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Estimating the Modified Weibull Parameters in Presence of Step-Stress Partially Accelerated Life Testing

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Abstract: In this paper, assuming that the lifetime of each item under normal condition follows the modified Weilbull distribution, partially accelerated life tests based on progressive Type-II censored samples are considered. The likelihood equations are to be solved numerically to obtain the maximum likelihood estimates. Based on normal approximation to the asymptotic distribution of maximum likelihood estimates, the approximate confidence intervals for the parameters are derived. It is difficult to get explicit form for Bayes estimates, so Markov chain Monte Carlo method is used to solve this problem, which gives us flexibility to construct the credible intervals for the parameters. Lindley approximation is also being disscussed to obtain Bayes estimators. Finally, an illustrative example and simulation studies are being considered.

Keywords: Modified Weibull distribution; Partially accelerated life testing; Progressive Type-II censoring; Maximum Likelihood Estimates; Asymptotic confidence intervals; Bayesian estimation.

Acronym

ALT	Accelerated Life Test	MLE	Maximum Likelihood Estimates
PALT	Partially ALT	ACIs	Approximate Confidence Intervals
MWD	Modified Weibull Distribution	SEL	Squared Error Loss Function
PDF	Probability Density Function	MCMC	Markov chain Monte Carlo
CDF	Cumulative Distribution Function	M-H	Metropolis-Hastings
SF	Survival Function	CRIs	credible intervals
SS-PALT	Step-stress PALT	AVG	Average Point Estimate
PRO-II-C	Progressive Type-II Censored	MSE	Mean Squared Error

1 Introduction

There is a difficulty in getting information about the lifetime of products with high quality during testing under normal conditions, so ALTs are used in manufacturing industries to get failure data in short period of time. According to Nelson [1] there exist three ALT methods. The first one is called constant-stress ALT, where the stress is being constant during the experiment, see Al-Hussaini and Abdel-Hamid [2], Bagdonavicius and Nikulin [3] and Kim and Bai [4] . The second one is called progressive-stress ALT, where the applied stress is increasing during time, see Wang and Fei [5] and Abdel-Hamid and Al-Hussaini [6]. The third one is called step-stress ALT, where the test condition changes at a certain time or after the termination of specific number of failures, see Balakrishnan et al. [7], Nelson [8] , Wu et al. [9] and Abdel-Hamid and Al-Hussaini [10]. Miler and Nelson [11] got the optimal step-stress ALT plans in the case where tested items have exponentially distributed lives and are observed until all items fail. For more researches on ALTs, see others Fan et al. [12], Ismail et al. [13], Ma and Meeker [14] and Tangi et al. [15]. We use PALT when the relationship between life and

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stress is unknown. An extra advantage of PALT appears in getting more failure data in a limited time without subjecting all the units to high levels of stress. In constant-stress PALT, each unit is run at either use or accelerated condition only, see Abushal and Soliman [16] and EL-Sagheer [17]. In step-stress PALT, items are tested at normal level, then the stress will be changed at a certain time. Many authors have dealt with this type, including Abdel-Hamid [18], Srivastava and Mittal [19] and EL-Sagheer [20]. A more general censoring scheme called PRO-II-C has been used in this paper in which the removal of prespecified number of units is done when an individual unit fails, this continues until fixed number of units failed, at this stage the remainder of the surviving units are removed, see Aggarwala and Balakrishnan [21], Bairamov and Eryilmaz [22], Ali Mousa and EL-Sagheer [23], Singh et al. [24] and EL-Sagheer [25]. Another generalization of Weibull distribution called MWD was introduced by Xie et al. [26] and a detailed statistical analysis was given in Tang et al. [27]. This distribution is in fact a generalization of the model studied by Chen [28]. If X follows a MWD, then the PDF, CDF and SF are given respectively by

$$f_1(x) = \lambda\alpha\left(\frac{x}{\beta}\right)^{\alpha-1}\exp\left\{\left(\frac{x}{\beta}\right)^{\alpha}+\lambda\beta\left(1-\exp\left[\left(\frac{x}{\beta}\right)^{\alpha}\right]\right)\right\}, \quad x>0,\ \alpha,\ \beta,\ \lambda>0, \tag{1}$$

$$F_1(x) = 1-\exp\left\{\lambda\beta\left(1-\exp\left\{\left(\frac{x}{\beta}\right)^{\alpha}\right\}\right)\right\}, \qquad x>0,\ \alpha,\beta,\lambda>0, \tag{2}$$

and

$$S_1(x) = \exp\left\{\lambda\beta\left(1-\exp\left\{\left(\frac{x}{\beta}\right)^{\alpha}\right\}\right)\right\}. \tag{3}$$

It is clear that the Exponential Power Distribution(α,β) is a special case of MWD with $\lambda=1$, so any results obtained in our study, will also be valid for the Exponential Power Distribution (α,β) and this was considered and studied by Smith and Bain [29]. The MWD model has been fitted to the failure times of Aarset data ,see Aarset [30] and the failure times of a sample of devices in a large system, see Gupta et al. [31]. The detailed description of the data set can be found in Meeker and Escobar [32].

This paper can be organized as follows: In Section 2 some basic assumptions and description of the model are presented. The derivation of the MLEs of the parameters as well as the corresponding ACIs are showed in Section 3. Section 4 discusses Bayesian estimation. A numerical example is presented in Section 5 . A simulation example is presented in Section 6 to compare the methods of estimation. Finally, some concluding remarks are considered in Section 7 to show the theoritical results.

2 Model description and assumptions

In SS-PALT all n units are run under normal conditions for a prefixed time τ and then they are run under accelerated conditions. Let n_1 be the number of failures that happen before time τ and $m-n_1$ be the number of failures happening after time τ where units are run at accelerated conditions. PRO-II-C is applied as follows, when the first failure occurs, $X^{R_1}_{1;m,n}, R_1$ items are randomly withdrawn from the remaining $n-1$ surviving items. At the second failure $X^{R_2}_{2;m,n}, R_2$ items from the remaining $n-2-R_1$ items are randomly withdrawn. The test continues until the m^{th} failure $X^{R_m}_{m;m,n}$. The remaining $R_m = n-m-\sum_{t=1}^{m-1} R_t$ items are withdrawn. According to Degroot and Goel [33], the lifetime say Z of a unit under SS-PALT can be defined as:

$$Z = \begin{cases} X, & if\ X<\tau, \\ \tau+\mu^{-1}(X-\tau), & if\ X\geq\tau, \end{cases} \tag{4}$$

where X is the lifetime of the item under use condition, τ is the time when stress is changed and μ is the acceleration factor. Assume that the lifetime of the unit follows MWD, then the PDF of the total lifetime of the unit is given by

$$f(x) = \begin{cases} 0, & x<0, \\ f_1(x), & 0<x<\tau, \\ f_2(x), & x\geq\tau, \end{cases} \tag{5}$$

where $f_1(x)$ is given by Eq. (1) and $f_2(x)$ can be obtained from Eqs. $(1),(4)$ and (5) using transformation variable technique and is given by:

$$f_2(x) = \lambda\mu\alpha\left(\xi(\beta,\mu)\right)^{\alpha-1}\exp\left\{\left(\xi(\beta,\mu)\right)^{\alpha}+\lambda\beta\left(1-\exp\left\{\left(\xi(\beta,\mu)\right)^{\alpha}\right\}\right)\right\}, \tag{6}$$

and

$$S_2(x) = \exp\left\{\lambda\beta\left(1-\exp\left\{(\xi(\beta,\mu))^{\alpha}\right\}\right)\right\}, \tag{7}$$

where

$$\xi(\beta,\mu) = \frac{\mu(x-\tau)+\tau}{\beta}.$$

3 Maximum likelihood estimation

If the failure times of the n independent items originally in the test are from a continuous population with PDF $f(x)$, the joint PDF for $X^{R_1}_{1;m,n} < X^{R_2}_{2;m,n} < \ldots < X^{R_{n_1}}_{n_1;m,n} < \tau < X^{R_{n_1+1}}_{n_1+1;m,n} < \ldots < X^{R_m}_{m;m,n}$ is given by

$$L(\alpha,\beta,\lambda,\mu|\underline{x}) = B\left\{\prod_{k=1}^{n_1} f_1(x_k)(S_1(x_k))^{R_k}\right\}.\left\{\prod_{k=n_1+1}^{m} f_2(x_k)(S_2(x_k))^{R_k}\right\}, \tag{8}$$

where

$$B = n(n-1-R_1)(n-2-R_1-R_2)\cdots\left(n-m+1-\sum_{t=1}^{m-1}R_t\right). \tag{9}$$

