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A Five Parameter Statistical Distribution with Application to Real Data

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Abstract: A five parameter lifetime statistical model is proposed in this research. The model is called the Kumuraswamy (KW) exponentiated linear exponential distribution. We derive some statistical properties such as quantile functions, median, moments, entropy, order statistics and many more. The estimation of the model parameters is provided by means of the maximum likelihood (MLEs) approach and the observed information matrix is also presented. Moreover, we perform simulation analysis in order to analyze the effectiveness of the parameters estimation. A set of real data is used to demonstrate the efficiency and effectiveness of the model proposed model with the well-known lifetime models.

Keywords: KW-G distribution, Exponentiated linear exponential, simulation, maximum likelihood, observed information matrix, real data.

1 Introduction

Recently, Mahmouda and Alamb [10] introduced a new lifetime model by exponentiating the exponential part of the cumulative distribution function (cdf) of the linear exponential distribution given by $G(x) = 1 - e^{-(\frac{\theta}{2}x^2+\lambda x)}$ contrary to what was done by Sarhana and Kundu [14] who exponentiated the whole cdf of the linear exponential distribution. Mahmouda and Alamb [10] called their model Exponentiated Linear Exponential distribution. They studied and investigated its mathematical and statistical properties and thereafter showed the flexibility of their model. The cdf and probability density function (pdf) of the exponentiated linear exponential distribution is

$$G(x) = 1 - e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}} \; and \; g(x) = \alpha(\theta x+\lambda)(\frac{\theta}{2}x^2+\lambda x)^{\alpha-1}e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}} \tag{1}$$

Here, we propose an extension of the exponentiated linear exponential distribution based on the family of KW Exponentiated (Kw-G) distributions proposed by Cordeiro and de Castro [9]. Some mathematical properties of this family was investigated by Nadarajah et al. [12]. The KW distribution is not that popular among statisticians and is being less examined in the literature. The cdf of the KW-G distribution for $(0 < x < 1)$ is $F(x) = 1 - (1 - x^a)^b$, where $a > 0$ and $b > 0$ are shape parameters, and its density function has the form $f(x) = abx^{a-1}(1 - x^a)^{b-1}$, it is increasing, a constant, decreasing, uni-modal depending on the values of parameters. The KW-G distribution does not seem to be very familiar to statisticians and before is not being investigated systematically in detail, nor has its relative interchangeability with the beta distribution been widely appreciated. However, in recent papers, Jones [7] examine the background and genesis of this distribution and, more importantly, made clear some similarities and differences between the beta and KW distributions. In this paper, we compound the works of KW [8] and the authors in [9] to introduce a new distribution. The distribution is constructed as: the baseline cumulative function of a random variable denoted by G is written as

$$F(x) = 1 - [1 - G(x)^a]^b \tag{2}$$

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where $a > 0$ and $b > 0$ are also shape parameters which capture the skewness and tail weights. Because of its tractable distribution function (2), the Kw-G distribution is applied efficiently to deal with censored data and its corresponding density function is

$$f(x) = abg(x)G(x)^{a-1}[1-G(x)^{a}]^{b-1}. \tag{3}$$

(3) possess a lots of characteristics for the class of beta-G distributions [4], but has some advantages in terms of tractability, since it does not involve any special function such as the beta function. Equivalently, as occurs with the beta-G family of distributions, special KW-G distributions may be derived as: the KW-normal distribution is obtained by taking $G(x)$ in (2) to be the normal cumulative function. Analogously, the KW-Weibull Cordeiro et al. [3], KW-exponentiated gamma Pascoa et al. [13], KW-Birnbaum-Saunders Saulo et al. [15], KW-Gumbel Cordeiro et al. [2], KW-Exponentiated failure rate [6], Kw-Modified inverse weibul [1] and Kw-Linear exponential distributions [11] can be attained by considering $G(x)$ as the cdf of the Weibull, exponentiated gamma, Birnbaum-Saunders, Gumbel, exponentiated linear failure rate, modified inverse weibull and Linear exponential distributions, respectively, and many others. Therefore, each new KW-G may be formulated via G distribution. The main aims of introducing this model is as follows: (i) The additional parameters introduced by the KW generalization is sought as a means to furnish a more flexible distribution. (ii) Some modelling phenomenon with non-monotone failure rates such as the bathtub-shaped and uni-modal failure rates, which are common in reliability and biological studies, take a reasonable parametric fit with this distribution. (iii) The KW-GLED distribution is expected to have immediate application in reliability and survival studies. (iv) KW-GLED distribution shows better fitting, more flexible in shape and easier to perform and formula for modelling lifetime data. Some interesting models were also proposed in [17],[18]. This paper is outlined as follows. In section 2, we define the Kw-GLED distribution, present its sub-models and provide expansions for its cumulative and density functions. In addition, we study the properties and limiting behavior of its pdf and hazard rate function. Mathematical and Statistical properties of this distribution are given in sections 3. Maximum likelihood estimation is performed and the observed information matrix is determined in section 4. In section 5 simulation studies is performed, we provide an application of the Kw-GLE distribution to real data set in section 6. Finally, conclusions are given in section 7.

2 The kW-GLE

The cdf and pdf of the KW exponentiated linear exponential distribution (KW-GLED) are obtained by substituting the cdf in (1) into the cdf and pdf in (2) and (3) respectively. The cdf of the KW-GLE is given by

$$F(x) = 1-\left(1-\left[1-e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}\right]^{a}\right)^{b} \tag{4}$$

and the corresponding pdf is given by

$$f(x) = ab\alpha(\theta x+\lambda)(\frac{\theta}{2}x^2+\lambda x)^{\alpha-1}e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}\left(1-\left[1-e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}\right]^{a}\right)^{b-1}\left(1-e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}\right)^{a-1} \tag{5}$$

the survival function is given by

$$s(x) = \left(1-\left[1-e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}\right]^{a}\right)^{b} \tag{6}$$

and the hazard function is

$$h(x) = \frac{ab\alpha(\theta x+\lambda)(\frac{\theta}{2}x^2+\lambda x)^{\alpha-1}e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}\left(1-e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}\right)^{a-1}}{\left(1-\left[1-e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}\right]^{a}\right)} \tag{7}$$

2.1 Properties of KW-GLED pdf and hazard

The limiting behavior, shapes of the pdf and hazard are studied here. Table (1) has shown the limiting behavior of the pdf and the hazard rate function of the KW-GLE distribution with different parameter values.

The pdf of the KW-GLED is decreasing for $a < 1$, $0 < \alpha < 1$ and uni-modal for $a > 1$, $\alpha > 1$.

