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Bivariate Exponentiated Modified Weibull Distribution

H. S. Shahen, A. H. El-Bassiouny and Mohamed Abouhawwash *

Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

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Abstract: In this paper, a new bivariate exponentiated modified Weibull distribution (BEMW) is introduced. It is a Marshall-Olkin type. Marginal and conditional distribution functions are studied. Furthermore, marginal moments are calculated. Also, joint hazard rate function and maximum likelihood estimates (MLEs) of the parameters are presented. An application of the BEMW distribution to an American football league data set is provided and the profiles of the log-likelihood function of parameters of BEMW distribution is plotted.

Keywords: Reliability, Marshall-Olkin, Weibull distribution, Exponentiated modified Weibull distribution, Survival Functions

1 Introduction

Elbatal [1] introduced the exponentiated modified Weibull distribution by powering a positive real number (α) to the cumulative modified Weibull distribution function. This distribution is flexible in accommodating all the forms of the hazard rate function can be used in various problems for modeling random lifetimes. Another important characteristic of this distribution is that it reduces to, the Weibull, exponentiated exponential (Gupta and Kundu [2]), exponentiated Weibull distribution (Mudholkar et al. [5,6]), generalized Rayleigh (Kundu and Rakab [3]), modified Weibull distribution (Lai et al.[4]) and some other distributions.

The aim of this paper is to introduce a new bivariate exponentiated modified Weibull (BEMW) distribution, whose marginals are EMW distributions. It is a Marshall-Olkin type. Many authors used this method to introduce a new bivariate distribution, see for example Marshall and Olkin [8], Kundu and Gupta [7], Sarhan and Balakrishnan [9], El-Bassiouny et al. [10], El-Gohary et al. [11] and El-Bassiouny et al. [12,13,14].

This article is organized as follows, a new bivariate exponentiated Modified Weibull (BEMW) distribution is given in Section 2. Also, various properties including the joint cumulative distribution function, the joint probability density function, marginal probability density functions, and conditional probability density functions are investigated in Section 2. The marginal expectation is provided in Section 3. Some reliability studies are obtained in Section 4. Section 5 is devoted to the maximum likelihood estimates of the parameters of the BEMW distribution. In Section 6, an application of the BEMW distribution to an American football league data set is provided. Finally, the results of this paper are concluded in Section 7.

2 Bivariate Exponentiated Modified Weibull distribution

In this section, we discuss the BEMW distribution. We start with the joint cumulative distribution function and derive the corresponding joint probability density function of this distribution. Let X be a random variable has exponentiated modified Weibull (EMW) distribution with parameters α, β, θ and $\gamma > 0$, then its cumulative distribution function (cdf) is given by

$$F(x) = \left(1 - e^{-\theta x - \gamma x^{\beta}}\right)^{\alpha},\ x > 0, \tag{1}$$

* Corresponding author e-mail: saleh1284@mans.edu.eg

and the probability density function (pdf) is given by

$$f(x) = \alpha\left(\theta + \gamma\beta x^{\beta-1}\right) e^{-\theta x-\gamma x^{\beta}}\left(1 - e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha-1}. \tag{2}$$

2.1 The Joint Cumulative Distribution Function

Suppose that $U_1 \sim EMW(\alpha_1,\beta,\theta,\gamma)$, $U_2 \sim EMW(\alpha_2,\beta,\theta,\gamma)$ and $U_3 \sim EMW(\alpha_3,\beta,\theta,\gamma)$ are independent random variables. Define $X_1 = \max\{U_1,U_3\}$ and $X_2 = \max\{U_2,U_3\}$. Then, the bivariate vector $(X_1,X_2) \sim BEMW(\alpha_1,\alpha_2,\alpha_3,\beta,\theta,\gamma)$.

In the following lemma, We study the joint cumulative distribution function of the random variables X_1 and X_2.

Lemma 1. *The joint cdf of X_1 and X_2 is given by*

$$F_{BEMW}(x_1,x_2) = \left(1 - e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1}\left(1 - e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2}\left(1 - e^{-\theta z-\gamma z^{\beta}}\right)^{\alpha_3} \tag{3}$$

where $z = \min(x_1,x_2)$.

Proof.

$$\begin{aligned} F_{BEMW}(x_1,x_2) &= P(X_1 \le x_1, X_2 \le x_2) \\ &= P(\max\{U_1,U_3\} \le x_1, \max\{U_2,U_3\} \le x_2) \\ &= P(U_1 \le x_1, U_2 \le x_2, U_3 \le \min(x_1,x_2)). \end{aligned}$$

Where, $U_i\ (i = 1,2,3)$ are independent random variables. Then, we obtain

$$F_{BEMW}(x_1,x_2) = P(U_1 \le x_1)P(U_2 \le x_2)P(U_3 \le \min(x_1,x_2))$$

$$\begin{aligned} &= F_{EMW}(x_1;\alpha_1,\beta,\theta,\gamma)F_{EMW}(x_2;\alpha_2,\beta,\theta,\gamma)F_{EMW}(z;\alpha_3,\beta,\theta,\gamma) \\ &= \left(1 - e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1}\left(1 - e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2}\left(1 - e^{-\theta z-\gamma z^{\beta}}\right)^{\alpha_3}. \end{aligned}$$

2.2 The Joint Probability Density Function

In this subsection, we study the joint probability density function of the random variables X_1 and X_2 in the following theorem.

