L has a form

$$\widehat{C_L} = \frac{\Gamma\left[\widehat{\alpha}-\frac{1}{\widehat{\beta}}\right]\Gamma\left[\frac{1}{\widehat{\beta}}\right] - \widehat{\beta}\widehat{\lambda}^{\,\widehat{\beta}}L\;\Gamma[\widehat{\alpha}]}{\widehat{\beta}\sqrt{\Gamma[\widehat{\alpha}]\Gamma[\widehat{\alpha}-\frac{2}{\widehat{\beta}}]\Gamma[\frac{2+\widehat{\beta}}{\widehat{\beta}}] - \Gamma[\widehat{\alpha}-\frac{1}{\widehat{\beta}}]^2\Gamma[1+\frac{1}{\widehat{\beta}}]^2}} \tag{16}$$

Following [12] and [14], the asymptotic normal distribution for the MLEs has been obtained as

$$\frac{\partial^2\, ln\,(L\,(\alpha,\beta,\lambda;X))}{\partial\alpha^2} = \frac{-m}{\alpha^2}\,, \tag{17}$$

$$\frac{\partial^2\, ln\,(L\,(\alpha,\beta,\lambda;X))}{\partial\alpha\partial\beta} = -\sum_{i=1}^{m}\frac{k(R_i+1)x_{\,i:m:n:k}^{\,\beta}ln(x_{\,i:m:n:k}^{\,\beta})}{\lambda+x_{\,i:m:n:k}^{\,\beta}}, \tag{18}$$

$$\frac{\partial^2\, ln\,(L\,(\alpha,\beta,\lambda;X))}{\partial\alpha\partial\lambda} = -\sum_{i=1}^{m}\frac{k\,(R_i+1)}{\lambda+x_{\,i:m:n:k}^{\,\beta}} + \frac{k}{\lambda}\sum_{i=1}^{m}(R_i+1)\,, \tag{19}$$

$$\frac{\partial^2\, ln\,(L\,(\alpha,\beta,\lambda;X))}{\partial\beta^2} = \frac{-m}{\beta^2} - \sum_{i=1}^{m}[1+\alpha\;k\;(R_i+1)]\,\lambda x_{\,i:m:n:k}^{\,\beta}[\frac{\ln(x_{i:m:n:k})}{\lambda+x_{\,i:m:n:k}^{\,\beta}}]^2, \tag{20}$$

$$\frac{\partial^2\, ln\,(L\,(\alpha,\beta,\lambda;X))}{\partial\beta\partial\lambda} = \sum_{i=1}^{m}[1+\alpha\;k\;(R_i+1)]\,[\frac{x_{\,i:m:n:k}^{\,\beta}ln(x_{\,i:m:n:k}^{\,\beta})}{(\lambda+x_{\,i:m:n:k}^{\,\beta})^2}\,, \tag{21}$$

$$\frac{\partial^2\, ln\,(L\,(\alpha,\beta,\lambda;X))}{\partial\lambda^2} = \sum_{i=1}^{m}\frac{[1+\alpha\;k\;(R_i+1)]}{(\lambda+x_{\,i:m:n:k}^{\,\beta})^2} - \frac{\alpha\;k}{\lambda^2}\sum_{i=1}^{m}(R_i+1). \tag{22}$$

According to [14] under some regularity conditions, the asymptotic normality of MLE of θ is

$$\widehat{\theta} \sim N\left(\theta, I\,(\theta)^{-1}\right). \tag{23}$$

where $I(\theta)$ is the Fisher information matrix. By considering the approximate information matrix $I_O(\widehat{\theta})$ which is defined by