Proof. By taking the log of (5) and differentiating we reach

$$(\log(f(x)))' = t(x)-\alpha(\frac{\theta}{2}x^2+\lambda x)^{\alpha-1}-a(b-1)\left(1-e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}\right)^{a-1}$$

Table 1: limiting behavior of the pdf and hazard rate function of the $KW-GLE$

	$a=1, \alpha=1$	$a>1, \alpha>1$	$0<a<1, 0<\alpha<1$
$\lim_{x\to 0} f(x)$	λb	0	∞
$\lim_{x\to\infty} f(x)$	0	0	0
$\lim_{x\to\infty} h(x)$	λb	0	∞
$\lim_{x\to 0} h(x)$	0	0	0

where

$$t(x)=\theta+\frac{(\alpha-1)}{(\frac{\theta}{2}x^2+\lambda x)^{\alpha-1}}+\frac{\alpha(a-1)(\frac{\theta}{2}x^2+\lambda x)^{\alpha-1}e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}}{1-e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}}$$

the function $t(x)$ is negative for $a<1$, $0<\alpha<1$ and so $f(x)<0$ for all $x>0$. Therefore, the pdf is decreasing for $a<1$, $0<\alpha<1$. However, when $a>1$, $\alpha>1$, $\theta>0$, $\lambda>0$, $b>0$ with mode at $x=\frac{1}{\theta}(-\lambda+\sqrt{\frac{\theta}{b}})$ and $-\lambda+\sqrt{\frac{\theta}{b}}>0$ and at $x=0$ if $-\lambda+\sqrt{\frac{\theta}{b}}<0$

The hazard rate function of the KW-GLE is an increasing (decreasing) function for $\alpha>1(<1)$ and $a>1(<1)$.

Proof. By taking the log of (5) and differentiating we get $\frac{\partial f(x)}{\partial x^2}>0$ where $\alpha>1(<1)$, which implies an increasing(decreasing) hazard rate function.

However, the hazard rate function is upside down bathtube shape (uni-modal) and bathtube shape. This can be illustrated graphically.

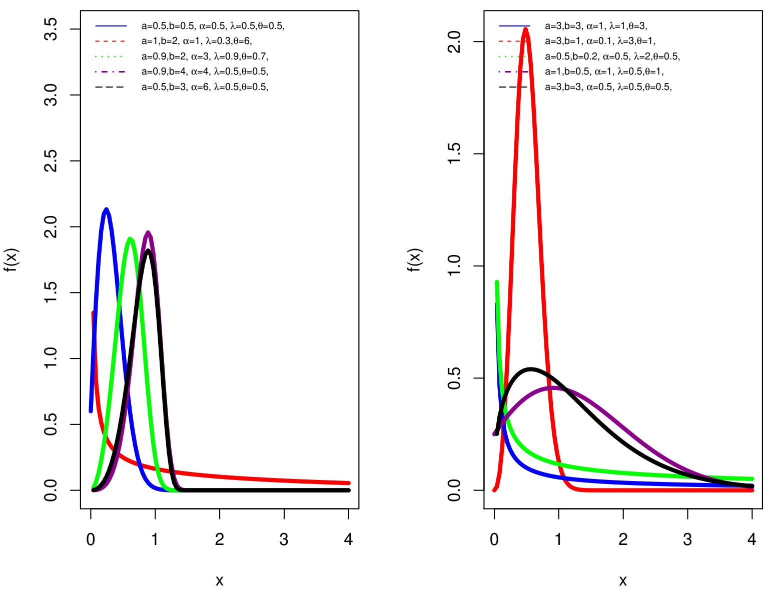


Fig. 1: Density of the KW-GLE for different parameter values

2.2 Special Cases

Table (2) has shown the submodels of the KW-GLE distribution with different parameter values.

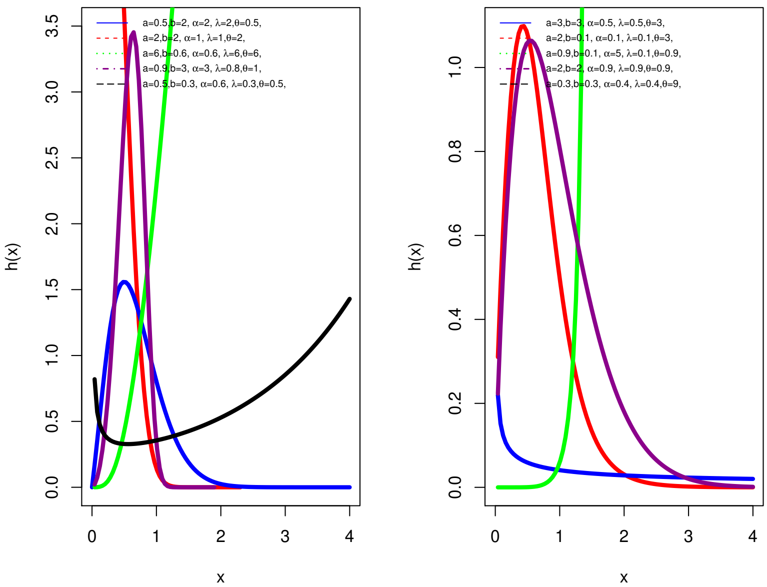


Fig. 2: hazard rate function of the KW-GLE for different parameter values

Table 2: Sub models of the $KW-GLE(a,b,\alpha,\lambda,\theta)$

	a	b	α	λ	θ	Distribution
[1]	1	1	1	λ	0	Exponential
[2]	1	1	1	0	θ	Rayleigh
[3]	1	1	1	λ	θ	Linear Failure Rate
[4]	a	1	1	0	θ	Exponentiated Rayleigh

	a	b	α	θ	λ	Distribution
[5]	a	1	1	λ	0	Exponentiated Exponential
[6]	a	1	1	λ	θ	Exponentiated Linear Failure Rate
[7]	1	1	α	λ	θ	Exponentiated linear Exponential
[8]	a	1	α	λ	θ	Exponentiated Exponentiated Linear Exponential
[9]	a	b	1	0	θ	KW Rayleigh

	a	b	α	θ	λ	Distribution
[10]	a	b	1	λ	0	KW Exponential
[11]	a	b	1	λ	θ	KW Linear Failure Rate

2.3 Expansion

We present in this subsection a representation of the cdf and pdf of the KW-GLE distribution. The relation (8) and (9) will be used throughout the work. If b is a positive real-integer and $|z|<1$, then

$$(1-z)^{b-1}=\sum_{j=0}^{\infty} w_j z^j \tag{8}$$

where

$$w_j=\frac{(-1)^j\Gamma(b)}{\Gamma(b-j)\Gamma(j+1)}.$$

and the exponentiated binomial theorem if β is a positive and $|z|<1$, then

$$(1-z)^{\beta-1}=\sum_{i=0}^{\infty}(-1)^i\binom{\beta-1}{i}z^i \tag{9}$$

using (9) in (4) we have the cdf of the KW-GLE as follows

$$F(x)=1-\sum_{j=0}^{\infty}(-1)^j\binom{b}{j}\left(e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}\right)^{aj} \tag{10}$$

also using (9) in (5) twice, we obtain that

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{k+j}\binom{b-1}{j}\binom{a(j+1)-1}{k}ab\alpha(\theta x+\lambda)(\frac{\theta}{2}x^2+\lambda x)^{\alpha-1}e^{-(k+1)(\frac{\theta}{2}x^2+\lambda x)^{\alpha}} \\ &= w_{j,k}f(x;\alpha,\lambda,\theta) \end{aligned} \tag{11}$$

where

$$w_{j,k}=ab\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{k+j}\binom{b-1}{j}\binom{a(j+1)-1}{k}e^{-k(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}$$

and

$$f(x;\alpha,\lambda,\theta)=\alpha(\theta x+\lambda)(\frac{\theta}{2}x^2+\lambda x)^{\alpha-1}e^{-(\frac{\theta}{2}x^2+\lambda x)^{\alpha}}$$

which is clearly the pdf of the exponentiated linear exponential distribution.