The likelihood function without normalized constant can be written as

$$L(\alpha,\beta,\lambda,\mu|\underline{x}) = (\alpha\lambda)^m\mu^{m-n_1}\prod_{k=1}^{n_1}\left(\frac{x_k}{\beta}\right)^{\alpha-1}\prod_{k=n_1+1}^{m}(\xi(\beta,\mu))^{\alpha-1}\exp\left\{\Phi_1(\alpha,\beta,\lambda,\mu)+\Phi_2(\alpha,\beta,\lambda,\mu)\right\}, \tag{10}$$

where

$$\Phi_1(\alpha,\beta,\lambda,\mu) = \sum_{k=1}^{n_1}\left(\frac{x_k}{\beta}\right)^{\alpha} + \lambda\beta(1+R_k)\left(1-\exp\left\{\left(\frac{x_k}{\beta}\right)^{\alpha}\right\}\right), \tag{11}$$

and

$$\Phi_2(\alpha,\beta,\lambda,\mu) = \sum_{k=n_1+1}^{m}(\xi(\beta,\mu))^{\alpha} + \lambda\beta(1+R_k)\left(1-\exp\left\{(\xi(\beta,\mu))^{\alpha}\right\}\right). \tag{12}$$

Therefore, the log-likelihood function without normalized constant $\ell(\alpha,\beta,\lambda,\mu|\underline{x})$ is given by

$$\begin{aligned}\ell(\alpha,\beta,\lambda,\mu|\underline{x}) =& (m-n_1)\log\mu + m\log\lambda\alpha + \Phi_1(\alpha,\beta,\lambda,\mu)+\Phi_2(\alpha,\beta,\lambda,\mu)\\ &+(\alpha-1)\left(\sum_{k=1}^{n_1}\log\left(\frac{x_k}{\beta}\right)+\sum_{k=n_1+1}^{m}\log\xi(\beta,\mu)\right).\end{aligned} \tag{13}$$

Then, upon differentiating Eq. (13) with respect to α, β, λ and μ respectively and equating each result to zero, we get

$$\frac{m}{\alpha} + \left(\sum_{k=1}^{n_1}\log\left(\frac{x_k}{\beta}\right)+\sum_{k=n_1+1}^{m}\log\xi(\beta,\mu)\right) + \frac{\partial\Phi_1(\alpha,\beta,\lambda,\mu)}{\partial\alpha} + \frac{\partial\Phi_2(\alpha,\beta,\lambda,\mu)}{\partial\alpha} = 0, \tag{14}$$

$$\frac{m(1-\alpha)}{\beta} + \frac{\partial\Phi_1(\alpha,\beta,\lambda,\mu)}{\partial\beta} + \frac{\partial\Phi_2(\alpha,\beta,\lambda,\mu)}{\partial\beta} = 0, \tag{15}$$

$$\frac{m}{\lambda} + \frac{\partial\Phi_1(\alpha,\beta,\lambda,\mu)}{\partial\lambda} + \frac{\partial\Phi_2(\alpha,\beta,\lambda,\mu)}{\partial\lambda} = 0, \tag{16}$$

and

$$\frac{m-n_1}{\mu} + \frac{\partial\Phi_2(\alpha,\beta,\lambda,\mu)}{\partial\mu} + (\alpha-1)\sum_{k=n_1+1}^{m}\frac{x_k-\tau}{\mu(x_k-\tau)+\tau} = 0. \tag{17}$$

A system of nonlinear simultaneous equations in four unknown variables α, β, λ and μ is resulted, but it is not an easy task to get an exact solution, so we tackle this situation by using a numerical method named Newton Raphson to obtain an approximate solution of the above nonlinear system see, EL-Sagheer [25].

3.1 Approximate confidence intervals

The asymptotic Fisher information matrix I of the maximum likelihood estimates is the negative second order partial derivatives of the log-likelihood function with respect to α, β, λ and μ. Let $\psi_1=\alpha$, $\psi_2=\beta$, $\psi_3=\lambda$ and $\psi_4=\mu$, then

$$I=\left(\frac{\partial^2\ell}{\partial\psi_i\partial\psi_j}\right)_{\downarrow(\widehat{\alpha},\widehat{\beta},\widehat{\lambda},\widehat{\mu})}, i,\ j=1,2,3,4. \tag{18}$$

Therefore, the asymptotic variance-covariance matrix can be written as follows

$$I^{-1}=\begin{bmatrix} -\frac{\partial^2\ell}{\partial\alpha^2} & -\frac{\partial^2\ell}{\partial\alpha\partial\beta} & -\frac{\partial^2\ell}{\partial\alpha\partial\lambda} & -\frac{\partial^2\ell}{\partial\alpha\partial\mu} \\ \frac{-\partial^2\ell}{\partial\beta\partial\alpha} & -\frac{\partial^2\ell}{\partial\beta^2} & -\frac{\partial^2\ell}{\partial\beta\partial\lambda} & -\frac{\partial^2\ell}{\partial\beta\partial\mu} \\ -\frac{\partial^2\ell}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ell}{\partial\lambda\partial\beta} & -\frac{\partial^2\ell}{\partial\lambda^2} & -\frac{\partial^2\ell}{\partial\lambda\partial\mu} \\ -\frac{\partial^2\ell}{\partial\mu\partial\alpha} & -\frac{\partial^2\ell}{\partial\mu\partial\beta} & -\frac{\partial^2\ell}{\partial\mu\partial\lambda} & -\frac{\partial^2\ell}{\partial\mu^2} \end{bmatrix}^{-1}_{\downarrow(\widehat{\alpha},\widehat{\beta},\widehat{\lambda},\widehat{\mu})} = \begin{bmatrix} var(\widehat{\alpha}) & cov(\widehat{\alpha},\widehat{\beta}) & cov(\widehat{\alpha},\widehat{\lambda}) & cov(\widehat{\alpha},\widehat{\mu}) \\ cov(\widehat{\beta},\widehat{\alpha}) & var(\widehat{\beta}) & cov(\widehat{\beta},\widehat{\lambda}) & cov(\widehat{\beta},\widehat{\mu}) \\ cov(\widehat{\lambda},\widehat{\alpha}) & cov(\widehat{\lambda},\widehat{\beta}) & var(\widehat{\lambda}) & cov(\widehat{\lambda},\widehat{\mu}) \\ cov(\widehat{\mu},\widehat{\alpha}) & cov(\widehat{\mu},\widehat{\beta}) & cov(\widehat{\mu},\widehat{\lambda}) & var(\widehat{\mu}) \end{bmatrix}, \tag{19}$$

where

$$\frac{\partial^2\ell}{\partial\alpha^2}=\frac{-m}{\alpha^2}+\frac{\partial^2\Phi_1(\alpha,\beta,\lambda,\mu)}{\partial\alpha^2}+\frac{\partial^2\Phi_2(\alpha,\beta,\lambda,\mu)}{\partial\alpha^2}, \tag{20}$$

$$\frac{\partial^2\ell}{\partial\beta^2}=\frac{m(\alpha-1)}{\beta^2}+\frac{\partial^2\Phi_1(\alpha,\beta,\lambda,\mu)}{\partial\beta^2}+\frac{\partial^2\Phi_2(\alpha,\beta,\lambda,\mu)}{\partial\beta^2}, \tag{21}$$