Theorem 1. *If the joint cdf of (X_1,X_2) is as in (3) then, the joint pdf of (X_1,X_2) is given by*

$$f_{BEMW}(x_1,x_2) = \begin{cases} f_1(x_1,x_2) & if\ x_1 < x_2 \\ f_2(x_1,x_2) & if\ x_2 < x_1 \\ f_3(x) & .\ if\ x_1 = x_2 = x \end{cases}$$

where

$$\begin{aligned} f_1(x_1,x_2) &= f_{EMW}(x_1;\alpha_1+\alpha_3,\beta,\theta,\gamma)f_{EMW}(x_2;\alpha_2,\beta,\theta,\gamma) \\ &= (\alpha_1+\alpha_3)\alpha_2\left(\theta + \gamma\beta x_1^{\beta-1}\right)e^{-\theta x_1-\gamma x_1^{\beta}}\left(1 - e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1+\alpha_3-1} \\ &\quad \times\left(\theta + \gamma\beta x_2^{\beta-1}\right)e^{-\theta x_2-\gamma x_2^{\beta}}\left(1 - e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2-1} \end{aligned} \tag{4}$$

$$\begin{aligned} f_2(x_1,x_2) &= f_{EMW}(x_1;\alpha_1,\beta,\theta,\gamma)f_{EMW}(x_2;\alpha_2+\alpha_3,\beta,\theta,\gamma) \\ &= (\alpha_2+\alpha_3)\alpha_1\left(\theta + \gamma\beta x_1^{\beta-1}\right)e^{-\theta x_1-\gamma x_1^{\beta}}\left(1 - e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1-1} \\ &\quad \times\left(\theta + \gamma\beta x_2^{\beta-1}\right)e^{-\theta x_2-\gamma x_2^{\beta}}\left(1 - e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2+\alpha_3-1} \end{aligned} \tag{5}$$

$$\begin{aligned} f_3(x) &= \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} f_{EMW}(x;\alpha_1+\alpha_2+\alpha_3,\beta,\theta,\gamma) \\ &= \alpha_3\left(\theta+\gamma\beta x^{\beta-1}\right)e^{-\theta x-\gamma x^{\beta}}\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_1+\alpha_2+\alpha_3-1} \end{aligned} \tag{6}$$

*Proof.*Let us first assume that $x_1 < x_2$. Then, $F_{BEMW}(x_1,x_2)$ in (3) becomes

$$F_{BEMW}(x_1,x_2) = \left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1+\alpha_3}\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2}.$$

Then, upon differentiating this function w.r.t. x_1 and x_2 we obtain the expression of $f_1(x_1,x_2)$ gives in (4). By the same way we obtain $f_2(x_1,x_2)$ when $x_2 < x_1$. But $f_3(x)$ cannot be derived in a similar way. For this reason, we use the following identity to derive $f_3(x)$

$$\int_0^{\infty}\int_0^{x_2} f_1(x_1,x_2)\,dx_1dx_2 + \int_0^{\infty}\int_0^{x_1} f_2(x_1,x_2)\,dx_2dx_1 + \int_0^{\infty} f_3(x)\,dx = 1$$

let

$$I_1 = \int_0^{\infty}\int_0^{x_2} f_1(x_1,x_2)\,dx_1dx_2 \quad \text{and} \quad I_2 = \int_0^{\infty}\int_0^{x_1} f_2(x_1,x_2)\,dx_2dx_1$$

then

$$\begin{aligned} I_1 &= \int_0^{\infty}\int_0^{x_2}(\alpha_1+\alpha_3)\left(\theta+\gamma\beta x_1^{\beta-1}\right)e^{-\theta x_1-\gamma x_1^{\beta}}\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1+\alpha_3-1} \\ &\quad \times\alpha_2\left(\theta+\gamma\beta x_2^{\beta-1}\right)e^{-\theta x_2-\gamma x_2^{\beta}}\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2-1} dx_1dx_2 \\ &= \int_0^{\infty}\alpha_2\left(\theta+\gamma\beta x_2^{\beta-1}\right)e^{-\theta x_2-\gamma x_2^{\beta}}\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_1+\alpha_2+\alpha_3-1} dx_2. \end{aligned} \tag{7}$$

Similarly

$$I_2 = \int_0^{\infty}\alpha_1\left(\theta+\gamma\beta x_1^{\beta-1}\right)e^{-\theta x_1-\gamma x_1^{\beta}}\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1+\alpha_2+\alpha_3-1} dx_1. \tag{8}$$

From (7) and (8), we get

$$\int_0^{\infty} f_3(x)\,dx = 1 - I_1 - I_2$$

$$\begin{aligned} &= \int_0^{\infty}(\alpha_1+\alpha_2+\alpha_3)\left(\theta+\gamma\beta x^{\beta-1}\right)e^{-\theta x-\gamma x^{\beta}}\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_1+\alpha_2+\alpha_3-1} dx \\ &\quad -\int_0^{\infty}\alpha_2\left(\theta+\gamma\beta x^{\beta-1}\right)e^{-\theta x-\gamma x^{\beta}}\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_1+\alpha_2+\alpha_3-1} dx \\ &\quad -\int_0^{\infty}\alpha_1\left(\theta+\gamma\beta x^{\beta-1}\right)e^{-\theta x-\gamma x^{\beta}}\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_1+\alpha_2+\alpha_3-1} dx. \end{aligned}$$

Then

$$\begin{aligned} f_3(x) &= \alpha_3\left(\theta+\gamma\beta x^{\beta-1}\right)e^{-\theta x-\gamma x^{\beta}}\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_1+\alpha_2+\alpha_3-1} \\ &= \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3} f_{EMW}(x;\alpha_1+\alpha_2+\alpha_3,\beta,\theta,\gamma). \end{aligned}$$

This completes the proof of the theorem.