$$I_O\left(\widehat{\theta}\right) = -\begin{bmatrix} \frac{\partial^2\;ln(L(\alpha,\beta,\lambda;X))}{\partial\alpha^2} & \frac{\partial^2\;ln(L(\alpha,\beta,\lambda;X))}{\partial\alpha\partial\beta} & \frac{\partial^2\;ln(L(\alpha,\beta,\lambda;X))}{\partial\alpha\partial\lambda} \\ \frac{\partial^2\;ln(L(\alpha,\beta,\lambda;X))}{\partial\beta\partial\alpha} & \frac{\partial^2\;ln(L(\alpha,\beta,\lambda;X))}{\partial\beta^2} & \frac{\partial^2\;ln(L(\alpha,\beta,\lambda;X))}{\partial\beta\partial\lambda} \\ \frac{\partial^2\;ln(L(\alpha,\beta,\lambda;X))}{\partial\lambda\partial\alpha} & \frac{\partial^2\;ln(L(\alpha,\beta,\lambda;X))}{\partial\lambda\partial\beta} & \frac{\partial^2\;ln(L(\alpha,\beta,\lambda;X))}{\partial\lambda^2} \end{bmatrix}_{\widehat{\theta}}$$

$$=\begin{bmatrix} v_{\alpha\alpha} & v_{\alpha\beta} & v_{\alpha\alpha} \\ v_{\beta\alpha} & v_{\beta\beta} & v_{\beta\lambda} \\ v_{\lambda\alpha} & v_{\lambda\beta} & v_{\lambda\lambda} \end{bmatrix}_{\widehat{\theta}} \tag{24}$$

Using the variance-covariance matrix $I_O\left(\widehat{\theta}\right)^{-1}$to estimate $I\left(\theta\right)^{-1}$.

Let $C_L \equiv C(\theta)$, and due to [15] the multivariate delta method stated that the asymptotic normal distribution of $C(\widehat{\theta})$ is

$$\widehat{C_L} \equiv C(\widehat{\theta}) \sim N(\ C_L, \Psi_\theta) \tag{25}$$

The approximate asymptotic variance-covariance matrix $\Psi_{\widehat{\theta}}$ of $C(\theta)$ to estimate Ψ_θ is defined as

$$\Psi_{\widehat{\theta}} = \left(\frac{\partial C(\theta)}{\partial\alpha} \ \frac{\partial C(\theta)}{\partial\beta} \ \frac{\partial C(\theta)}{\partial\lambda} \right) I_O\left(\theta\right)^{-1} \begin{pmatrix} \frac{\partial C(\theta)}{\partial\alpha} \\ \frac{\partial C(\theta)}{\partial\beta} \\ \frac{\partial C(\theta)}{\partial\lambda} \end{pmatrix}_{\theta=\widehat{\theta}} \tag{26}$$

5 Testing Process for the Lifetime Performance Index

Constructing a statistical testing concerning the lifetime performance index is to judge whether it adheres to the required level. Consider c^* is the target value and assuming that the required index value of lifetime performance C_L is larger than c^*.Hence, the construction of the hypothesis testing is as follows

$$H_o: \quad C_L \leq c^*$$

against

$$H_1 \ : \quad C_L > c^*$$

Due to [6], taking $\widehat{C_L}$ to be asymptotic normal distribution (25), and the MLE of C_L is used as the test statistic, the required rejection region can be obtained as $\left\{ \widehat{C_L} \ where \ \widehat{C_L} > C_O \right\}$ where C_O is the critical value. It can be obtained at a specified significance level α^* from the formula

$$P\left(\frac{\widehat{C_L} - \ C_L}{\sqrt{\Psi_{\widehat{\theta}}}} = \frac{C_O - c^*}{\sqrt{\Psi_{\widehat{\theta}}}} \right) = 1-\alpha^*$$

where, $\frac{\widehat{C_L} - \ C_L}{\sqrt{\Psi_{\widehat{\theta}}}} \sim N(0,1)$. Then, $\frac{C_O - c^*}{\sqrt{\Psi_{\widehat{\theta}}}} = z_{\alpha^*}$ and the critical value is

$$C_O = c^* + z_{\alpha^*}\sqrt{\Psi_{\widehat{\theta}}} \tag{27}$$

Moreover, the $100\left(1-\alpha^*\right)\%$ one sided confidence interval of C_L is

$$C_L \geq \widehat{C_L} - z_{\alpha^*}\sqrt{\Psi_{\widehat{\theta}}}$$

and the $100\left(1-\alpha^*\right)\%$ lower confidence bound for C_L is

$$\underline{LB} = \widehat{C_L} - z_{\alpha^*}\sqrt{\Psi_{\widehat{\theta}}} \tag{28}$$