$$\frac{\partial^2\ell}{\partial\lambda^2}=\frac{-m}{\lambda^2}, \tag{22}$$

$$\frac{\partial^2\ell}{\partial\lambda\partial\mu}=\sum_{k=n_1+1}^{m}-\beta(1+R_k)\exp\left\{(\xi(\beta,\mu))^{\alpha}\right\}\alpha(\xi(\beta,\mu))^{\alpha-1}\left(\frac{(x_k-\tau)}{\beta}\right), \tag{23}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\lambda\partial\alpha}=&\sum_{k=1}^{n_1}-\beta(1+R_k)\log\xi(\beta,\mu)\exp\left\{(\xi(\beta,\mu))^{\alpha}\right\}(\xi(\beta,\mu))^{\alpha}\\&-\sum_{k=n_1+1}^{m}\beta(1+R_k)\log\left(\frac{x_k}{\beta}\right)\exp\left\{\left(\frac{x_k}{\beta}\right)^{\alpha}\right\}\left(\frac{x_k}{\beta}\right)^{\alpha},\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\lambda\partial\beta}=&\sum_{k=1}^{n_1}(1+R_k)\left(\left(1-\exp\left\{\left(\frac{x_k}{\beta}\right)^{\alpha}\right\}\right)+\alpha\exp\left\{\left(\frac{x_k}{\beta}\right)^{\alpha}\right\}\left(\frac{x_k}{\beta}\right)^{\alpha}\right)\\&+\sum_{k=n_1+1}^{m}(1+R_k)\left((1-\exp\left\{(\xi(\beta,\mu))^{\alpha}\right\})+\alpha\exp\left\{(\xi(\beta,\mu))^{\alpha}\right\}(\xi(\beta,\mu))^{\alpha}\right),\end{aligned} \tag{25}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\alpha\partial\mu}=&\sum_{k=n_1+1}^{m}\alpha(\xi(\beta,\mu))^{\alpha-1}\frac{(x_k-\tau)}{\beta}\log\xi(\beta,\mu)\left(1-\lambda\beta(1+R_k)\exp\left\{(\xi(\beta,\mu))^{\alpha}\right\}\right)\\&+\sum_{k=n_1+1}^{m}(\xi(\beta,\mu))^{\alpha}\frac{(x_k-\tau)}{\mu(x_k-\tau)+\tau}\left(1-\lambda\beta(1+R_k)\exp\left\{(\xi(\beta,\mu))^{\alpha}\right\}\right)\\&-\sum_{k=n_1+1}^{m}(\xi(\beta,\mu))^{\alpha}\log\xi(\beta,\mu)\left(\lambda\beta\alpha(1+R_k)\exp\left\{(\xi(\beta,\mu))^{\alpha}\right\}(\xi(\beta,\mu))^{\alpha-1}\left(\frac{(x_k-\tau)}{\beta}\right)\right)\\&+\frac{(x_k-\tau)}{\mu(x_k-\tau)+\tau},\end{aligned} \tag{26}$$

$$\frac{\partial^2\ell}{\partial\beta\partial\mu}=\sum_{k=n_1+1}^{m}(\xi(\beta,\mu))^{\alpha-1}\left(\frac{(x_k-\tau)}{\beta}\right)\left(\frac{-\alpha^2}{\beta}+\lambda\alpha(1+R_k)(\alpha-1)\exp\left\{(\xi(\beta,\mu))^{\alpha}\right\}\right), \tag{27}$$

$$\begin{aligned}\frac{\partial^2\ell}{\partial\beta\partial\alpha} =& \frac{-m}{\alpha^2}+\sum_{k=1}^{n_1}\log^2\left(\frac{x_k}{\beta}\right)\left(\frac{x_k}{\beta}\right)^{\alpha}\left(1-\lambda\beta\left(1+R_k\right)\exp\left\{\left(\frac{x_k}{\beta}\right)^{\alpha}\right\}\right)\\ &+\sum_{k=1}^{n_1}\left(\frac{x_k}{\beta}\right)^{\alpha}\log\left(\frac{x_k}{\beta}\right)\left(\alpha\left(\frac{x_k}{\beta}\right)^{\alpha}-1\right)\left(\lambda\left(1+R_k\right)\exp\left\{\left(\frac{x_k}{\beta}\right)^{\alpha}\right\}\right)\\ &+\sum_{k=n_1+1}^{m}\log^2\left(\xi\left(\beta,\mu\right)\right)\left(\xi\left(\beta,\mu\right)\right)^{\alpha}\left(1-\lambda\beta\left(1+R_k\right)\exp\left\{\left(\xi\left(\beta,\mu\right)\right)^{\alpha}\right\}\right)\\ &+\sum_{k=n_1+1}^{m}\left(\xi\left(\beta,\mu\right)\right)^{\alpha}\log\left(\xi\left(\beta,\mu\right)\right)\left(\alpha\left(\xi\left(\beta,\mu\right)\right)^{\alpha}-1\right)\left(\lambda\left(1+R_k\right)\exp\left\{\left(\xi\left(\beta,\mu\right)\right)^{\alpha}\right\}\right),\end{aligned} \tag{28}$$

and

$$\frac{\partial^2\ell}{\partial\mu^2}=\frac{n_1-m}{\mu^2}+\frac{\partial^2\Phi_2\left(\alpha,\beta,\lambda,\mu\right)}{\partial\mu^2}+\left(1-\alpha\right)\sum_{k=n_1+1}^{m}\frac{\left(x_k-\tau\right)^2}{\left(\mu\left(x_k-\tau\right)+\tau\right)^2}. \tag{29}$$

The common method to construct the confidence interval for the parameters α,β,λ and μ is to use the asymptotic normal distribution of MLEs and this is showed by Vander Weil and Meeker [34]. Therefore, $(1-\gamma)100\%$ ACIs for the parameters α, β, λ and μ become

$$\widehat{\alpha}\pm Z_{\gamma/2}\sqrt{var(\widehat{\alpha})},\quad \widehat{\beta}\pm Z_{\gamma/2}\sqrt{var(\widehat{\beta})},\quad \widehat{\lambda}\pm Z_{\gamma/2}\sqrt{var(\widehat{\lambda})},\quad \widehat{\mu}\pm Z_{\gamma/2}\sqrt{var(\widehat{\mu})}, \tag{30}$$

where $Z_{\gamma/2}$ is the percentile of the standard normal distribution with right-tail probability $\gamma/2$.

4 Bayesian estimation

In this section, the Bayes estimates and the corresponding CRIs for the parameters α,β,λ and μ are obtained. In some situations where we do not have sufficient prior information about our parameters, so we can use non-informative uniform distribution as a prior distribution for the parameters. Hence, the prior distributions for our parameters α, β, λ and μ are

$$\begin{aligned}\pi(\alpha)&\propto\alpha^{-1} \quad ,\alpha>0,\\ \pi(\beta)&\propto\beta^{-1} \quad ,\beta>0,\\ \pi(\lambda)&\propto\lambda^{-1} \quad ,\lambda>0,\end{aligned} \tag{31}$$

and

$$\pi(\mu)\propto\mu^{-1} \quad ,\mu>1.$$

The joint prior of the four parameters can be expressed by :

$$\pi(\alpha,\beta,\lambda,\mu)\propto(\alpha,\beta,\lambda,\mu)^{-1}\,,\alpha>0,\ \beta>0,\ \lambda>0,\ \ \mu>1 \tag{32}$$

The joint posterior density function of α,β,λ and μ given the data ,denoted by $\pi^*(\alpha,\beta,\lambda\mu|\underline{x})$ can be written as

$$\pi^*(\alpha,\beta,\lambda,\mu|\underline{x})=\frac{L(\alpha,\beta,\lambda,\mu|\underline{x})\times\pi(\alpha,\beta,\lambda,\mu)}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\beta,\lambda,\mu|\underline{x})\times\pi(\alpha,\beta,\lambda,\mu)d\alpha d\beta d\lambda d\mu}. \tag{33}$$