2.3 Marginal Probability Density Functions

The following theorem gives the marginal probability density functions of X_1 and X_2.

Theorem 2. *The marginal probability density functions of* X_i $(i=1,2)$ *is given by*

$$\begin{aligned} f_{X_i}(x_i) &= (\alpha_i+\alpha_3)\left(\theta+\gamma\beta x_i^{\beta-1}\right)e^{-\theta x_i-\gamma x_i^{\beta}}\left(1-e^{-\theta x_i-\gamma x_i^{\beta}}\right)^{\alpha_i+\alpha_3-1} \\ &= f_{EMW}(x_i;\alpha_i+\alpha_3,\beta,\theta,\gamma), \quad x_i>0,\ i=1,2. \end{aligned} \tag{9}$$

Proof. The marginal cumulative distribution function of X_i, say $F(x_i)$, as follows:

$$\begin{aligned} F(x_i) &= P(X_i \le x_i) \\ &= P(\max\{U_i,U_3\} \le x_i) \\ &= P(U_i \le x_i, U_3 \le x_i) \end{aligned}$$

since, the random variables U_i $(i=1,2)$ and U_3 are mutually independent, then

$$\begin{aligned} F(x_i) &= P(U_i \le x_i)\,P(U_3 \le x_i) \\ &= F_{EMW}(x_i;\alpha_i+\alpha_3,\beta,\theta,\gamma) \\ &= \left(1-e^{-\theta x_i-\gamma x_i^{\beta}}\right)^{\alpha_i+\alpha_3}. \end{aligned} \tag{10}$$

Differentiating w.r.t. x_i we obtain the formula given in (9).

2.4 Conditional Probability Density Functions

Given the marginal probability density functions of X_1 and X_2 we can now derive the conditional probability density functions as presented in the following theorem

Theorem 3. *The conditional probability density functions of* X_i, *given* $X_j=x_j$, $f(x_i|x_j)$, $i,j=1,2;\ i\ne j$, *is given by*

$$f_{X_i|X_j}(x_i|x_j)=\begin{cases} f^{(1)}_{X_i|X_j}(x_i|x_j) & if\ x_j<x_i, \\ f^{(2)}_{X_i|X_j}(x_i|x_j) & if\ x_i<x_j, \\ f^{(3)}_{X_i|X_j}(x_i|x_j) & if\ x_i=x_j=x, \end{cases}$$

where

$$f^{(1)}_{X_i|X_j}(x_i|x_j)=\frac{(\alpha_i+\alpha_3)\,\alpha_j\left(\theta+\gamma\beta x_i^{\beta-1}\right)e^{-\theta x_i-\gamma x_i^{\beta}}\left(1-e^{-\theta x_i-\gamma x_i^{\beta}}\right)^{\alpha_i+\alpha_3-1}}{(\alpha_j+\alpha_3)\left(1-e^{-\theta x_j-\gamma x_j^{\beta}}\right)^{\alpha_3}}$$

$$f^{(2)}_{X_i|X_j}(x_i|x_j)=\alpha_i\left(\theta+\gamma\beta x_i^{\beta-1}\right)e^{-\theta x_i-\gamma x_i^{\beta}}\left(1-e^{-\theta x_i-\gamma x_i^{\beta}}\right)^{\alpha_i-1}$$

$$f^{(3)}_{X_i|X_j}(x_i|x_j)=\frac{\alpha_3}{\alpha_i+\alpha_3}\left(1-e^{-\theta x_i-\gamma x_i^{\beta}}\right)^{\alpha_i}.$$

Proof. The proof follows immediately by substituting the joint probability density function of (X_1,X_2) given in (4), (5) and (6) and the marginal probability density function of X_i $(i=1,2)$ given in (9), using the relation

$$f_{X_i|X_j}(x_i|x_j)=\frac{f_{X_i,X_j}(x_i,x_j)}{f_{X_j}(x_j)},\ i=1,2.$$

3 The Marginal Expectation

In this section, the rth moments of X_i $(i=1,2)$ are computed.

Theorem 4. *The rth moment of X_i $(i=1,2)$ is given by*

$$E\left(X_i^r\right)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha_i+\alpha_3-1}{j}(-1)^{j+k}(\alpha_i+\alpha_3)\frac{((j+1)\gamma)^k}{k!}$$
$$\times\left(\theta\left(\frac{1/\theta}{(j+1)}\right)^{r+\beta k+1}\Gamma(r+\beta k+1)+\gamma\beta\left(\frac{1/\theta}{(j+1)}\right)^{r+\beta k+\beta}\Gamma(r+\beta k+\beta)\right) \tag{11}$$

Proof.