The testing of the lifetime performance index of the POLO distribution is summarized in the following steps:

SETP 1: Finding the MLE of $\alpha,\ \beta,\ and\ \lambda$ parameters of POLO distribution under the progressive first-failure type II censoring sample $X_{1:m:n:k},\ X_{2:m:n:k},\ldots\ldots, X_{m:m:n:k}$ and the censoring scheme $R = (R_1,\ R_2,\ldots\ldots, R_m)$ from Equations (13), (14), and (15). Then apply the goodness of fit test based on Gini statistic following [16].

STEP 2: Determine the performance index c^*, where the lower lifetime limit L is pre-determined. Then constructing the statistical test concerning the lifetime performance as $H_o: \quad C_L \leq c^*$versus $H_1 \ : \quad C_L > c^*$.

STEP 3: Specify the significance level α^*.

STEP 4: Obtaining the $100\left(1-\alpha^*\right)\%$ lower confidence interval $[\underline{LB},\infty)$ for the lifetime performance index C_L as (28).

STEP 5: Finally, the decision is taken as : if $c^* \notin [\underline{LB}\ ,\infty]$, then reject H_o. It physically means that there is a significant indication that the lifetime performance index meets the required level.

To illustrate the testing procedure, consider the following Examples. The above testing steps are followed step- by-step.

6 Numerical Examples

Example 6.1. Real lifetime data (Ball Bearing Data)

The data presented the number of millions of revolutions before failing for 30 ball bearings in a life endurance test [2]. A progressive first-failure censoring scheme was conducted with $k = 1$, $m = 10$, and $R_i = (R_1 \ldots\ldots R_m) = (0,0,0,2,3,3,3,3,3,3)$. The observations in hundreds of millions were given in Table 3.

Table 3: Progressive first-failure censored sample for ball bearing set.

i	1	2	3	4	5	6	7	8	9	10
$X_{i:m:n:k}$	0.1788	0.2892	0.33	0.4152	0.4212	0.4560	0.5184	0.5196	0.5556	1.0512
R_i	0	0	0	2	3	3	3	3	3	3

Then, the proposed testing procedure of C_L based on a confidence interval is stated as follows:

STEP 1: Consider the progressive first - failure type II censoring $\{x_{i:10:30:1},\ i = 1,2,\ldots,10\} = \{0.1788,\ 0.2892,\ 0.33,\ 0.4152,\ 0.4212,\ 0.4560,\ 0.5184,\ 0.5196,\ 0.5556,\ 1.0512\}$ with the above scheme then finding the MLE estimates of POLO distribution, $\alpha,\ \beta,\ and\ \lambda,$ using Equations (13), (14), and (15). The obtained results of the parameter estimates are $\widehat{\alpha} = 1.00706$, $\widehat{\beta} = 4.92447$, $\widehat{\lambda} = 0.250471$ and to test whether the failure times of the ball bearing follows POLO distribution with the p.d.f. $f(x) = 1.23\, x^{3.92447}\left(0.250471 + x^{4.92447}\right)^{-2.00706}$, $x > 0$,

Gini statistics [16] for the progressive first failure censoring will be used as in the procedure of testing as follows:

At $\alpha^* = 0.05$, significance level, consider the test hypothesis

$$H_0 : X \sim POLO(1.00706,\ 4.92447, 0.250471)\ V.S.$$

$$H_1 : X \nsim\ POLO(1.00706,\ 4.92447, 0.250471)$$

The Gini statistic is given as follows:

$$G_m = \frac{\sum_{i=1}^{m-1} i\, W_{i+1}}{(m-1)\sum_{i=1}^{m} W_i}$$

where, $W_i = (m-i+1)(Z_i - Z_{i-1})$, $Z_0 = 0,\ i = 1,2,\ldots,m,\ Z_1 = n\, Y_i$, $Z_i = [n - \sum_{j=1}^{i-1}(R_j+1)(Y_i - Y_{i-1})$, $i = 2,3,..,10$ and the data transformation is $Y_i = \ln\left(1 + \frac{X_{i:10:30:1}^{4.924471}}{0.250471}\right)$, $i = 1,2,..,10$. For $m = 3,\ldots 20.$, the rejection region $\{G_m > \xi_{1-\frac{\alpha^*}{2}}$ or $G_m < \xi_{\frac{\alpha^*}{2}}\}$ where the critical value $\xi_{\frac{\alpha^*}{2}}$ is the $100\left(1 - \frac{\alpha^*}{2}\right)$ % percentile of the Gini statistic. See [16].