The Bayes estimates of any function of the parameters, say $z\left(\alpha,\beta,\lambda,\mu\right)$ using SEL function is

$$\begin{aligned}\tilde{z}_{BS}\left(\alpha,\beta,\lambda,\mu\right)&=E_{\alpha,\beta,\lambda,\mu|\underline{x}}\left[z\left(\alpha,\beta,\lambda,\mu\right)\right]\\ &=\frac{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty z\left(\alpha,\beta,\lambda,\mu\right)\times L(\alpha,\beta,\lambda,\mu|\underline{x})\times\pi(\alpha,\beta,\lambda,\mu)d\alpha d\beta d\lambda d\mu}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\beta,\lambda,\mu|\underline{x})\times\pi(\alpha,\beta,\lambda,\mu)d\alpha d\beta d\lambda d\mu},\end{aligned} \tag{34}$$

while the Bayes estimate of $z\left(\alpha,\beta,\lambda,\mu\right)$ using LINEX loss function is

$$\tilde{z}_{BL}\left(\alpha,\beta,\lambda,\mu\right)=\frac{-1}{\varepsilon}\log\left[E_{\alpha,\beta,\lambda,\mu|data}\left[e^{-\varepsilon z(\alpha,\beta,\lambda,\mu)}\right]\right],\ \ \varepsilon\neq 0,$$

and

$$E_{\alpha,\beta,\lambda,\mu|\underline{x}}\left[e^{-\varepsilon z(\alpha,\beta,\lambda,\mu)}\right]=\frac{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{-\varepsilon z(\alpha,\beta,\lambda,\mu)}\times L(\alpha,\beta,\lambda,\mu|\underline{x})\times\pi(\alpha,\beta,\lambda,\mu)d\alpha d\beta d\lambda d\mu}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\beta,\lambda,\mu|\underline{x})\times\pi(\alpha,\beta,\lambda,\mu)d\alpha d\beta d\lambda d\mu}. \tag{35}$$

It is not possible to compute (33), (34) and (35) analytically. So, Lindley approximation and MCMC technique are being used to obtain the Bayes estimates for α, β, λ and μ .

4.1 Lindley Approximation

Lindley approximation was introduced by Lindley [35] and its importance appears in a sense that the Bayes estimators can be approximated in a form containing no integrals. Let us consider the ratio of two integrals

$$I=\frac{\int_{(\alpha,\beta,\lambda,\mu)}w\left(\alpha,\beta,\lambda,\mu\right)e^{\ell(\alpha,\beta,\lambda,\mu)+\rho(\alpha,\beta,\lambda,\mu)}d\left(\alpha,\beta,\lambda,\mu\right)}{\int_{(\alpha,\beta,\lambda,\mu)}e^{\ell(\alpha,\beta,\lambda,\mu)+\rho(\alpha,\beta,\lambda,\mu)}d\left(\alpha,\beta,\lambda,\mu\right)}, \tag{36}$$

where $w(\alpha,\beta,\lambda,\mu)$ is any function of α or β or λ or μ, $\ell(\alpha,\beta,\lambda,\mu)$ is the log-likelihood function and $\rho\left(\alpha,\beta,\lambda,\mu\right)=\log\pi(\alpha,\beta,\lambda,\mu)$. Then the ratio of the two integrals can be calculated as follows

$$\begin{aligned}I=&\,w\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\mu}\right)+\hat{w}_1\hat{a}_1+\hat{w}_2\hat{a}_2+\hat{w}_3\hat{a}_3+\hat{w}_4\hat{a}_4+\hat{a}_5+\hat{a}_6+\hat{A}\left(\hat{w}_1\hat{\sigma}_{11}+\hat{w}_2\hat{\sigma}_{12}+\hat{w}_3\hat{\sigma}_{13}+\hat{w}_4\hat{\sigma}_{14}\right)\\&+\hat{B}\left(\hat{w}_1\hat{\sigma}_{21}+\hat{w}_2\hat{\sigma}_{22}+\hat{w}_3\hat{\sigma}_{23}+\hat{w}_4\hat{\sigma}_{24}\right)+\hat{C}\left(\hat{w}_1\hat{\sigma}_{31}+\hat{w}_2\hat{\sigma}_{32}+\hat{w}_3\hat{\sigma}_{33}+\hat{w}_4\hat{\sigma}_{34}\right)\\&+\hat{D}\left(\hat{w}_1\hat{\sigma}_{41}+\hat{w}_2\hat{\sigma}_{42}+\hat{w}_3\hat{\sigma}_{43}+\hat{w}_4\hat{\sigma}_{44}\right),\end{aligned} \tag{37}$$

where $\hat{\alpha},\hat{\beta},\hat{\lambda}$,and $\hat{\mu}$ are the MLE of the parameters α, β, λ and μ, respectively.

$$\hat{a}_i=\hat{\rho}_1\hat{\sigma}_{i1}+\hat{\rho}_2\hat{\sigma}_{i2}+\hat{\rho}_3\hat{\sigma}_{i3}+\hat{\rho}_4\hat{\sigma}_{i4},\ \ \text{for } i=1,2,3,4,$$

$$\hat{a}_5=\hat{\sigma}_{12}\hat{w}_{12}+\hat{\sigma}_{13}\hat{w}_{13}+\hat{\sigma}_{14}\hat{w}_{14}+\hat{\sigma}_{23}\hat{w}_{23}+\hat{\sigma}_{24}\hat{w}_{24}+\hat{\sigma}_{34}\hat{w}_{34},$$

$$\hat{a}_6=\frac{1}{2}(\hat{\sigma}_{11}\hat{w}_{11}+\hat{\sigma}_{22}\hat{w}_{22}+\hat{\sigma}_{33}\hat{w}_{33}+\hat{\sigma}_{44}\hat{w}_{44}),$$

$$\hat{A}=\hat{\sigma}_{11}\hat{\ell}_{111}+2\hat{\sigma}_{12}\hat{\ell}_{121}+2\hat{\sigma}_{13}\hat{\ell}_{131}+2\hat{\sigma}_{14}\hat{\ell}_{141}+2\hat{\sigma}_{23}\hat{\ell}_{231}+2\hat{\sigma}_{24}\hat{\ell}_{241}+2\hat{\sigma}_{34}\hat{\ell}_{341}+\hat{\sigma}_{22}\hat{\ell}_{221}+\hat{\sigma}_{33}\hat{\ell}_{331}+\hat{\sigma}_{44}\hat{\ell}_{441},$$

$$\hat{B}=\hat{\sigma}_{11}\hat{\ell}_{112}+2\hat{\sigma}_{12}\hat{\ell}_{122}+2\hat{\sigma}_{13}\hat{\ell}_{132}+2\hat{\sigma}_{14}\hat{\ell}_{142}+2\hat{\sigma}_{23}\hat{\ell}_{232}+2\hat{\sigma}_{24}\hat{\ell}_{242}+2\hat{\sigma}_{34}\hat{\ell}_{342}+\hat{\sigma}_{22}\hat{\ell}_{222}+\hat{\sigma}_{33}\hat{\ell}_{332}+\hat{\sigma}_{44}\hat{\ell}_{442},$$

$$\hat{C}=\hat{\sigma}_{11}\hat{\ell}_{113}+2\hat{\sigma}_{12}\hat{\ell}_{123}+2\hat{\sigma}_{13}\hat{\ell}_{133}+2\hat{\sigma}_{14}\hat{\ell}_{143}+2\hat{\sigma}_{23}\hat{\ell}_{233}+2\hat{\sigma}_{24}\hat{\ell}_{243}+2\hat{\sigma}_{34}\hat{\ell}_{343}+\hat{\sigma}_{22}\hat{\ell}_{223}+\hat{\sigma}_{33}\hat{\ell}_{333}+\hat{\sigma}_{44}\hat{\ell}_{443},$$