$$E\left(X_i^r\right)=\int_0^{\infty}x_i^r f_{X_i}(x_i)\,dx_i$$

$$=\int_0^{\infty}x_i^r(\alpha_i+\alpha_3)\left(\theta+\gamma\beta x_i^{\beta-1}\right)e^{-\theta x_i-\gamma x_i^{\beta}}\left(1-e^{-\theta x_i-\gamma x_i^{\beta}}\right)^{\alpha_i+\alpha_3-1}dx_i$$
$$=\sum_{j=0}^{\infty}\binom{\alpha_i+\alpha_3-1}{j}(-1)^j(\alpha_i+\alpha_3)\int_0^{\infty}x_i^r\left(\theta+\gamma\beta x_i^{\beta-1}\right)$$
$$\times e^{-(j+1)\left(\theta x_i+\gamma x_i^{\beta}\right)}dx_i$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha_i+\alpha_3-1}{j}(-1)^{j+k}(\alpha_i+\alpha_3)\frac{((j+1)\gamma)^k}{k!}\int_0^{\infty}x_i^{r+\beta k}$$
$$\times e^{-(j+1)\theta x_i}\left(\theta+\gamma\beta x_i^{\beta-1}\right)dx_i$$
$$=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{\alpha_i+\alpha_3-1}{j}(-1)^{j+k}\frac{(\alpha_i+\alpha_3)((j+1)\gamma)^k}{k!}\left(\left(\frac{1/\theta}{(j+1)}\right)^{r+\beta k+1}\right.$$
$$\left.\times\ \theta\Gamma(r+\beta k+1)+\gamma\beta\left(\frac{1/\theta}{(j+1)}\right)^{r+\beta k+\beta}\Gamma(r+\beta k+\beta)\right).$$

4 Reliability Studies

In this section, we compute the joint reliability function of (X_1,X_2), the joint hazard rate function of (X_1,X_2), the cdf of the random variable $U=\max\{X_1,\ X_2\}$ and the cdf of the random variable $V=\min\{X_1,X_2\}$.

4.1 Joint Survival Function

Theorem 5. *The joint reliability function of (X_1,X_2) is given by*

$$\overline{F}_{X_1,X_2}(x_1,x_2)=\begin{cases}\overline{F}_1(x_1,x_2) & if\ x_1<x_2\\ \overline{F}_2(x_1,x_2) & if\ x_2<x_1\\ \overline{F}_3(x_1,x_2) & if\ x_1=x_2=x\end{cases} \tag{12}$$

where

$$\overline{F}_1(x_1,x_2)=1-\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2+\alpha_3}-\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1+\alpha_3}$$
$$\times\left[1-\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2}\right]$$
$$\overline{F}_2(x_1,x_2)=1-\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1+\alpha_3}-\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2+\alpha_3}$$
$$\times\left[1-\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1}\right]$$

and

$$\overline{F}_3(x_1,x_2) = 1-\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_3}\left[\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_1}+\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_2}-\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_1+\alpha_2}\right].$$

*Proof.*The joint reliability function of (X_1,X_2) can be obtained by

$$\overline{F}_{X_1,X_2}(x_1,x_2) = 1-F_{X_1}(x_1)-F_{X_2}(x_2)+F_{X_1,X_2}(x_1,x_2). \tag{13}$$

Substituting from (3) and (10) in (13), we get

$$\overline{F}_{X_1,X_2}(x_1,x_2) = 1-\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1+\alpha_3}-\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2+\alpha_3}$$

$$+\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1}\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2}\left(1-e^{-\theta z-\gamma z^{\beta}}\right)^{\alpha_3}$$

where $z=\min(x_1,x_2)$.
First case: if $X_1<X_2$, then $Z=\min(X_1,X_2)=X_1$, hence,

$$\begin{aligned}\overline{F}_1(x_1,x_2) &= 1-\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1+\alpha_3}-\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2+\alpha_3}\\ &\quad+\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1}\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2}\times\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_3}\\ &= 1-\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2+\alpha_3}-\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1+\alpha_3}\times\left[1-\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2}\right].\end{aligned} \tag{14}$$

Second case: if $X_2<X_1$, then $Z=\min(X_1,X_2)=X_2$, hence,

$$\begin{aligned}\overline{F}_2(x_1,x_2) &= 1-\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1+\alpha_3}-\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2+\alpha_3}\\ &\quad+\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1}\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2}\times\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_3}\\ &= .1-\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1+\alpha_3}-\left(1-e^{-\theta x_2-\gamma x_2^{\beta}}\right)^{\alpha_2+\alpha_3}\times\left[1-\left(1-e^{-\theta x_1-\gamma x_1^{\beta}}\right)^{\alpha_1}\right].\end{aligned} \tag{15}$$

third case: if $X_1=X_2=X$, then $Z=X$,hence,

$$\overline{F}_3(x_1,x_2) = 1-\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_3}\left[\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_1}+\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_2}-\left(1-e^{-\theta x-\gamma x^{\beta}}\right)^{\alpha_1+\alpha_2}\right]. \tag{16}$$

From (14), (15) and (16), the proof is complete.

4.2 Joint Hazard Rate Function

Theorem 6.*The joint hazard rate function of (X_1,X_2) can be obtained by*

$$h_{X_1,X_2}(x_1,x_2) = \begin{cases} h_1(x_1,x_2) & if\ x_1<x_2\\ h_2(x_1,x_2) & if\ x_2<x_1\\ h_3(x_1,x_2) & if\ x_1=x_2=x\end{cases}$$

where

$$h_1(x_1,x_2) = \left((\alpha_1+\alpha_3)\left(\theta+\gamma\beta x_1^{\beta-1}\right)e^{-\theta x_1-\gamma x_1^\beta}\left(1-e^{-\theta x_1-\gamma x_1^\beta}\right)^{\alpha_1+\alpha_3-1}\right.$$
$$\left.\times\alpha_2\left(\theta+\gamma\beta x_2^{\beta-1}\right)e^{-\theta x_2-\gamma x_2^\beta}\left(1-e^{-\theta x_2-\gamma x_2^\beta}\right)^{\alpha_2-1}\right)$$
$$\div\left(1-\left(1-e^{-\theta x_2-\gamma x_2^\beta}\right)^{\alpha_2+\alpha_3}-\left(1-e^{-\theta x_1-\gamma x_1^\beta}\right)^{\alpha_1+\alpha_3}-\left(1-\left(1-e^{-\theta x_2-\gamma x_2^\beta}\right)^{\alpha_2}\right)\right)$$