3 Important properties

This portion will provide some basic statistical properties of the model under consideration.

3.1 Median and quantile

When we invert (4), the quantile function $(for\ 0<q<1)$ is reached as

$$x_q=\frac{-\lambda\pm\sqrt{\lambda^2+2\theta[-ln(1-(1-(1-q)^{\frac{1}{b}})^{\frac{1}{a}})]^{\frac{1}{\alpha}}}}{\theta}$$

since θ is positive, we have our quantile function as

$$x_q=\frac{-\lambda+\sqrt{\lambda^2+2\theta[-ln(1-(1-(1-q)^{\frac{1}{b}})^{\frac{1}{a}})]^{\frac{1}{\alpha}}}}{\theta} \tag{12}$$

and the median is given by

$$x_q=\frac{-\lambda+\sqrt{\lambda^2+2\theta[-ln(1-(1-(\frac{1}{2})^{\frac{1}{b}})^{\frac{1}{a}})]^{\frac{1}{\alpha}}}}{\theta} \tag{13}$$

3.2 Moments

Here, we present the rth moment of the KW-GLE distribution.

Theorem 31*the rth moment of the KW-GLED is given by*

$$E(X^r)=\sum_{i=0}^{r}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j}\Gamma(a)}{\Gamma(a-j)j!}\binom{r}{i}\frac{\Gamma(\frac{r-i+2}{2})}{\Gamma(\frac{r-i+2}{2}-k)k!}2^{\frac{r-i}{2}-k}$$
$$\times\frac{\lambda^{2k+i}}{\theta^{\frac{i-r}{2}+k}}\Gamma\left(\frac{r-i}{2\alpha}-\frac{k}{\alpha}+1,(b+j)^{-1}\right)$$

proof. See Appendix B1

Letting $r=1$ and $r=3$ from (31), we obtain mean and third moment of the KW-GLED respectively.

Theorem 32*Let X be distributed according to the $KW-GLED$ then*

$$E\left[\left(\frac{\theta}{2}x^2+\lambda x\right)^r\right]=\sum_{j=0}^{r}\sum_{i=0}^{2j}\binom{r}{j}\binom{2j}{i}\frac{(-1)^{r-j}\lambda^{2(r-j)+i}}{\theta^{2(r-j)+i}}\mu_{2j-i}$$
$$=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(a)}{\Gamma(a-j)j!}\Gamma\left(\frac{\alpha+r}{\alpha},(b+j)^{-1}\right)$$

Proof. See Appendix B2.

We obtain the second and fourth moments from (32). Substituting $r=1$ in (32), we can obtain the following

$$E\left(\frac{\theta}{2}x^2+\lambda x\right)=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(a)}{\Gamma(a-j)j!}\Gamma\left(\frac{\alpha+1}{\alpha},(b+j)^{-1}\right)$$

therefore

$$\frac{\theta}{2}E(X^2)+\lambda E(X)=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(a)}{\Gamma(a-j)j!}\Gamma\left(\frac{\alpha+1}{\alpha},(b+j)^{-1}\right)$$

and consequently

$$\mu_2=\frac{2}{\theta}\left\{\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(a)}{\Gamma(a-j)j!}\Gamma\left(\frac{\alpha+1}{\alpha},(b+j)^{-1}\right)-\lambda\mu\right\}$$

Similarly by substituting $r=2$ in (32) also, we will have the following

$$E\left[\left(\frac{\theta}{2}x^2+\lambda x\right)^2\right]=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(a)}{\Gamma(a-j)j!}\Gamma\left(\frac{\alpha+2}{\alpha},(b+j)^{-1}\right)$$

therefore

$$\frac{\theta}{4}E(X^4)+\lambda\theta E(X^3)+\lambda^2E(X^2)=\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(a)}{\Gamma(a-j)j!}\Gamma\left(\frac{\alpha+2}{\alpha},(b+j)^{-1}\right)$$

hence

$$\mu_4=\frac{4}{\theta^2}\left\{\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(a)}{\Gamma(a-j)j!}\Gamma\left(\frac{\alpha+2}{\alpha},(b+j)^{-1}\right)-\lambda\theta\mu_3-\lambda^2\mu_2\right\}$$

The variance of the KW-GLE is computed by

$$Var(X)=E(X^2)-(E(X))^2$$

where

$$E(X^2)=\mu_2=\frac{2}{\theta}\left\{\sum_{j=0}^{\infty}\frac{(-1)^j\Gamma(a)}{\Gamma(a-j)j!}\Gamma\left(\frac{\alpha+1}{\alpha},(b+j)^{-1}\right)-\lambda\mu\right\}$$

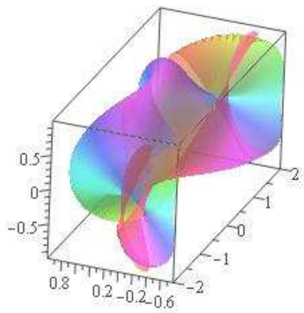
and $E(X)$ is the mean of the $KW-GLED$.

The kurtosis and skewness for the governing model can be reached as. Figure 3 depicts some physical features of the skewness and kurtosis respectively.

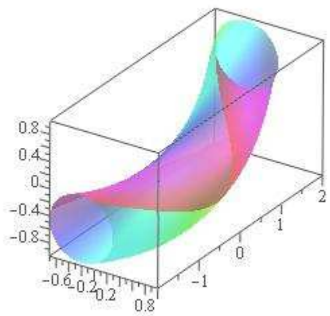
$$\gamma_3 = \frac{\mu^{(3)} - 3\mu\mu^{(2)} + 2\mu^3}{\left(\mu^{(2)} - \mu^2\right)^{\frac{3}{2}}} \tag{14}$$

$$\gamma_4 = \frac{\mu^{(4)} - 4\mu\mu^{(3)} + 6\mu^2\mu^{(2)} - 3\mu^4}{\left(\mu^{(2)} - \mu^2\right)^2} \tag{15}$$

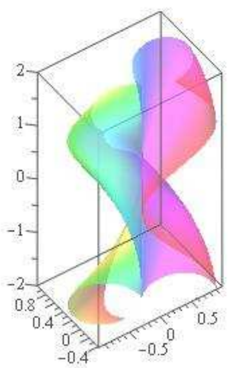
where $\mu^{(2)}$, $\mu^{(3)}$ and $\mu^{(4)}$ are second, third and fourth moments respectively



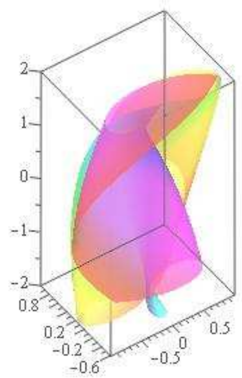
(a) Skewwness.



(b) Skewness.



(c) kurtosis.



(d) kurtosis.

Fig. 3: Skewness and kurtosis of KW-GLE in α and a when $b = 0.9, \theta = 1, \lambda = 0.4$..