and

$$\hat{D}=\hat{\sigma}_{11}\hat{\ell}_{114}+2\hat{\sigma}_{12}\hat{\ell}_{124}+2\hat{\sigma}_{13}\hat{\ell}_{134}+2\hat{\sigma}_{14}\hat{\ell}_{144}+2\hat{\sigma}_{23}\hat{\ell}_{234}+2\hat{\sigma}_{24}\hat{\ell}_{244}+2\hat{\sigma}_{34}\hat{\ell}_{344}+\hat{\sigma}_{22}\hat{\ell}_{224}+\hat{\sigma}_{33}\hat{\ell}_{334}+\hat{\sigma}_{44}\hat{\ell}_{444},$$

where subscripts 1,2,3 and 4 on the right-hand side stand for α, β, λ and μ, respectively

$$\hat{\rho}_i=\left(\frac{\partial\rho}{\partial\Omega_i}\right)_{\downarrow(\hat{\Omega}_1,\hat{\Omega}_2,\hat{\Omega}_3,\hat{\Omega}_4)},\ \text{for } i\ =1,2,3,4\ \text{ and }\ \Omega_1=\alpha,\ \Omega_2=\beta,\ \Omega_3=\lambda,\ \Omega_4=\mu\ .$$

$$\hat{w}_{ij}=\left(\frac{\partial^2 w\left(\Omega_1,\Omega_2,\Omega_3,\Omega_4\right)}{\partial\Omega_i\partial\Omega_j}\right)_{\downarrow(\hat{\Omega}_1,\hat{\Omega}_2,\hat{\Omega}_3,\hat{\Omega}_4)},\ \ \text{for } i,j\ =1,2,3,4.$$

$$\hat{\ell}_{ij} = \left(\frac{\partial^2 \ell(\Omega_1,\Omega_2,\Omega_3,\Omega_4)}{\partial\Omega_i\partial\Omega_j}\right)_{\downarrow(\hat{\Omega}_1,\hat{\Omega}_2,\hat{\Omega}_3,\hat{\Omega}_4)}, \quad \text{for } i,j = 1,2,3,4.$$

$$\hat{\sigma}_{ij} = \frac{-1}{\hat{\ell}_{ij}}, \text{ for } i,i = 1,2,3,4.$$

$$\hat{\ell}_{ijk} = \left(\frac{\partial^2 \ell(\Omega_1,\Omega_2,\Omega_3,\Omega_4)}{\partial\Omega_i\partial\Omega_j\partial\Omega_k}\right)_{\downarrow(\hat{\Omega}_1,\hat{\Omega}_2,\hat{\Omega}_3,\hat{\Omega}_4)}, \quad \text{for } i,j,k = 1,2,3,4.$$

If $w\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\mu}\right) = \hat{\alpha}$, then the Bayes estimate of the parameter α under SEL function from (37) is

$$\begin{aligned}\hat{\alpha}_{Blind-SEL} = {} & \hat{\alpha} + \hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{w}_4\hat{a}_4 + \hat{a}_5 + \hat{a}_6 + \hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13} + \hat{w}_4\hat{\sigma}_{14}) \\ & + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23} + \hat{w}_4\hat{\sigma}_{24}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33} + \hat{w}_4\hat{\sigma}_{34}) \\ & + \hat{D}(\hat{w}_1\hat{\sigma}_{41} + \hat{w}_2\hat{\sigma}_{42} + \hat{w}_3\hat{\sigma}_{43} + \hat{w}_4\hat{\sigma}_{44}), \end{aligned} \tag{38}$$

while, if $w\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\mu}\right) = \hat{\beta}$, then the Bayes estimate of the parameter β under SEL function from (37) is :

$$\begin{aligned}\hat{\alpha}_{Blind-SEL} = {} & \hat{\beta} + \hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{w}_4\hat{a}_4 + \hat{a}_5 + \hat{a}_6 + \hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13} + \hat{w}_4\hat{\sigma}_{14}) \\ & + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23} + \hat{w}_4\hat{\sigma}_{24}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33} + \hat{w}_4\hat{\sigma}_{34}) \\ & + \hat{D}(\hat{w}_1\hat{\sigma}_{41} + \hat{w}_2\hat{\sigma}_{42} + \hat{w}_3\hat{\sigma}_{43} + \hat{w}_4\hat{\sigma}_{44}), \end{aligned} \tag{39}$$

while, if $w\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\mu}\right) = \hat{\lambda}$, then the Bayes estimate of the parameter λ under SEL function from (37) is

$$\begin{aligned}\hat{\alpha}_{Blind-SEL} = {} & \hat{\lambda} + \hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{w}_4\hat{a}_4 + \hat{a}_5 + \hat{a}_6 + \hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13} + \hat{w}_4\hat{\sigma}_{14}) \\ & + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23} + \hat{w}_4\hat{\sigma}_{24}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33} + \hat{w}_4\hat{\sigma}_{34}) \\ & + \hat{D}(\hat{w}_1\hat{\sigma}_{41} + \hat{w}_2\hat{\sigma}_{42} + \hat{w}_3\hat{\sigma}_{43} + \hat{w}_4\hat{\sigma}_{44}), \end{aligned} \tag{40}$$

and if $w\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\mu}\right) = \hat{\mu}$, then the Bayes estimate of the parameter μ under SEL function from (37) is

$$\begin{aligned}\hat{\alpha}_{Blind-SEL} = {} & \hat{\mu} + \hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{w}_4\hat{a}_4 + \hat{a}_5 + \hat{a}_6 + \hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13} + \hat{w}_4\hat{\sigma}_{14}) \\ & + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23} + \hat{w}_4\hat{\sigma}_{24}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33} + \hat{w}_4\hat{\sigma}_{34}) \\ & + \hat{D}(\hat{w}_1\hat{\sigma}_{41} + \hat{w}_2\hat{\sigma}_{42} + \hat{w}_3\hat{\sigma}_{43} + \hat{w}_4\hat{\sigma}_{44}), \end{aligned} \tag{41}$$

If $w\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\mu}\right) = e^{-\varepsilon\hat{\alpha}}$, then the Bayes estimate of the parameter α under LINEX loss function from (37) is

$$\begin{aligned}\hat{\alpha}_{Blind-SEL} = {} & e^{-\varepsilon\hat{\alpha}} + \hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{w}_4\hat{a}_4 + \hat{a}_5 + \hat{a}_6 + \hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13} + \hat{w}_4\hat{\sigma}_{14}) \\ & + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23} + \hat{w}_4\hat{\sigma}_{24}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33} + \hat{w}_4\hat{\sigma}_{34}) \\ & + \hat{D}(\hat{w}_1\hat{\sigma}_{41} + \hat{w}_2\hat{\sigma}_{42} + \hat{w}_3\hat{\sigma}_{43} + \hat{w}_4\hat{\sigma}_{44}), \end{aligned} \tag{42}$$

while, if $w\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\mu}\right) = e^{-\varepsilon\hat{\beta}}$, then the Bayes estimate of the parameter β under LINEX loss function from (37) is

$$\begin{aligned}\hat{\alpha}_{Blind-SEL} = {} & e^{-\varepsilon\hat{\beta}} + \hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{w}_4\hat{a}_4 + \hat{a}_5 + \hat{a}_6 + \hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13} + \hat{w}_4\hat{\sigma}_{14}) \\ & + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23} + \hat{w}_4\hat{\sigma}_{24}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33} + \hat{w}_4\hat{\sigma}_{34}) \\ & + \hat{D}(\hat{w}_1\hat{\sigma}_{41} + \hat{w}_2\hat{\sigma}_{42} + \hat{w}_3\hat{\sigma}_{43} + \hat{w}_4\hat{\sigma}_{44}), \end{aligned} \tag{43}$$

while, if $w\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\mu}\right) = e^{-\varepsilon\hat{\lambda}}$, then the Bayes estimate of the parameter λ under LINEX loss function from (37) is