$$h_2(x_1,x_2) = \left((\alpha_2+\alpha_3)\,\alpha_1\left(\theta+\gamma\beta x_1^{\beta-1}\right)e^{-\theta x_1-\gamma x_1^\beta}\left(1-e^{-\theta x_1-\gamma x_1^\beta}\right)^{\alpha_1-1}\right.$$
$$\left.\times\left(\theta+\gamma\beta x_2^{\beta-1}\right)e^{-\theta x_2-\gamma x_2^\beta}\left(1-e^{-\theta x_2-\gamma x_2^\beta}\right)^{\alpha_2+\alpha_3-1}\right)\div$$
$$\left(1-\left(1-e^{-\theta x_1-\gamma x_1^\beta}\right)^{\alpha_1+\alpha_3}-\left(1-e^{-\theta x_2-\gamma x_2^\beta}\right)^{\alpha_2+\alpha_3}\times\left(1-\left(1-e^{-\theta x_1-\gamma x_1^\beta}\right)^{\alpha_1}\right)\right)$$

and

$$h_3(x,x) = \left(\alpha_3\left(\theta+\gamma\beta x^{\beta-1}\right)e^{-\theta x-\gamma x^\beta}\left(1-e^{-\theta x-\gamma x^\beta}\right)^{\alpha_1+\alpha_2+\alpha_3-1}\right)$$
$$\div\left[1-\left(1-e^{-\theta x-\gamma x^\beta}\right)^{\alpha_3}\left(\left(1-e^{-\theta x-\gamma x^\beta}\right)^{\alpha_1}+\left(1-e^{-\theta x-\gamma x^\beta}\right)^{\alpha_2}-\left(1-e^{-\theta x-\gamma x^\beta}\right)^{\alpha_1+\alpha_2}\right)\right].$$

Proof. This can be easily deduced by using $h_{X_1,X_2}(x_1,x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{\overline{F}_{X_1,X_2}(x_1,x_2)}$.

Lemma 2. *The $cdfs$ of the random variables $U=\max\{X_1,X_2\}$ and $V=\min\{X_1,X_2\}$ are given by*

$$F_U(u) = \left(1-e^{-\theta u-\gamma u^\beta}\right)^{\alpha_1+\alpha_2+\alpha_3},$$
$$F_V(v) = \left(1-e^{-\theta v-\gamma v^\beta}\right)^{\alpha_1+\alpha_3}+\left(1-e^{-\theta v-\gamma v^\beta}\right)^{\alpha_2+\alpha_3}-\left(1-e^{-\theta v-\gamma v^\beta}\right)^{\alpha_1+\alpha_2+\alpha_3}.$$

Proof. The cdf of the random variable $U=\max\{X_1,X_2\}$ is

$$\begin{aligned}
F_U(u) &= P[U\le u]\\
&= P[\max\{X_1,X_2\}\le u]\\
&= P[X_1\le u,\ X_2\le u]\\
&= P[\max\{U_1,U_3\}\le u,\ \max\{U_2,U_3\}\le u]\\
&= P[U_1\le u, U_2\le u, U_3\le u]\\
&= P[U_1\le u]\,P[U_2\le u]\,P[U_3\le u]\\
&= F_{EMW}(u;\alpha_1)\,F_{EMW}(u;\alpha_2)\,F_{EMW}(u;\alpha_3)\\
&= \left(1-e^{-\theta u-\gamma u^\beta}\right)^{\alpha_1+\alpha_2+\alpha_3}.
\end{aligned}$$

The cdf of the random variables $V=\min\{X_1,X_2\}$ is given by

$$\begin{aligned}
F_V(v) &= P[V\le v]\\
&= P[\min\{X_1,X_2\}\le v]\\
&= 1-P[\min\{X_1,X_2\}>v]\\
&= 1-P[X_1>v,\ X_2>v]\\
&= 1-\overline{F}(v,v)\\
&= F_{X_1}(v)+F_{X_2}(v)-F_{X_1,X_2}(v,v)\\
&= \left(1-e^{-\theta v-\gamma v^\beta}\right)^{\alpha_1+\alpha_3}+\left(1-e^{-\theta v-\gamma v^\beta}\right)^{\alpha_2+\alpha_3}-\left(1-e^{-\theta v-\gamma v^\beta}\right)^{\alpha_1+\alpha_2+\alpha_3}.
\end{aligned}$$

5 Maximum Likelihood Estimators

Kundu and Gupta [7] used the method of maximum likelihood to estimate the unknown parameters of the bivariate generalized exponential distribution. In the same way we use the method of maximum likelihood to estimate the unknown parameters of the *BEMW* distribution.

Suppose $((x_{11},x_{21}),\ ...,\ (x_{1n},x_{2n}))$ is a random sample from *BEMW* distribution.Consider the following notation

$I_1 = \{i;\ x_{1i} < x_{2i}\},\ I_2 = \{i;\ x_{1i} > x_{2i}\},\ I_3 = \{i;\ x_{1i} = x_{2i} = x_i\},\ I = I_1 \cup I_2 \cup I_3,\ |I_1| = n_1,\ |I_2| = n_2,\ |I_3| = n_3$, and $n_1 + n_2 + n_3 = n$.