3.3 Entropy

An entropy can be regarded as the measurement of variation of the ambiguity. There are two famous entropies which the Renhi and Shannon entropies. The Renyi entropy of a random variable with probability density function f(.) is define as

$$I_R(r) = \frac{1}{1-r} log \int_0^\infty f^r(x)dx$$

for $r > 0$ and $r \neq 1$ the Renhi entropy of the KW-GLED is given by

$$I_R(r) = -log\alpha + \frac{1}{1-r} log \left\{ \sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+j} \frac{\Gamma(r(a-1)+1)}{\Gamma(r(a-1)+1-j)j!} \frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r+1}{2}-i)i!} \lambda^{2i} \frac{2^{\frac{r-1}{2}-i}}{\theta^{\frac{r-1}{2}+i}} \Gamma(u,v) \right\}$$

Proof. See Appendix B3

The Shannon entropy is defined by $E[-\log f(x)]$, this is a special case of the Renhi entropy when $r \uparrow 1$.

3.4 Order Statistics

The density function $f_{i,n}(x)$ of the i^{th} order statistic for $i = 1,2,...,n$ from data values $x_1,...,x_n$ having the KW-GLED can be expressed as

$$f_{i,n}(x) = \frac{n!}{(i-1)(n-i)} [G(x)]^{i-1}[1-G(x)]^{n-i} g(x)$$

where $G(x)$ and $g(x)$are cdf and pdf in (1) respectively. Then,

$$f_{i,n}(x) = \alpha(\theta x+\lambda)(\frac{\theta}{2}x^2+\lambda x)^{\alpha-1} \left[e^{-(\frac{\theta}{2}x^2+\lambda x)^\alpha}\right]^b \sum_{j=0}^{n-i}(-1)^j \left(\sum_{i=0}^{\infty} \left[1-e^{-(\frac{\theta}{2}x^2+\lambda x)^\alpha}\right]\right)^{i+j-1}$$

Using the equation of [?] for a power series raised to a positive integer k given by

$$\left(\sum_{j=0}^{\infty}\right)^k = \sum_{j=0}^{\infty} c_{k,j}u^j \quad (16)$$

where the coefficients $c_{k,j}$ (for k=1,2,...) can be determined from the recurrence equation

$$c_{k,j} = (ja_0)^{-1} \sum_{m=1}^{j} [m(k+1)-j] a_m c_{k,j-m}$$

and $c_{k,0} = a_0^k$. Hence, $c_{k,j}$ follows directly from $c_{k,0}$,...,$c_{k,j-1}$ and therefore from a_0,...,a_k, after some algebra, we obatain from (16)

$$f_{i,n}(x) = \sum_{j=0}^{n-i}(-1)^j \binom{n-i}{j} \sum_{k=0}^{\infty} c_{i+j-1,k} f_{\theta_{i,j,k},j}(x) \quad (17)$$

where $f_{\theta_{i,j,k},j}(x)$ is the KGLED density function with the parameter vector $\theta_{i,j,k} = (a_{i,j,k},b,\alpha,\lambda,\theta)^T$. The result in (17) gives the density function of the KW-GLED order statistics as a linear combination of KGLED density functions.

4 Statistical Inference

4.1 Estimation

Let $X_1,...,X_n$ be a random sample with observed values $x_1,...,x_n$ from the KW-GLED with parameters $\alpha,\lambda,\theta,a,and,b$. Let $\Theta = (a,b,\alpha,,\theta,\lambda)^T$ be the parameter vector. The log likelihood function is given by

$$l_n \equiv l_n(x;\Theta) = nlog(\alpha) + \sum_{i=0}^{n} log(\theta x_i+\lambda) + (\alpha-1)\sum_{i=0}^{n} log\left(\frac{\theta}{2}x^2+\lambda x\right)$$
$$+b\sum_{i=0}^{n} log\left(e^{-(\frac{\theta}{2}x^2+\lambda x)^\alpha}\right) + (a-1)\sum_{i=0}^{n} log\left(1-e^{-(\frac{\theta}{2}x^2+\lambda x)^\alpha}\right)$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equation obtained by differentiating $l_n(y;\Theta)$ above. The components of the score vector $U(y;\Theta)$ are given by

$$U_\lambda(x;\Theta) = \frac{\partial}{\partial\lambda}l_n(y;\theta) = \sum_{i=0}^{n}\frac{1}{\theta x_i+\lambda} + (\alpha-1)\sum_{i=1}^{n}\frac{x_i}{(\frac{\theta}{2}x_i^2+\lambda x_i)}$$
$$+\alpha(a-1)\sum_{i=1}^{n}\frac{x_i(\frac{\theta}{2}x_i^2+\lambda x_i)^{\alpha-1}e^{-(\frac{\theta}{2}x_i^2+\lambda x_i)^\alpha}}{(1-e^{-(\frac{\theta}{2}x_i^2+\lambda x_i)^\alpha})} - \alpha b\sum_{i=1}^{n}x_i(\frac{\theta}{2}x_i^2+\lambda x_i)^{\alpha-1}$$

$$U_\theta(x;\Theta) = \frac{\partial}{\partial\theta}l_n(y;\theta) = \sum_{i=0}^{n}\frac{x_i}{\theta x_i+\lambda} + \frac{(\alpha-1)}{2}\sum_{i=1}^{n}\frac{x_i^2}{(\frac{\theta}{2}x_i^2+\lambda x_i)} - \frac{\alpha b}{2}\sum_{i=1}^{n}x_i^2(\frac{\theta}{2}x_i^2+\lambda x_i)^{\alpha-1}$$
$$+\frac{\alpha(a-1)}{2}\sum_{i=1}^{n}\frac{x_i^2(\frac{\theta}{2}x_i^2+\lambda x_i)^{\alpha-1}e^{-(\frac{\theta}{2}x_i^2+\lambda x_i)^\alpha}}{(1-e^{-(\frac{\theta}{2}x_i^2+\lambda x_i)^\alpha})}$$

$$U_a(x;\Theta) = \frac{\partial}{\partial a}l_n(y;\theta) = \sum_{i=1}^{n}log\left(1-e^{-(\frac{\theta}{2}x_i^2+\lambda x_i)^\alpha}\right)$$

$$U_b(y;\Theta) = \frac{\partial}{\partial b}l_n(y;\theta) = \sum_{i=1}^{n}\left(\frac{\theta}{2}x_i^2+\lambda x_i\right)^\alpha$$

$$U_\alpha(y;\Theta) = \frac{\partial}{\partial\alpha}l_n(y;\theta) = \frac{n}{\alpha} + \sum_{i=1}^{n}log\left(\frac{\theta}{2}x_i^2+\lambda x_i\right)$$

The MLEs does not have an explicit form of normal equations. As a result of this, numerical methods will be used to obtain the MLEs. Simulations will be used to assess the performance and consistency of the MLEs.