$$\begin{aligned}\hat{\alpha}_{Blind-SEL} = {} & e^{-\varepsilon\hat{\lambda}} + \hat{w}_1\hat{a}_1 + \hat{w}_2\hat{a}_2 + \hat{w}_3\hat{a}_3 + \hat{w}_4\hat{a}_4 + \hat{a}_5 + \hat{a}_6 + \hat{A}(\hat{w}_1\hat{\sigma}_{11} + \hat{w}_2\hat{\sigma}_{12} + \hat{w}_3\hat{\sigma}_{13} + \hat{w}_4\hat{\sigma}_{14}) \\ & + \hat{B}(\hat{w}_1\hat{\sigma}_{21} + \hat{w}_2\hat{\sigma}_{22} + \hat{w}_3\hat{\sigma}_{23} + \hat{w}_4\hat{\sigma}_{24}) + \hat{C}(\hat{w}_1\hat{\sigma}_{31} + \hat{w}_2\hat{\sigma}_{32} + \hat{w}_3\hat{\sigma}_{33} + \hat{w}_4\hat{\sigma}_{34}) \\ & + \hat{D}(\hat{w}_1\hat{\sigma}_{41} + \hat{w}_2\hat{\sigma}_{42} + \hat{w}_3\hat{\sigma}_{43} + \hat{w}_4\hat{\sigma}_{44}), \end{aligned} \tag{44}$$

and if $w\left(\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\mu}\right)=e^{-\varepsilon\hat{\mu}}$, then the Bayes estimate of the parameter μ under LINEX loss function from (37) is

$$\begin{aligned}\hat{\alpha}_{Blind-SEL} &= e^{-\varepsilon\hat{\mu}}+\hat{w}_1\hat{a}_1+\hat{w}_2\hat{a}_2+\hat{w}_3\hat{a}_3+\hat{w}_4\hat{a}_4+\hat{a}_5+\hat{a}_6+\hat{A}\left(\hat{w}_1\hat{\sigma}_{11}+\hat{w}_2\hat{\sigma}_{12}+\hat{w}_3\hat{\sigma}_{13}+\hat{w}_4\hat{\sigma}_{14}\right)\\ &+\hat{B}\left(\hat{w}_1\hat{\sigma}_{21}+\hat{w}_2\hat{\sigma}_{22}+\hat{w}_3\hat{\sigma}_{23}+\hat{w}_4\hat{\sigma}_{24}\right)+\hat{C}\left(\hat{w}_1\hat{\sigma}_{31}+\hat{w}_2\hat{\sigma}_{32}+\hat{w}_3\hat{\sigma}_{33}+\hat{w}_4\hat{\sigma}_{34}\right)\\ &+\hat{D}\left(\hat{w}_1\hat{\sigma}_{41}+\hat{w}_2\hat{\sigma}_{42}+\hat{w}_3\hat{\sigma}_{43}+\hat{w}_4\hat{\sigma}_{44}\right).\end{aligned} \tag{45}$$

4.2 MCMC technique

An important sub-classes of MCMC methods are Gibbs sampling and more general Metropolis-within-Gibbs samplers to generate samples from the posterior density function $\pi^*(\alpha,\beta,\lambda,\mu|x)$ and in turn compute the Bayes estimates and also construct the corresponding CRIs based on the generated posterior samples. The expression for the posterior is directly proportional to the product of the likelihood and the prior and this can be written as :

$$\begin{aligned}\pi^*(\alpha,\beta,\lambda,\mu|\underline{x}) &\propto \mu^{m-n_1-1}\lambda^{m-1}\alpha^{m-1}\beta^{-1}\\ &\times\prod_{k=1}^{n_1}\left(\frac{x_k}{\beta}\right)^{\alpha-1}\prod_{k=n_1+1}^{m}\left(\xi\left(\beta,\mu\right)\right)^{\alpha-1}\exp\left\{\Phi_1\left(\alpha,\beta,\lambda,\mu\right)+\Phi_2\left(\alpha,\beta,\lambda,\mu\right)\right\}\end{aligned} \tag{46}$$

The conditional posterior densities of λ, μ, α and β are as follows :

$$\pi_1^*(\lambda|\alpha,\beta,\mu,\underline{x})\sim Gamma\left(m\ ,\Phi_1\left(\alpha,\beta,\lambda,\mu\right)+\Phi_2\left(\alpha,\beta,\lambda,\mu\right)\right)\ , \tag{47}$$

$$\begin{aligned}\pi_2^*(\mu|\alpha,\beta,\lambda,\underline{x}) &\propto \mu^{m-n_1-1}\prod_{k=n_1+1}^{m}\left(\xi\left(\beta,\mu\right)\right)^{\alpha-1}\\ &\times\exp\left\{\sum_{k=n_1+1}^{m}\left(\xi\left(\beta,\mu\right)\right)^{\alpha}-\lambda\beta\left(1+R_k\right)\exp\left\{\left(\xi\left(\beta,\mu\right)\right)^{\alpha}\right\}\right\},\end{aligned} \tag{48}$$

$$\pi_3^*(\alpha|\beta,\lambda,\mu,\underline{x})\propto\alpha^{m-1}\prod_{k=1}^{n_1}\left(\frac{x_k}{\beta}\right)^{\alpha-1}\prod_{k=n_1+1}^{m}\left(\xi\left(\beta,\mu\right)\right)^{\alpha-1}\exp\left\{\Phi_1\left(\alpha,\beta,\lambda,\mu\right)+\Phi_2\left(\alpha,\beta,\lambda,\mu\right)\right\}, \tag{49}$$

and

$$\pi_4^*(\beta|\alpha,\lambda,\mu,\underline{x})\propto\beta^{-1}\prod_{k=1}^{n_1}\left(\frac{x_k}{\beta}\right)^{\alpha-1}\prod_{k=n_1+1}^{m}\left(\xi\left(\beta,\mu\right)\right)^{\alpha-1}\exp\left\{\Phi_1\left(\alpha,\beta,\lambda,\mu\right)+\Phi_2\left(\alpha,\beta,\lambda,\mu\right)\right\}. \tag{50}$$

It can be seen that Eq. (47) follows gamma distribution, so samples of λ can be easily generated using any gamma generating routine.The posteriors of μ,α and β in Eqs. (48), (49) and (50) respectively are unknown, but upon plotting each of them, their graphs are similar to normal distribution, see Fig. (1), (2) and (3), so the Metropolis-Hastings method with normal proposal distribution is used to generate from each of them. For more details about Metropolis-Hastings algorithm, see [36].The algorithm of Gibbs sampling within M-H is as follows :

Step 1:Start with an $(\beta^{(0)}=\hat{\beta},\ \alpha^{(0)}=\hat{\alpha},\ \lambda^{(0)}=\hat{\lambda},\ \mu^{(0)}=\hat{\mu})$.
Step 2:Set $t=1$.
Step 3:Generate $\lambda^{(t)}$ from $\pi_1^*(\lambda|\alpha^{(t-1)},\beta^{(t-1)},\mu^{(t-1)},x)$.
Step 4:Using M-H method, generate $\mu^{(t)}$ from $\pi_2^*(\mu|\alpha^{(t-1)},\beta^{(t-1)},\lambda^{(t)},x)$ with the N($\mu^{(t-1)},var(\widehat{\mu})$) proposal distribution.
Step 5:Using M-H method, generate $\alpha^{(t)}$ from $\pi_3^*(\alpha|\beta,\lambda,\mu,x)$ with the N($\alpha^{(t-1)},var(\widehat{\alpha})$) proposal distribution.
Step 6:Using M-H method, generate $\beta^{(t)}$ from $\pi_4^*(\beta|\alpha,\lambda,\mu,x)$ with the N($\beta^{(t-1)},var(\widehat{\beta})$) proposal distribution.
Step 7:Compute $\alpha^{(t)}$, $\beta^{(t)}$, $\lambda^{(t)}$ and $\mu^{(t)}$.
Step 8:Set $t=t+1$.
Step 9:Repeat Steps $3-6$ N times.