For the above data, the Gini statistic is obtained as

$$G_{10} = 0.657584.$$

and $\xi_{0.025} = 0.31232 < G_{10} = 0.657584 < \xi_{0.975} = 0.68768$, hence, H_0 cannot be rejected at level of significance $\alpha^* = 0.05$. That is, there is an evidence to indicate that the failure time for the ball bearing in endurance test follows $POLO(1.00706,\ 4.92447, 0.250471)$ distribution.

STEP 2: The lower lifetime limit is assumed to be 0.3236569, i.e. if the lifetime of the ball bearing exceeds 0.3236569 then the ball bearing is defined as a conforming product. To deal with the product purchasers' concerns about the lifetime performance, the conforming rate P_r of products is required to exceed 80%. Referring to Table (1), the C_L value is required to exceed 0.8. Thus, the performance index value is set at $c^* = 0.75$ and the testing of hypothesis : $H_o :\ C_L \leq 0.75$ versus $H_1\ :\ C_L > 0.75$.

STEP 3: Specify a significance level $\alpha^* = 0.05$.

STEP 4: Using Equations (16), (24), and (26), the lower confidence interval bound

$$\underline{LB} = \widehat{C_L} - z_{\alpha^*}\sqrt{\Psi_{\widehat{\theta}}}$$

$$= 2.49515 - (1.645)\sqrt{0.724039} = 1.0954116$$

Hence, the 95% one-sided confidence interval for C_L is $[\underline{LB}, \infty) = [1.0954116,\ \infty)$

STEP 5: Because of the performance index $c^* = 0.75 \notin [\underline{LB}, \infty) = [1.0954116,\ \infty)$, $H_o :\ \ C_L \leq 0.75$ is rejected. Thus, the lifetime performance index of the 30 ball bearing meets the required level. Furthermore, from (16) and (26)

$\widehat{C_L} = 2.49515 > C_0 = c^* + z_{\alpha^*}\sqrt{\Psi_{\widehat{\theta}}} = 0.75 + (1.645)\sqrt{0.724039} \approx 2.1497384$. So we reject $H_o: \quad C_L \leq 0.75$ and the ball bearing operation does meet the required level.

Example 6.2: Simulated Data Set

A progressive first - failure type II censored data with $n = 40,\ m = 25,\ k = 1$ were generated from POLO distribution and the censoring scheme is shown in Table(4).

Table 4: The simulated progressive first failure censored data

i	1	2	3	4	5	6	7	8	9
$X_{i:m:n:k}$	0.284191	0.31489	0.51344	0.529163	0.594185	0.69818	0.712654	0.761804	0.771
R_i	0	1	1	1	1	1	1	1	1
i	10	11	12	13	14	15	16	17	18
$X_{i:m:n:k}$	0.775582	0.78807	0.813992	0.860315	0.896915	0.922534	0.936087	0.97435	1.008
R_i	1	1	1	1	1	0	0	0	0
i	19	20	21	22	23	24	25		
$X_{i:m:n:k}$	1.05524	1.0664	1.10677	1.13489	1.14347	1.19325	1.33049		
R_i	0	0	0	0	0	0	2		

Then, the testing procedure of C_L based on a confidence interval is specified as follows:

STEP 1: By considering the censoring data above in Table (4), the MLE of the POLO distribution parameters is obtained from Equations (13), (14) and (15) and the results are: $\widehat{\alpha} = 24.656,\ \widehat{\beta} = 14.1082,\ \widehat{\lambda} = 0.92693$. Applying Gini statistic to verify whether the simulated data comes from POLO distribution with the p.d.f $f(x) = 53.567767\, x^{13.1082}\left(0.92693 + x^{14.1082}\right)^{-25.656},\ x > 0,$