For interval estimation, we consider the observe information matrix. The 5×5 units matrix $J = J(\theta)$ is expressed as

$$\mathbf{J} = \begin{bmatrix} J_{\lambda\lambda} & \cdots & J_{\lambda\alpha} \\ \vdots & \ddots & \vdots \\ J_{\alpha\lambda} & \cdots & J_{\alpha\alpha} \end{bmatrix}$$

One can see that $\hat{\theta}$ is consistent estimator of θ therefore, the authenticity of the asymptotic normality will remain unchanged when the fisher information matrix I is changed by the observed fisher information attained at $\hat{\theta}J(\theta)$: Owing to this, a $\gamma 100\%$ approximate asymtotic interval for each component parameter $\hat{\theta}_\mathbf{l}$ of $\hat{\theta}$ is expressed as

$$\left(\hat{\theta}_\mathbf{l} - \mathbf{Z}_{\frac{1+\gamma}{2}}\sqrt{\mathbf{J}^{\hat{\theta}_\mathbf{l}\hat{\theta}_\mathbf{l}}}, \hat{\theta}_\mathbf{l} + \mathbf{Z}_{\frac{1+\gamma}{2}}\sqrt{\mathbf{J}^{\hat{\theta}_\mathbf{l}\hat{\theta}_\mathbf{l}}}\right)$$

where $\mathbf{J}^{\hat{\theta}_l\hat{\theta}_l}$ exhibits diagonals element for $\mathbf{J}(\hat{\theta})^{-1}$ associating with all the parameter $l = (a,b,\alpha,\lambda,\theta)$ and $Z_{\frac{1+\gamma}{2}}$ represents quantile $\frac{1+\gamma}{2}$ of the standard normal distribution. The likelihood ration (LR) statistics is applied for testing Kw-GLE with other existing distributions. By considering the partition $\theta = (\theta_1^T, \theta_2^T)^T$, tests of hypothesis of the kind $H_0 : \theta_1 = \theta_1^{(0)}$ vs $H_1 : \theta_1 \neq \theta_1^{(0)}$ is observed by LR statistics that can be expressed as $w = 2\left\{l(\hat{\theta}) - (l\tilde{\theta})\right\}$, where $\hat{\theta}$, $\tilde{\theta}$ are the MLEs of θ under H_1 and H_0 respectively. Under the null hypothesis, $w \xrightarrow{d} \chi_q^2$, where q is the dimension of the vector θ_1 of interest. The LR test rejects H_0 if $w > \xi_q$, where ξ_q is the upper $100_\gamma\%$ point of the χ_q^2 distribution.

5 Simulation

Herein, we provide an assessment for the effieciency of the MLEs by simulating the parameters. We analyze the proposed estimator of θ=(a, b , α , θ, λ) of the proposed MLEs. 10000 samples of $n = 30, 100, 300,$ and 500 size are proposed from KW-GLED. For each scenario of the five different set values of θ. Samples from KW-GLED are attained by using the (12). The accuracy of the approximation of the standard error of the MLEs have been assessed with the aid of the fisher information matrix.

The following are observed from the Table (3) below:

1. The numerical method is proven to be stable of the MLEs approach since convergence is achieved.
2. Original values and the averages estimate are almost the same.
3. There is a consistent performance with the MLEs estimate.
4. When the sample size increases the standard errors of the MLEs decreases

Table 3: The averages of 10000 MLEs and mean of the simulated standard errors for $KW-GLED$

		AE					SD				
n	$(a, b, \alpha, \theta, \lambda)$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$sd(\hat{\alpha})$	$sd(\hat{\theta})$	$sd(\hat{\lambda})$	$sd(\hat{a})$	$sd(\hat{b})$
30	(0.6, 0.6, 0.5, 1.0, 1.0)	0.935	0.684	0.768	1.278	1.541	0.761	2.636	2.057	2.906	1.931
	(3.0, 0.5, 0.5, 1.0, 1.0)	5.853	0.531	0.719	3.469	1.139	5.406	0.793	0.796	10.251	1.557
	(2.0, 0.5, 0.5, 1.0, 1.0)	4.164	0.448	0.891	2.481	1.251	3.636	0.845	1.112	7.932	1.707
	(2.0, 2.0, 2.0, 1.0, 1.0)	3.886	1.887	2.898	2.323	1.718	3.482	4.867	3.515	7.167	2.290
100	(0.6, 0.6, 0.5, 1.0, 1.0)	0.763	0.663	0.691	0.943	1.182	0.491	2.505	1.975	1.033	0.997
	(3.0, 0.5, 0.5, 1.0, 1.0)	5.931	0.373	0.721	1.681	1.196	4.825	0.732	0.611	5.331	1.461
	(2.0, 0.5, 0.5, 1.0, 1.0)	3.707	0.515	0.866	1.638	1.034	2.858	0.971	1.076	5.411	1.154
	(2.0, 2.0, 2.0, 1.0, 1.0)	3.137	2.213	2.469	1.364	1.534	2.451	4.814	2.798	4.153	1.837
300	(0.6, 0.6, 0.5, 1.0, 1.0)	0.664	0.586	0.601	0.895	1.063	0.324	1.538	1.679	0.455	0.578
	(3.0, 0.5, 0.5, 1.0, 1.0)	5.531	0.349	0.715	1.067	1.031	4.039	0.707	0.511	2.737	1.094
	(2.0, 0.5, 0.5, 1.0, 1.0)	3.305	0.421	0.718	0.961	1.041	2.158	0.772	0.721	1.902	0.955
	(2.0, 2.0, 2.0, 1.0, 1.0)	2.678	2.521	2.195	1.018	1.346	1.742	4.741	2.128	1.538	1.429
500	(0.6, 0.6, 0.5, 1.0, 1.0)	0.607	0.523	0.564	0.909	0.997	0.221	0.901	0.775	0.325	0.344
	(3.0, 0.5, 0.5, 1.0, 1.0)	5.109	0.306	0.679	0.842	1.019	3.291	0.637	0.403	0.869	0.873
	(2.0, 0.5, 0.5, 1.0, 1.0)	3.011	0.408	0.641	0.844	0.957	1.686	0.716	0.491	0.831	0.603
	(2.0, 2.0, 2.0, 1.0, 1.0)	2.404	2.556	2.052	0.945	1.138	1.2124	4.093	1.575	0.498	0.831

6 Application

Herein, we demonstrate the applicability of the KW-GLED by means of a real data. The data set represents the strengths of $1.5cm$ glass fibres and they are taken from Smith and Naylor [**?**]. We compare the fit of the KW-GLED with those of the beta exponentiated exponential (BGE), beta exponential (BE), exponentiated exponential (GE), exponentiated exponential poison (EEP), generalization of exponential poison (GEP) and exponential poison (EP) distributions. For each distribution, the unknown parameters are estimated by the method of maximum likelihood. The maximum likelihood estimates and the corresponding AIC and BIC values are shown in Tables (4). We can see that the smallest AIC and BIC are obtained from the KW-GLED. The fitted probability density functions and the fitted cumulative distribution function are shown in Figures (**??**) and (**??**) respectively. So, from the figures, we can conclude that the KW-GLED is the most appropriate model for the data set out of the considered distributions. The MLEs of the KW-GLE parameters a, b, α, λ and θ are computed by maximizing the objective function using R software. The estimated values of the parameters,log-likelihood statistic, Akaike Information Criterion, $AIC = 2p - 2log(L)$, and Bayesian Information Criterion, $BIC = Plog(n) - 2log(L)$ where $L = L(\theta)$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters for the data.