The likelihood function of the sample of size n is given by:

$$l(\alpha_1,\alpha_2,\alpha_3,\beta,\theta,\gamma) = \prod_{i=1}^{n_1} f_1(x_{1i},x_{2i}) \prod_{i=1}^{n_2} f_2(x_{1i},x_{2i}) \prod_{i=1}^{n_3} f_3(x_i)$$

The log-likelihood function can be expressed as

$$L(\alpha_1,\alpha_2,\alpha_3,\beta,\theta,\gamma) = \ln l(\alpha_1,\alpha_2,\alpha_3,\beta,\theta,\gamma)$$

$$\begin{aligned}
&= n_1 \ln(\alpha_1+\alpha_3) + n_1 \ln \alpha_2 + \sum_{i=1}^{n_1} \ln\left(\theta + \gamma\beta x_{1i}^{\beta-1}\right) - \sum_{i=1}^{n_1}\left(\theta x_{1i} + \gamma x_{1i}^{\beta}\right) \\
&\quad + (\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1} \ln\left(1 - e^{-\theta x_{1i} - \gamma x_{1i}^{\beta}}\right) + \sum_{i=1}^{n_1} \ln\left(\theta + \gamma\beta x_{2i}^{\beta-1}\right) \\
&\quad + \sum_{i=1}^{n_1} \ln\left(\theta + \gamma\beta x_{2i}^{\beta-1}\right) - \sum_{i=1}^{n_1}\left(\theta x_{2i} + \gamma x_{2i}^{\beta}\right) + (\alpha_2 - 1) \\
&\quad \times \sum_{i=1}^{n_1} \ln\left(1 - e^{-\theta x_{2i} - \gamma x_{2i}^{\beta}}\right) + n_2 \ln \alpha_1 + \sum_{i=1}^{n_2} \ln\left(\theta + \gamma\beta x_{1i}^{\beta-1}\right) \\
&\quad - \sum_{i=1}^{n_2}\left(\theta x_{1i} + \gamma x_{1i}^{\beta}\right) + (\alpha_1 - 1)\sum_{i=1}^{n_2} \ln\left(1 - e^{-\theta x_{1i} - \gamma x_{1i}^{\beta}}\right) \\
&\quad + n_2 \ln(\alpha_2+\alpha_3) + \sum_{i=1}^{n_2} \ln\left(\theta + \gamma\beta x_{2i}^{\beta-1}\right) - \sum_{i=1}^{n_2}\left(\theta x_{2i} + \gamma x_{2i}^{\beta}\right) \\
&\quad + (\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2} \ln\left(1 - e^{-\theta x_{2i} - \gamma x_{2i}^{\beta}}\right) + n_3 \ln \alpha_3 - \sum_{i=1}^{n_3}\left(\theta x_i + \gamma x_i^{\beta}\right) \\
&\quad + \sum_{i=1}^{n_3} \ln\left(\theta + \gamma\beta x_i^{\beta-1}\right) + (\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3} \ln\left(1 - e^{-\theta x_i - \gamma x_i^{\beta}}\right).
\end{aligned}$$

Differentiating the log-likelihood with respect to $\alpha_1, \alpha_2, \alpha_3, \beta, \theta$ and γ respectively, and setting the results equal to zero, we have

$$\begin{aligned}
\frac{\partial L}{\partial \alpha_1} &= \frac{n_1}{\alpha_1+\alpha_3} + \sum_{i=1}^{n_1} \ln\left(1 - e^{-\theta x_{1i} - \gamma x_{1i}^{\beta}}\right) + \frac{n_2}{\alpha_1} \\
&\quad + \sum_{i=1}^{n_2} \ln\left(1 - e^{-\theta x_{1i} - \gamma x_{1i}^{\beta}}\right) + \sum_{i=1}^{n_3} \ln\left(1 - e^{-\theta x_i - \gamma x_i^{\beta}}\right)
\end{aligned} \tag{17}$$

$$\begin{aligned}
\frac{\partial L}{\partial \alpha_2} &= \frac{n_1}{\alpha_2} + \sum_{i=1}^{n_1} \ln\left(1 - e^{-\theta x_{2i} - \gamma x_{2i}^{\beta}}\right) + \frac{n_2}{\alpha_2+\alpha_3} \\
&\quad + \sum_{i=1}^{n_2} \ln\left(1 - e^{-\theta x_{2i} - \gamma x_{2i}^{\beta}}\right) + \sum_{i=1}^{n_3} \ln\left(1 - e^{-\theta x_i - \gamma x_i^{\beta}}\right)
\end{aligned} \tag{18}$$

$$\begin{aligned}
\frac{\partial L}{\partial \alpha_3} &= \frac{n_1}{\alpha_1+\alpha_3} + \sum_{i=1}^{n_1} \ln\left(1 - e^{-\theta x_{1i} - \gamma x_{1i}^{\beta}}\right) + \frac{n_2}{\alpha_2+\alpha_3} \\
&\quad + \sum_{i=1}^{n_2} \ln\left(1 - e^{-\theta x_{2i} - \gamma x_{2i}^{\beta}}\right) + \frac{n_3}{\alpha_3} + \sum_{i=1}^{n_3} \ln\left(1 - e^{-\theta x_i - \gamma x_i^{\beta}}\right)
\end{aligned} \tag{19}$$