Step 10:Obtain the Bayes MCMC point estimate of η_j where $\eta_1 = \alpha$, $\eta_2 = \beta$, $\eta_3 = \lambda$ and $\eta_4 = \mu$ for $j = 1,2,3$ and 4 as

$$E(\eta_j|data) = \frac{1}{N-M}\sum_{k=M+1}^{N} \eta_j^{(k)}$$

and the estimates for the aforementioned parameters under LINEX loss function as

$$\hat{\eta}_j = \frac{-1}{\varepsilon}\log\left[\frac{1}{N-M}\sum_{k=M+1}^{N}\exp\left\{-\varepsilon\eta_j^{(k)}\right\}\right],$$

where M is the burn-in period .

Step 11:To compute CRIs of η_j , order $\eta_j^{(M+1)}, \eta_j^{(M+2)}, ..., \eta_j^{(N)}$ as $\eta_{j(1)} < \eta_{j(2)} < ... < \eta_{j(N-M)}$. Then the $100(1-2\gamma)\%$ symmetric CRI becomes

$$\left(\eta_{j(\gamma(N-M))}, \eta_{j((1-\gamma)(N-M))}\right) .$$

5 Numerical computations

In this section , a numerical example is presented. PROG-II censored sample from MWD under SSPALT is generated using the algorithm described in Balakrishnan and Sandhu [37] and using tranformation (4) with $\tau = 0.3$. A simulation data for PROG-II censored sample under SSPALT from the MWD with parameters $(\alpha, \beta, \lambda, \mu)$ =(2, 1, 1.5, 1.5), $\tau = 0.3$ and by using progressive censoring schemes $n = 30$, $m = 15$ and $R = \{3, 0, 3, 0, 2, 0, 2, 0, 3, 0, 1, 0, 1, 0, 0\}$ are presented in Table 1. The MLEs, Bayesian Estimates using Lindley's approximation SEL, Bayesian Estimates using Lindley's approximation Linex Loss function and Bayes point estimates have been calculated for the parameters α, β, λ and μ and listed in Table 2. 95% ACI, and CRIs for the parameters α, β, λ and μ are computed and listed in Table 3. In MCMC procedure, the chain has been run 12,000 and the first 2000 have been canceled as burn-in .

Table 1. SSPALT simulation data generated from MWD

Failure times before $\tau = 0.3$	Failure times after $\tau = 0.3$				
0.2963	0.3342	0.3576	0.3806	0.4371	0.5496
	0.6649	0.6856	0.7119	0.7885	0.8295
	0.8700	0.9393	1.2083	1.2489	

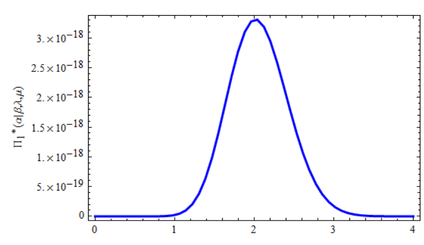

Fig. 1: Conditional Posterior density function of α

Table 2. Differents Point estimates for parameters α, β, λ, μ.

(.)	(.)MLE	(.)Lindley				(.)MCMC			
		SEL	Linex			SEL	Linex		
			c=-1	c=0.001	c=1		c=-1	c=0.001	c=1
α	2.457	2.332	2.422	2.332	2.262	1.107	1.100	1.107	1.106
β	1.195	1.268	1.273	1.268	1.262	0.899	0.896	0.899	0.896
λ	1.151	1.023	1.064	1.023	0.992	1.01	1.011	1.001	1.001
μ	1.006	0.945	0.950	0.945	0.942	1.701	1.703	1.702	1.701

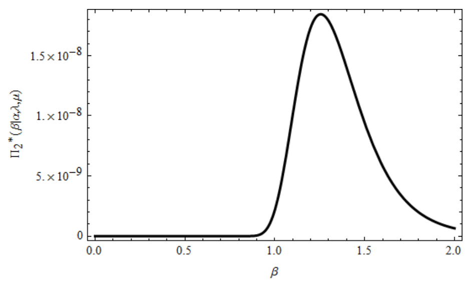

Fig. 2: Conditional Posterior density function of β

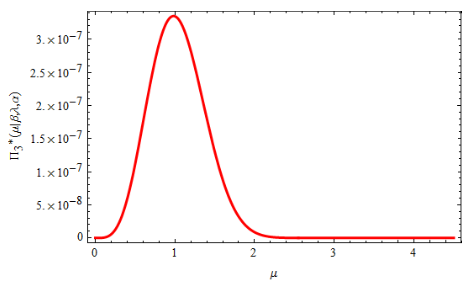

Fig. 3: Conditional Posterior density function of μ

Table 3. 95% Confidence intervals for parameters α, β, λ and μ

Method	α	Length	β	Length
ACI	[-1.3265, 6.2416]	7.5680	[-1.8966, 4.2878]	6.1844
CRI	[1.0996, 1.1016]	0.0020	[0.9991, 1.0005]	0.0014
Method	λ	Length	μ	Length
ACI	[-3.0259, 5.3285]	8.3543	[-2.2379, 4.251]	6.4889
CRI	[1.6997, 1.7009]	0.00121	[0.8989, 0.9001]	0.0011

6 Simulation study

In this section, we made a simulation study using (MATHEMATICA ver. 8.0) and computed the mean square error (MSE) for the parameters α, β, λ and μ. The procedure was performed 1000 times and the population parameters were $(\alpha, \beta, \lambda, \mu)$ =(2, 1, 1, 1.2) with different sample sizes n and m. The comparison between the different methods of the resulting estimators α, β, λ and μ has been considered in their MSEs and this is illustrated at Tables 4, 5, 6 and 7.

$$MSE = \frac{1}{N-M} \sum_{i=1}^{N-M} \left(\hat{\xi}_k^{(i)} - \xi_k \right)^2,$$

where $N-M$ is the number of simulations, k = 1, 2, 3 and $(\xi_1 = \alpha,\ \xi_2 = \beta,\ \xi_3 = \mu)$. In our study three different censoring schemes are considered :

Scheme I : $R_1 = n - m$, $R_i = 0$ for $i \neq 1$.

Scheme II : $R_{\frac{m}{2}} = R_{\frac{m}{2}+1} = \frac{n-m}{2}$, $R_i = 0$ for $i \neq \frac{m}{2}$ and $i \neq \frac{m}{2} + 1$.

Scheme III : $R_m = n - m$, $R_i = 0$ for $i \neq m$.

Table 4. The MSE of the parameter α **at** $\tau = 0.4$

(n,m)	CS	MLE	Lindley				MCMC			
			SEL	Linex			SEL	Linex		
				c=-1	c=0.001	c=1		c=-1	c=0.001	c=1
$(\mathbf{30},\mathbf{15})$	I	0.339	0.177	0.174	0.177	0.165	0.079	0.078	0.079	0.073
	II	0.347	0.185	0.183	0.185	0.175	0.083	0.082	0.08	0.082
	III	0.347	0.196	0.195	0.196	0.187	0.088	0.084	0.07	0.080
$(\mathbf{30},\mathbf{20})$	I	0.320	0.157	0.156	0.158	0.156	0.073	0.071	0.069	0.070
	II	0.322	0.159	0.157	0.159	0.150	0.075	0.073	0.07	0.072
	III	0.331	0.168	0.167	0.167	0.163	0.077	0.077	0.078	0.076
$(\mathbf{50},\mathbf{25})$	I	0.306	0.135	0.134	0.136	0.130	0.065	0.064	0.065	0.063
	II	0.308	0.137	0.136	0.138	0.134	0.069	0.066	0.067	0.062
	III	0.310	0.139	0.139	0.137	0.135	0.071	0.071	0.069	0.070
$(\mathbf{50},\mathbf{40})$	I	0.283	0.113	0.112	0.113	0.110	0.061	0.060	0.06	0.060
	II	0.291	0.123	0.122	0.133	0.121	0.061	0.061	0.059	0.061
	III	0.298	0.133	0.132	0.123	0.130	0.064	0.061	0.064	0.060
$(\mathbf{70},\mathbf{50})$	I	0.266	0.105	0.102	0.104	0.100	0.056	0.054	0.058	0.051
	II	0.274	0.107	0.102	0.106	0.100	0.058	0.054	0.0591	0.051
	III	0.282	0.112	0.110	0.113	0.110	0.061	0.060	0.0644	0.060
$(\mathbf{90},\mathbf{75})$	I	0.204	0.093	0.093	0.094	0.091	0.039	0.033	0.0366	0.031
	II	0.221	0.099	0.091	0.098	0.090	0.048	0.040	0.049	0.040
	III	0.221	0.104	0.102	0.106	0.102	0.051	0.050	0.053	0.050