6.1 Data illustration

The data set is: 0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2 ,0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

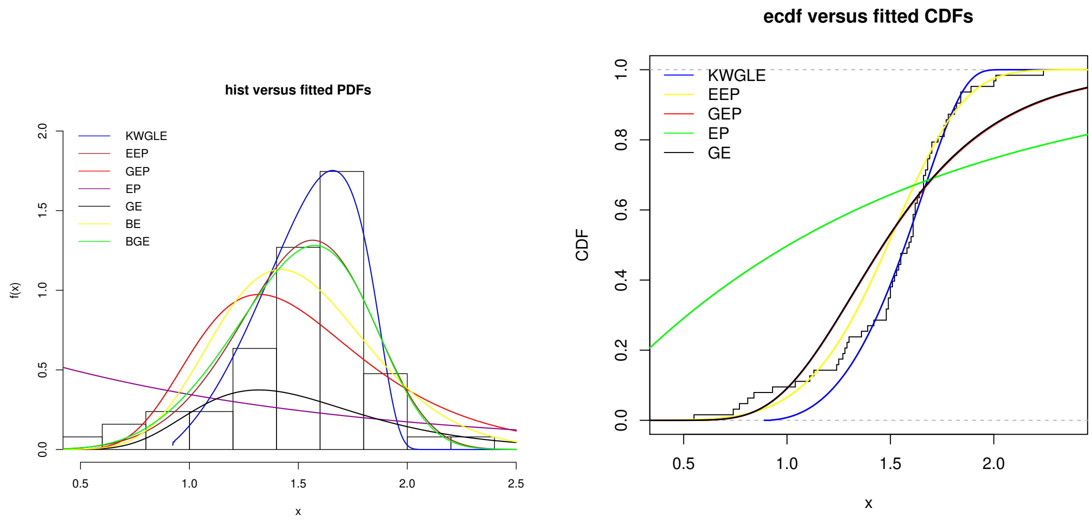
Table 4: MLEs of the strengths of $1.5cm$ glass fibres, measured at the National Physical Laboratory, England.

	a	b	α	λ	θ	$l(\theta)$	AIC	BIC
BGE(a,b,α,λ)	0.4125	93.4655	22.6124	0.92271	–	−15.5995	39.199	47.897
BE(a,b,λ)	17.7786	22.7222	–	0.3898	–	−24.1270	54.254	60.777
EEP(α,λ,θ)	–	–	6.4712	16619.29	0.1549	−15.6000	37.2	43.600
GEP(a,b,α,λ)	–	–	31.9563	280.6076	0.0094	−31.5500	69.1	75.500
EP(α,λ)	–	–	–	983.5960	0.0007	−88.8500	181.7	186.0
GE(α,λ)	–	–	31.3032	2.6105	–	−31.3834	66.7668	71.1156
KWGLE$(a,b,\alpha,\lambda,\theta)$	0.210000	1.887788	8.999999	−0.444591	1.000000	−10.8870	31.774	42.646

We compare the proposed the model under consideration with other existing models as shown in Table (5).

Table 5: LR statistics for the data

Model	Hypothesis	Statistic *LR*	p-value
BGE vs *KWGLE*	$H_0 : \theta = 1$ vs $H_1 : H_0$ is false	9.425	2.1405×10^{-3}
BE vs *KWGLE*	$H_0 : \theta = \alpha = 1$ vs $H_1 : H_0$ is false	26.48	1.8×10^{-6}
EEP vs *KWGLE*	$H_0 : \theta = \alpha = 1$ vs $H_1 : H_0$ is false	9.426	8.9778×10^{-3}
GEP vs *KWGLE*	$H_0 : \theta = \alpha = 1$ vs $H_1 : H_0$ is false	41.326	0.0001
EP vs *KWGLE*	$H_0 : a = b = \alpha = 1$ vs $H_1 : H_0$ is false	155.926	0.0004
GE vs *KWGLE*	$H_0 : a = b = \theta = 1$ vs $H_1 : H_0$ is false	40.9928	0.0002



(a) fitted densities for the data. (b) fitted cdf for the data.

Fig. 4: Fitted pdf and cdf for the data.

7 Conclusion

In this paper, we propose a new statistical distribution called the Kw-GLED. The proposed model performed much better that the other models due to the better fitting in a real data. The pdf of the proposed model is uni-modal, decreasing, and a constant depending on the parameter values. The hazard rate function possesses a bathtube shape and it also increasing, decreasing. Moreover, statistical properties are analyzed and the parameters of the Kw-GLED are estimated using the approach of MLEs and the information matrix is obtained . The hypothesis test via LR test and the simulation analysis were conducted so that the effectiveness of the estimation of parameters can be observed. The applicability and efficiency of the model were illustrated by means of a real data. The proposed model has proven to be more flexible for fitting lifetime data in reliability, biology and other areas.

Appendix A

The elements of the 5×5 information matrix are given by let $p_i = \frac{\theta}{2}x_i^2 + \lambda x_i$

$$J_{\lambda\lambda} = \frac{\partial^2}{\partial\lambda\partial\lambda}l_n(y;\theta) = -\sum_{i=1}^{n}\frac{1}{(\theta x_i+\lambda)^2} - (\alpha-1)\sum_{i=1}^{n}\frac{x_i^2}{p_i^2}$$
$$+\alpha(a-1)\sum_{i=1}^{n}\frac{x_i^2\left(\alpha p_i^{2(\alpha-1)}e^{-p_i^\alpha}+(\alpha-1)p_i^{\alpha-2}e^{-p_i^\alpha}(1-e^{-p_i^\alpha})\right)}{(1-e^{-p_i^\alpha})^2}$$
$$-\alpha(\alpha-1)b\sum_{i=1}^{n}x_i^2 p_i^{(\alpha-2)}$$

$$J_{\theta\theta} = \frac{\partial^2}{\partial\lambda\partial\theta}l_n(y;\theta) = \sum_{i=1}^{n}\frac{x_i^2}{(\theta x_i+\lambda)^2} + \frac{(\alpha-1)}{4}\sum_{i=1}^{n}\frac{x_i^4}{p_i^2} - \frac{\alpha(\alpha-1)b}{4}\sum_{i=1}^{n}x_i^4 p_i^{\alpha-2}$$
$$+\frac{\alpha(a-1)}{4}\sum_{i=1}^{n}\frac{x_i^4\left[(\alpha-1)p_i^{\alpha-2}e^{-p_i^\alpha}(1-e^{-p_i^\alpha})-\alpha p_i^{2(\alpha-1)}e^{-p_i^\alpha}\right]}{(1-e^{-p_i^\alpha})^2}$$

$$J_{aa} = \frac{\partial^2}{\partial a\partial a}l_n(y;\theta) = 0$$
$$J_{bb} = \frac{\partial^2}{\partial b\partial b}l_n(y;\theta) = 0$$
$$J_{\alpha\alpha} = \frac{\partial^2}{\partial\alpha\partial\alpha} = -\frac{n}{\alpha^2}$$

$$J_{\lambda\theta} = J_{\theta\lambda} = -\sum_{i=1}^{n}\frac{x_i}{(\theta x_i+\lambda)^2} - \frac{(\alpha-1)}{2}\sum_{i=1}^{n}\frac{x_i^3}{p_i^2}$$
$$-\frac{\alpha(\alpha-1)b}{2}\sum_{i=1}^{n}x_i^3 p_i^{\alpha-2}$$