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} = & \sum_{i=1}^{n_1} \frac{1}{\theta+\gamma\beta x_{1i}^{\beta-1}} - \sum_{i=1}^{n_1} x_{1i} + (\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1} \frac{x_{1i}e^{-\theta x_{1i}-\gamma x_{1i}^{\beta}}}{1-e^{-\theta x_{1i}-\gamma x_{1i}^{\beta}}} \\
& + \sum_{i=1}^{n_1} \frac{1}{\theta+\gamma\beta x_{2i}^{\beta-1}} - \sum_{i=1}^{n_1} x_{2i} + (\alpha_2-1)\sum_{i=1}^{n_1} \frac{x_{2i}e^{-\theta x_{2i}-\gamma x_{2i}^{\beta}}}{1-e^{-\theta x_{2i}-\gamma x_{2i}^{\beta}}} \\
& + \sum_{i=1}^{n_2} \frac{1}{\theta+\gamma\beta x_{1i}^{\beta-1}} - \sum_{i=1}^{n_2} x_{1i} + (\alpha_1-1)\sum_{i=1}^{n_2} \frac{x_{1i}e^{-\theta x_{1i}-\gamma x_{1i}^{\beta}}}{1-e^{-\theta x_{1i}-\gamma x_{1i}^{\beta}}} \\
& + \sum_{i=1}^{n_2} \frac{1}{\theta+\gamma\beta x_{2i}^{\beta-1}} - \sum_{i=1}^{n_2} x_{2i} + (\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2} \frac{x_{2i}e^{-\theta x_{2i}-\gamma x_{2i}^{\beta}}}{1-e^{-\theta x_{2i}-\gamma x_{2i}^{\beta}}} \\
& + \sum_{i=1}^{n_3} \frac{1}{\theta+\gamma\beta x_{i}^{\beta-1}} - \sum_{i=1}^{n_3} x_{i} + (\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3} \frac{x_{i}e^{-\theta x_{i}-\gamma x_{i}^{\beta}}}{1-e^{-\theta x_{i}-\gamma x_{i}^{\beta}}}
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \beta} = & \sum_{i=1}^{n_1} \frac{\gamma x_{1i}^{\beta-1}+\gamma\beta x_{1i}^{\beta-1}\ln(x_{1i})}{\theta+\gamma\beta x_{1i}^{\beta-1}} - \sum_{i=1}^{n_1} \gamma x_{1i}^{\beta}\ln(x_{1i}) + (\alpha_1+\alpha_3-1) \\
& \times \sum_{i=1}^{n_1} \frac{\gamma x_{1i}^{\beta}\ln(x_{1i})e^{-\theta x_{1i}-\gamma x_{1i}^{\beta}}}{1-e^{-\theta x_{1i}-\gamma x_{1i}^{\beta}}} + \sum_{i=1}^{n_1} \frac{\gamma x_{2i}^{\beta-1}+\gamma\beta x_{2i}^{\beta-1}\ln(x_{2i})}{\theta+\gamma\beta x_{2i}^{\beta-1}} \\
& - \sum_{i=1}^{n_1} \gamma x_{2i}^{\beta}\ln(x_{2i}) + (\alpha_2-1)\sum_{i=1}^{n_1} \frac{\gamma x_{2i}^{\beta}\ln(x_{2i})e^{-\theta x_{2i}-\gamma x_{2i}^{\beta}}}{1-e^{-\theta x_{2i}-\gamma x_{2i}^{\beta}}} \\
& + \sum_{i=1}^{n_2} \frac{\gamma x_{1i}^{\beta-1}+\gamma\beta x_{1i}^{\beta-1}\ln(x_{1i})}{\theta+\gamma\beta x_{1i}^{\beta-1}} - \sum_{i=1}^{n_2} \gamma x_{1i}^{\beta}\ln(x_{1i}) + (\alpha_1-1) \\
& \times \sum_{i=1}^{n_2} \frac{\gamma x_{1i}^{\beta}\ln(x_{1i})e^{-\theta x_{1i}-\gamma x_{1i}^{\beta}}}{1-e^{-\theta x_{1i}-\gamma x_{1i}^{\beta}}} + \sum_{i=1}^{n_2} \frac{\gamma x_{2i}^{\beta-1}+\gamma\beta x_{2i}^{\beta-1}\ln(x_{2i})}{\theta+\gamma\beta x_{2i}^{\beta-1}} \\
& - \sum_{i=1}^{n_2} \gamma x_{2i}^{\beta}\ln(x_{2i}) + (\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2} \frac{\gamma x_{2i}^{\beta}\ln(x_{2i})e^{-\theta x_{2i}-\gamma x_{2i}^{\beta}}}{1-e^{-\theta x_{2i}-\gamma x_{2i}^{\beta}}} \\
& + \sum_{i=1}^{n_3} \frac{\gamma x_{i}^{\beta-1}+\gamma\beta x_{i}^{\beta-1}\ln(x_{i})}{\theta+\gamma\beta x_{i}^{\beta-1}} - \sum_{i=1}^{n_3} \gamma x_{i}^{\beta}\ln(x_{i}) \\
& + (\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3} \frac{\gamma x_{i}^{\beta}\ln(x_{i})e^{-\theta x_{i}-\gamma x_{i}^{\beta}}}{1-e^{-\theta x_{i}-\gamma x_{i}^{\beta}}}
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \gamma} = & \sum_{i=1}^{n_1} \frac{\beta x_{1i}^{\beta-1}}{\theta+\gamma\beta x_{1i}^{\beta-1}} - \sum_{i=1}^{n_1} x_{1i}^{\beta} + (\alpha_1+\alpha_3-1)\sum_{i=1}^{n_1} \frac{x_{1i}^{\beta}e^{-\theta x_{1i}-\gamma x_{1i}^{\beta}}}{1-e^{-\theta x_{1i}-\gamma x_{1i}^{\beta}}} \\
& + \sum_{i=1}^{n_1} \frac{\beta x_{2i}^{\beta-1}}{\theta+\gamma\beta x_{2i}^{\beta-1}} - \sum_{i=1}^{n_1} x_{2i}^{\beta} + (\alpha_2-1)\sum_{i=1}^{n_1} \frac{x_{2i}^{\beta}e^{-\theta x_{2i}-\gamma x_{2i}^{\beta}}}{1-e^{-\theta x_{2i}-\gamma x_{2i}^{\beta}}} \\
& + \sum_{i=1}^{n_2} \frac{\beta x_{1i}^{\beta-1}}{\theta+\gamma\beta x_{1i}^{\beta-1}} - \sum_{i=1}^{n_2} x_{1i}^{\beta} + (\alpha_1-1)\sum_{i=1}^{n_2} \frac{x_{1i}^{\beta}e^{-\theta x_{1i}-\gamma x_{1i}^{\beta}}}{1-e^{-\theta x_{1i}-\gamma x_{1i}^{\beta}}} \\
& + \sum_{i=1}^{n_2} \frac{\beta x_{2i}^{\beta-1}}{\theta+\gamma\beta x_{2i}^{\beta-1}} - \sum_{i=1}^{n_2} x_{2i}^{\beta} + (\alpha_2+\alpha_3-1)\sum_{i=1}^{n_2} \frac{x_{2i}^{\beta}e^{-\theta x_{2i}-\gamma x_{2i}^{\beta}}}{1-e^{-\theta x_{2i}-\gamma x_{2i}^{\beta}}} \\
& + \sum_{i=1}^{n_3} \frac{\beta x_{i}^{\beta-1}}{\theta+\gamma\beta x_{i}^{\beta-1}} - \sum_{i=1}^{n_3} x_{i}^{\beta} + (\alpha_1+\alpha_2+\alpha_3-1)\sum_{i=1}^{n_3} \frac{x_{i}^{\beta}e^{-\theta x_{i}-\gamma x_{i}^{\beta}}}{1-e^{-\theta x_{i}-\gamma x_{i}^{\beta}}}
\end{aligned} \tag{22}
$$