Table 5. The MSE of the parameter β at $\tau = 0.4$

(n,m)	CS	MLE	Lindley				MCMC			
			SEL	Linex			SEL	Linex		
				c=-1	c=0.001	c=1		c=-1	c=0.001	c=1
$(\mathbf{30},\mathbf{15})$	I	0.424	0.185	0.183	0.186	0.180	0.085	0.083	0.085	0.083
	II	0.433	0.196	0.191	0.196	0.190	0.086	0.086	0.088	0.081
	III	0.454	0.199	0.196	0.199	0.195	0.087	0.085	0.087	0.084
$(\mathbf{30},\mathbf{20})$	I	0.407	0.143	0.143	0.144	0.140	0.074	0.073	0.073	0.073
	II	0.408	0.144	0.144	0.143	0.141	0.076	0.073	0.075	0.072
	III	0.423	0.159	0.157	0.159	0.152	0.079	0.074	0.079	0.074
$(\mathbf{50},\mathbf{25})$	I	0.356	0.122	0.122	0.123	0.122	0.064	0.061	0.064	0.061
	II	0.391	0.139	0.138	0.138	0.138	0.065	0.064	0.066	0.063
	III	0.401	0.141	0.141	0.142	0.140	0.068	0.066	0.068	0.062
$(\mathbf{50},\mathbf{40})$	I	0.348	0.119	0.117	0.118	0.111	0.057	0.052	0.058	0.052
	II	0.350	0.118	0.119	0.120	0.114	0.060	0.060	0.061	0.060
	III	0.352	0.120	0.121	0.121	0.120	0.061	0.062	0.062	0.063
$(\mathbf{70},\mathbf{50})$	I	0.287	0.112	0.112	0.113	0.110	0.033	0.033	0.032	0.032
	II	0.299	0.115	0.115	0.116	0.114	0.043	0.040	0.043	0.041
	III	0.341	0.117	0.111	0.118	0.111	0.050	0.050	0.049	0.051
$(\mathbf{90},\mathbf{75})$	I	0.236	0.095	0.093	0.088	0.092	0.030	0.033	0.032	0.030
	II	0.246	0.102	0.101	0.103	0.100	0.031	0.030	0.029	0.034
	III	0.247	0.105	0.103	0.104	0.100	0.033	0.032	0.031	0.038

Table 6. The MSE of the parameter λ at $\tau = 0.4$

(n,m)	CS	MLE	Lindley				MCMC			
			SEL	Linex			SEL	Linex		
				c=-1	c=0.001	c=1		c=-1	c=0.001	c=1
(30,15)	I	0.316	0.185	0.184	0.184	0.182	0.082	0.080	0.083	0.080
	II	0.323	0.186	0.185	0.186	0.184	0.090	0.093	0.091	0.092
	III	0.324	0.218	0.217	0.217	0.210	0.098	0.096	0.099	0.091
(30,20)	I	0.292	0.170	0.170	0.171	0.170	0.073	0.071	0.074	0.071
	II	0.293	0.181	0.181	0.181	0.180	0.078	0.076	0.078	0.072
	III	0.296	0.184	0.182	0.183	0.182	0.079	0.078	0.079	0.073
(50,25)	I	0.273	0.161	0.161	0.162	0.161	0.062	0.062	0.064	0.061
	II	0.281	0.162	0.163	0.163	0.163	0.065	0.063	0.066	0.060
	III	0.291	0.166	0.121	0.168	0.166	0.069	0.068	0.068	0.061
(50,40)	I	0.262	0.125	0.132	0.124	0.120	0.055	0.053	0.053	0.051
	II	0.269	0.138	0.151	0.139	0.131	0.056	0.054	0.056	0.054
	III	0.272	0.151	0.113	0.152	0.151	0.059	0.056	0.059	0.055
(70,50)	I	0.235	0.113	0.114	0.112	0.110	0.042	0.041	0.042	0.041
	II	0.244	0.116	0.113	0.117	0.114	0.052	0.052	0.055	0.050
	III	0.255	0.117	0.115	0.115	0.112	0.054	0.053	0.056	0.058
(90,75)	I	0.216	0.097	0.096	0.098	0.091	0.033	0.031	0.033	0.037
	II	0.224	0.103	0.103	0.102	0.102	0.036	0.033	0.035	0.031
	III	0.228	0.107	0.106	0.108	0.100	0.037	0.037	0.038	0.032

Table 7. The MSE of the parameter μ at $\tau = 0.4$

(n,m)	CS	MLE	Lindley				MCMC			
			SEL	Linex			SEL	Linex		
				c=-1	c=0.001	c=1		c=-1	c=0.001	c=1
(30,15)	I	0.314	0.199	0.190	0.199	0.191	0.076	0.071	0.076	0.070
	II	0.315	0.214	0.211	0.213	.0210	0.086	0.083	0.086	0.081
	III	0.316	0.224	0.223	0.224	0.221	0.088	0.085	0.089	0.084
(30,20)	I	0.306	0.172	0.172	0.172	0.171	0.068	0.066	0.066	0.061
	II	0.308	0.180	0.180	0.180	0.180	0.069	0.065	0.068	0.060
	III	0.310	0.186	0.185	0.189	0.145	0.070	0.070	0.071	0.070
(50,25)	I	0.283	0.146	0.145	0.145	0.151	0.057	0.056	0.055	0.052
	II	0.285	0.152	0.152	0.153	0.152	0.059	0.057	0.058	0.050
	III	0.288	0.154	0.155	0.155	0.130	0.063	0.060	0.064	0.060
(50,40)	I	0.272	0.130	0.130	0.131	0.133	0.053	0.053	0.054	0.050
	II	0.273	0.133	0.132	0.131	0.134	0.054	0.054	0.055	0.051
	III	0.279	0.136	0.133	0.133	0.139	0.056	0.056	0.056	0.054
(70,50)	I	0.260	0.117	0.114	0.116	0.110	0.045	0.043	0.045	0.043
	II	0.266	0.119	0.115	0.118	0.115	0.049	0.045	0.048	0.044
	III	0.269	0.123	0.122	0.122	0.120	0.048	0.047	0.046	0.046
(90,75)	I	0.250	0.104	0.101	0.105	0.100	0.043	0.043	0.044	0.042
	II	0.253	0.107	0.107	0.106	0.101	0.044	0.044	0.043	0.040
	III	0.254	0.112	0.111	0.114	0.111	0.048	0.049	0.049	0.044

7 Concluding Remarks

In this paper, The maximun likelihood for the parameters of the MWD is considered. This paper also studied the construction of CIs for the parameters α, β, λ and μ. MCMC method and Lindley approximation are used to tackle the problem of not getting an explicit form for the bayes estimate. A numerical example using simulated data is presented to show how the methods work based on progressive type-II censored data. Finally, a simulation study was performed to compare the performance of the proposed methods for different sample sizes and different censored schemes. From the results, we observe the following :

1. From Tables 4, 5, 6 and 7, as sample size *n* increases, the MSEs decrease and the Bayes estimates have the smallest MSEs for the parameters α, β, λ and μ.
2. For fixed values of the sample *n* and *m*, Scheme I performs better than other Schemes in the sense of having smaller MSEs.
3. From Tables 4, 5, 6 and 7, the estimated values for all parameter using Bayesian-Lindley Linex at c = 0.001 are approximately equal to the estimated values of all parameters using Bayesian-Lindley SEL.
4. From Tables 4, 5, 6 and 7, the estimated values for all parameters using Bayesian-MCMC Linex at c = 0.001 are approximately equal to the estimated value of the parameter using Bayesian-MCMC SEL.

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