$$J_{\lambda a} = J_{a\lambda} = \alpha\sum_{i=1}^{n}\frac{p_i^{(\alpha-1)}e^{-p_i^\alpha}}{(1-e^{-p_i^\alpha})^2}$$
$$J_{\lambda b} = J_{b\lambda} = -\alpha\sum_{i=1}^{n}x_i p_i^{(\alpha-1)}$$
$$J_{\lambda\alpha} = J_{\alpha\lambda} = \sum_{i=1}^{n}\frac{x_i}{p_i} - b\sum_{i=1}^{n}x_i p_i^{(\alpha-1)}$$
$$J_{\theta a} = J_{a\theta} = \frac{\alpha}{2}\sum_{i=1}^{n}\frac{x_i^2 p_i^{(\alpha-1)}e^{-p_i^\alpha}}{(1-e^{-p_i^\alpha})}$$
$$J_{\theta b} = J_{b\theta} = -\frac{\alpha}{2}\sum_{i=1}^{n}x_i^2 p_i^{(\alpha-1)}$$

$$J_{\theta\alpha} = J_{\alpha\theta} = \frac{1}{2}\sum_{i=1}^{n}\frac{x_i^2}{(p_i)} - \frac{b}{2}\sum_{i=1}^{n}x_i^2 p_i^{\alpha-1} + \frac{(a-1)}{2}\sum_{i=1}^{n}\frac{x_i^2 p_i^{(\alpha-1)}e^{-p_i^\alpha}}{(1-e^{-p_i^\alpha})}$$

$$J_{ab} = J_{ba} = 0$$

$$J_{a\alpha} = J_{\alpha a} = 0$$

$$J_{b\alpha} = J_{\alpha b} = 0$$

Appendix B

B1**proof.**

$$E(X^r) = \int_0^\infty x^r f(x)dx$$
$$= \alpha \int_0^\infty x^r(\lambda+\theta x)\left(\frac{\theta}{2}x^2+\lambda x\right)^{\alpha-1}\left[e^{-(\frac{\theta}{2}x^2+\lambda x)^\alpha}\right]^b\left(1-e^{-(\frac{\theta}{2}x^2+\lambda x)^\alpha}\right)^{a-1}dx$$

let $u=(\frac{\theta}{2}x^2+\lambda x)^\alpha$,$du=\alpha(\theta x+\lambda)(\frac{\theta}{2}x^2+\lambda x)^{\alpha-1}dx$,and $x=\frac{1}{\theta}\sqrt{2u^{\frac{1}{\alpha}}\theta+\lambda^2}-\frac{\lambda}{\theta}$, therefore

$$E(X^r) = \int_0^\infty \frac{1}{\theta^r}\left[\sqrt{2u^{\frac{1}{\alpha}}\theta+\lambda^2}-\lambda\right]^r [e^{-u}]^b[1-e^{-u}]^{a-1}du$$

where

$$[1-e^{-u}]^{a-1} = \sum_{j=0}^\infty \frac{(-1)^j\Gamma(a)}{\Gamma(a-j)j!}[e^{-u}]^j.$$

Using binomial expansion for

$$\left[\sqrt{2u^{\frac{1}{\alpha}}\theta+\lambda^2}-\lambda\right]^r = \sum_{i=0}^r(-1)^i\binom{r}{i}\lambda^i\left(2u^{\frac{1}{\alpha}}\theta+\lambda^2\right)^{\frac{r-i}{2}}$$

since $\lambda\geq 0$, $x>0$, $\alpha>0$, $\theta\geq 0$ and $u^{\frac{1}{\alpha}}=(\frac{\theta}{2}x^2+\lambda x)$ then clearly $\frac{\lambda^2}{2u^{\frac{1}{\alpha}}\theta}<1$ and

$$\left(2u^{\frac{1}{\alpha}}\theta+\lambda^2\right)^{\frac{r-i}{2}} = (2u^{\frac{1}{\alpha}}\theta)^{\frac{r-i}{2}}\left(1+\frac{\lambda^2}{2u^{\frac{1}{\alpha}}\theta}\right)^{\frac{r-i}{2}}$$

Therefore

$$\left(1+\frac{\lambda^2}{2u^{\frac{1}{\alpha}}\theta}\right)^{\frac{r-i}{2}} = \sum_{k=0}^\infty \frac{\Gamma(\frac{r-i+2}{2})}{\Gamma(\frac{r-i+2}{2}-k)k!}\lambda^{2k}(2\theta)^{-k}u^{-\frac{k}{\alpha}}$$

therefore

$$(2u^{\frac{1}{\alpha}}\theta)^{\frac{r-i}{2}}\left(1+\frac{\lambda^2}{2u^{\frac{1}{\alpha}}\theta}\right)^{\frac{r-i}{2}} = \sum_{k=0}^\infty \frac{\Gamma(\frac{r-i+2}{2})}{\Gamma(\frac{r-i+2}{2}-k)k!}\lambda^{2k}(2\theta)^{\frac{r-i}{2}-k}u^{\frac{r-i}{2\alpha}-\frac{k}{\alpha}}$$

$$E(X^r) = \sum_{i=0}^r\sum_{j=0}^\infty\sum_{k=0}^\infty \frac{(-1)^{i+j}\Gamma(a)}{\Gamma(a-j)j!\theta^r}\binom{r}{i}\frac{\Gamma(\frac{r-i+2}{2})}{\Gamma(\frac{r-i+2}{2}-k)k!}\frac{\lambda^{2k+i}}{\theta^{\frac{i-r}{2}+k}}2^{\frac{r-i}{2}-k}\int_0^\infty u^{\frac{r-i}{2\alpha}-\frac{k}{\alpha}}e^{-u(b+j)}du$$
$$= \sum_{i=0}^r\sum_{j=0}^\infty\sum_{k=0}^\infty \frac{(-1)^{i+j}\Gamma(a)}{\Gamma(a-j)j!}\binom{r}{i}\frac{\Gamma(\frac{r-i+2}{2})}{\Gamma(\frac{r-i+2}{2}-k)k!}2^{\frac{r-i}{2}-k}$$
$$\times\frac{\lambda^{2k+i}}{\theta^{\frac{i-r}{2}+k}}\Gamma\left(\frac{r-i}{2\alpha}-\frac{k}{\alpha}+1,(b+j)^{-1}\right)$$

B2**Proof.**

$$E\left[\left(\frac{\theta}{2}x^2+\lambda x\right)^r\right] = \int_0^\infty\left(\frac{\theta}{2}x^2+\lambda x\right)^r f(x)dx$$
$$= \alpha\int_0^\infty\left(\frac{\theta}{2}x^2+\lambda x\right)^r(\lambda+\theta x)\left(\frac{\theta}{2}x^2+\lambda x\right)^{\alpha-1}$$
$$\times\left[e^{-(\frac{\theta}{2}x^2+\lambda x)^\alpha}\right]^b\left(1-e^{-(\frac{\theta}{2}x^2+\lambda x)^\alpha}\right)^{a-1}dx$$

let $u=(\frac{\theta}{2}x^2+\lambda x)$,$du=(\theta x+\lambda)dx$, therefore

$$E\left[\left(\frac{\theta}{2}x^2+\lambda x\right)^r\right] = \alpha\int_0^\infty u^{r+\alpha-1}[e^{-u^\alpha}]^b[1-e^{-u^\alpha}]^{a-1}du$$

where

$$[1-e^{-u^{\alpha}}]^{a-1}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(a)}{\Gamma(a-j)j!}[e^{-u^{\alpha}}]^{j}$$

which shows that

$$E\left[\left(\frac{\theta}{2}x^{2}+\lambda x\right)^{r}\right]=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(a)}{\Gamma(a-j)j!}\alpha\int_{0}^{\infty}u^{r+\alpha-1}[e^{-u^{\alpha}}]^{b+j}du$$
$$=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(a)}{\Gamma(a-j)j!}\Gamma\left(\frac{\alpha+r}{\alpha},(b+j)^{-1}\right)$$