At $\alpha^* = 0.05$, significance level, consider the test hypothesis

$$H_0 : X \sim POLO\,(24.656,\ 14.1082,\ 0.92693)$$

$$V.S. \qquad H_1 : X \nsim POLO\,(24.656,\ 14.1082,\ 0.92693)$$

Using the transformation $Y_i = \ln\left(1 + \frac{X_{i:25:40:1}{}^{14.1082}}{0.92693}\right),\ i = 1,2,..,25$, the Gini statistic is calculated as

$$G_{25} = 0.53626$$

Due to [6], for $m > 20$, the rejection region is $\left\{\left|(G_m - 0.5)\left[12(m-1)\right]^{\frac{1}{2}}\right| > z_{\frac{\alpha^*}{2}}\right.$ where the critical value $z_{\frac{\alpha^*}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\alpha^*/2$.

Hence, $\left\{\left|(G_{25} - 0.5)\left[12(25-1)\right]^{\frac{1}{2}}\right| = 0.615347 < z_{0.025} = 1.96\right.$, hence, H_0 cannot be rejected at the level of significance 0.05. That is, there is an evidence to indicate that the simulated data come from the POLO distribution with the p.d.f.

$$f(x) = 53.567767\, x^{13.1082}\left(0.92693 + x^{14.1082}\right)^{-25.656},\ x > 0,$$

where, $\widehat{\theta} = (24.656,\ 14.1082,\ 0.92693)^T$.

STEP 2: Assuming the lifetime limit is 0.23245, i.e. if the lifetime exceeds 0.23245 then the product is defined as a conforming product. To deal with the product purchasers' concerns about the lifetime performance, the conforming rate P_r of products is required to exceed 80%. Referring to Table (2), the C_L value is required to exceed 0.8. Thus, the performance index value is set at $c^* = 0.8$ and the testing of hypothesis: $H_o: \quad C_L \leq 0.8$ versus $H_1 : \quad C_L > 0.8$.

STEP 3: Identify a significance level $\alpha^* = 0.05$.

STEP 4: Using Equations (16), (24), and (26), the lower confidence interval bound

$$\underline{LB} = \widehat{C_L} - z_{\alpha^*}\sqrt{\Psi_{\widehat{\theta}}}$$

$$= 10.194 - (1.645)\sqrt{0.163879} = 9.5280716$$

Hence, the 95% one-sided confidence interval for C_L is $[\underline{LB}, \infty) = [9.5280716,\ \infty)$.

STEP 5: Accordingly, the performance index $c^* = 0.8 \notin [\underline{LB}, \infty) = [9.5280716,\ \infty)$ then,

$H_o: \quad C_L \leq 0.8$ is rejected. Thus, the lifetime performance index of the product meets the required level. In addition, from (16) and (26) $\widehat{C_L} = 10.194 > C_0 = c^* + z_{\alpha^*}\sqrt{\Psi_{\widehat{\theta}}} = 0.8 + (1.645)\sqrt{0.163879} \approx 1.4659284$. Therefore, the decision is to reject $H_o: \quad C_L \leq 0.8$ and the lifetime performance index of the product meets the required level.

7 Conclusion

The process of controlling and improving the performance of a product is an important issue in business and many organizations. To meet the required customer's level of quality, the statistical methods and tests have been performed. Therefore, the lifetime performance index is one of the most widely indices which used to determine whether the product quality meets the required level. This paper is concerned with statistical inference of the lifetime performance index C_L based on a progressive first-failure type II censoring sample data from the Power Lomax Distribution (POLO). The main results are; specifying the conforming rate to the corresponding C_L , constructing the MLE of C_L under the three-parameter POLO distribution with the progressive first-failure type II right censoring sample by using the multivariate delta method. Moreover, the confidence interval of C_L is obtained. Finally, the testing of hypothesis concerning C_L is performed for evaluating the lifetime performance of products. The theoretical results are applied to two different examples indicating the desired aim of this work

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