Set $\frac{\theta}{2}x^{2}+\lambda x=0$ applying the completing the square method we will have

$$\left[\left(x+\frac{\lambda}{\theta}\right)^{2}-\left(\frac{\lambda}{\theta}\right)^{2}\right]^{r}=0$$

now expanding $\left[\left(x+\frac{\lambda}{\theta}\right)^{2}-\left(\frac{\lambda}{\theta}\right)^{2}\right]^{r}$ binomially we will have the following

$$\left[\left(x+\frac{\lambda}{\theta}\right)^{2}-\left(\frac{\lambda}{\theta}\right)^{2}\right]^{r}=\sum_{j=0}^{r}\binom{r}{j}\left(x+\frac{\lambda}{\theta}\right)^{2j}(-1)^{r-j}\left(\frac{\lambda}{\theta}\right)^{2(r-j)}$$

applying binomial expansion on $\left(x+\frac{\lambda}{\theta}\right)^{2j}$ we will have

$$\left(x+\frac{\lambda}{\theta}\right)^{2j}=\sum_{i=0}^{2j}\binom{2j}{i}\left(\frac{\lambda}{\theta}\right)^{i}x^{2j-i}$$

Consequently

$$\left[\left(x+\frac{\lambda}{\theta}\right)^{2}-\left(\frac{\lambda}{\theta}\right)^{2}\right]^{r}=\sum_{j=0}^{r}\sum_{i=0}^{2j}\binom{r}{j}\binom{2j}{i}\frac{(-1)^{r-j}\lambda^{2(r-j)+i}}{\theta^{2(r-j)+i}}X_{2j-i}$$

$$E\left(\sum_{j=0}^{r}\sum_{i=0}^{2j}\binom{r}{j}\binom{2j}{i}\frac{(-1)^{r-j}\lambda^{2(r-j)+i}}{\theta^{2(r-j)+i}}X_{2j-i}\right)=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(a)}{\Gamma(a-j)j!}\Gamma\left(\frac{\alpha+r}{\alpha},(b+j)^{-1}\right)$$

$$\sum_{j=0}^{r}\sum_{i=0}^{2j}\binom{r}{j}\binom{2j}{i}\frac{(-1)^{r-j}\lambda^{2(r-j)+i}}{\theta^{2(r-j)+i}}\mu_{2j-i}=\sum_{j=0}^{\infty}\frac{(-1)^{j}\Gamma(a)}{\Gamma(a-j)j!}\Gamma\left(\frac{\alpha+r}{\alpha},(b+j)^{-1}\right)$$

B3**proof.**

$$\int_{0}^{\infty}f^{r}(x)dx=\alpha^{r}\int_{0}^{\infty}(\lambda+\theta x)^{r}\left(\frac{\theta}{2}x^{2}+\lambda x\right)^{r(\alpha-1)}\left[e^{-(\lambda x+\frac{\theta}{2}x^{2})^{\alpha}}\right]^{br}\left[1-e^{-(\lambda x+\frac{\theta}{2}x^{2})^{\alpha}}\right]^{r(a-1)}dx$$

let

$$u=(\lambda x+\frac{\theta}{2}x^{2})^{\alpha}$$

$$du=\alpha(\lambda+\theta x)(\lambda x+\frac{\theta}{2}x^{2})^{\alpha-1}$$

$$x=\frac{1}{\theta}\sqrt{2\theta u^{\frac{1}{\alpha}}+\lambda^{2}}-\frac{\lambda}{\theta}$$

then

$$\int_{0}^{\infty}f^{r}(x)dx=\alpha^{r-1}\int_{0}^{\infty}\left(\sqrt{2\theta u^{\frac{1}{\alpha}}+\lambda^{2}}\right)^{r-1}(u^{\frac{1}{\alpha}})^{(\alpha-1)(r+1)}\left[e^{-u}\right]^{rb}(1-e^{-u})^{r(a-1)}du$$

If $r(a-1)+1>0$ is a real non integer we expand $(1-e^{-u})^{r(a-1)+1-1}$ using power series expansion

$$(1-e^{-u})^{r(a-1)+1-1}=\sum_{j=0}^{\infty}(-1)^j\frac{\Gamma(r(a-1)+1)}{\Gamma(r(a-1)+1-j)j!}(e^{-u})^j$$

Therefore
since $\lambda\geq 0$, $x>0$, $\alpha>0$, $\theta\geq 0$ and $u^{\frac{k}{\alpha}}=(\frac{\theta}{2}x^2+\lambda x)$ then clearly $\frac{\lambda^2}{2u^{\frac{1}{\alpha}}\theta}<1$ Therefore

$$\left(1+\frac{\lambda^2}{2u^{\frac{1}{\alpha}}\theta}\right)^{\frac{r-1}{2}}=\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r+1}{2}-i)i!}\lambda^{2i}(2\theta)^{-i}u^{-\frac{i}{\alpha}}$$

This shows that

$$\left(2u^{\frac{1}{\alpha}}\theta\right)^{\frac{r-1}{2}}\left(1+\frac{\lambda^2}{2u^{\frac{1}{\alpha}}\theta}\right)^{\frac{r-1}{2}}=\sum_{i=0}^{\infty}(-1)^i\frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r+1}{2}-i)i!}\lambda^{2i}(2\theta)^{-i}u^{-\frac{i}{\alpha}}\left(2u^{\frac{1}{\alpha}}\theta\right)^{-\frac{r-1}{2}}$$

Therefore

$$\int_0^{\infty}f^r(x)dx=\alpha^{r-1}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+j}\frac{\Gamma(r(a-1)+1)}{\Gamma(r(a-1)+1-j)j!}\frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r+1}{2}-i)i!}\lambda^{2i}\frac{2^{\frac{r-1}{2}-i}}{\theta^{\frac{r-1}{2}+i}}\Gamma(u,v)$$

where $u=-\frac{(r-1)}{2\alpha}-\frac{i}{\alpha}+\frac{(\alpha-1)(r+1)}{\alpha}+1$, $v=br+j$ This shows that

$$I_R(r)=\frac{1}{1-r}log\left[\alpha^{r-1}\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+j}\frac{\Gamma(r(a-1)+1)}{\Gamma(r(a-1)+1-j)j!}\frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r+1}{2}-i)i!}\lambda^{2i}\frac{2^{\frac{r-1}{2}-i}}{\theta^{\frac{r-1}{2}+i}}\Gamma(u,v)\right]$$

This implies that

$$I_R(r)=-log\alpha+\frac{1}{1-r}log\left\{\sum_{j=0}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+j}\frac{\Gamma(r(a-1)+1)}{\Gamma(r(a-1)+1-j)j!}\frac{\Gamma(\frac{r+1}{2})}{\Gamma(\frac{r+1}{2}-i)i!}\lambda^{2i}\frac{2^{\frac{r-1}{2}-i}}{\theta^{\frac{r-1}{2}+i}}\Gamma(u,v)\right\}$$

where $u=-\frac{(r-1)}{2\alpha}-\frac{i}{\alpha}+\frac{(\alpha-1)(r+1)}{\alpha}+1$, $v=br+j$

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