The maximum likelihood estimates $\widehat{\alpha}_1$, $\widehat{\alpha}_2$, $\widehat{\alpha}_3$, $\widehat{\theta}$, $\widehat{\beta}$ and $\widehat{\gamma}$ of the unknown parameters α_1, α_2, α_3, θ, β and γ respectively, are obtained by solving Equations (17) - (22).

6 Data Analysis

In this section, a real data set is used to compare the fits of the Bivariate Generalized Gompertz (BGG) distribution, Bivariate Exponentiated Modified Weibull Extension (BEMWE) distribution, Bivariate Exponentiated Generalized Weibull Gompertz (BEGWG) distribution and Bivariate Exponentiated Modified Weibull (BEMW) distribution. The data set (see Table 1) was first analyzed by Csorgo and Welsh [15] and represents the American Football (National Football League) League data and they are obtained from the matches played on three consecutive weekends in 1986. It is a bivariate data set, and the variables X_1 and X_2 are as follows: X_1 represents the $'$game time$'$ to the first points scored by kicking the ball between goal posts, and represents the $'$game time$'$ to the first points scored by moving the ball into the end zone. These times are of interest to a casual spectator who wants to know how long one has to wait to watch a touchdown or to a spectator who is interested only at the beginning stages of a game. Also all the data points are divided by 100 just for computational purposes.

The variables X_1 and X_2 have the following structure: (i) $X_1 < X_2$ means that the first score is a field goal, (ii) $X_1 = X_2$ means the first score is a converted touchdown, (iii) $X_1 > X_2$ means the first score is an unconverted touchdown or safety.

Table1. American Football League (NFL) data

X_1	X_2	X_1	X_2	X_1	X_2	X_1	X_2
2.05	3.98	8.53	14.57	2.90	2.90	1.38	1.38
9.05	9.05	31.13	49.88	7.02	7.02	10.53	10.53
0.85	0.85	14.58	20.57	6.42	6.42	12.13	12.13
3.43	3.43	5.78	25.98	8.98	8.98	14.58	14.58
7.78	7.78	13.80	49.75	10.15	10.15	11.82	11.82
10.57	14.28	7.25	7.25	8.87	8.87	5.52	11.27
7.05	7.05	4.25	4.25	10.40	10.25	19.65	10.70
2.58	2.58	1.65	1.65	2.98	2.98	17.83	17.83
7.23	9.68	6.42	15.08	3.88	6.43	10.85	38.07
6.85	34.58	4.22	9.48	0.75	0.75		
32.45	42.35	15.53	15.33	11.63	17.37		

The required numerical evaluations are carried out using the Package of Mathcad software. Table 2 provide the MLEs of the model parameters.The model selection is carried out using the *AIC* (Akaike information criterion) and the *CAIC* (consistent Akaike information criteria):

$$AIC = -2L + 2q,\ CAIC = -2L + \frac{2qn}{n-q-1}.$$

Where L denotes the log-likelihood function evaluated at the maximum likelihood estimates, q is the number of parameters and n is the sample size.

Table 2: MLEs for American Football League (NFL) data

Model	MLEs
BEGWGD $(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_3, 0.1, 0.2, 0.2, 0.5)$	$\widehat{\alpha}_1 = 0.0323, \widehat{\alpha}_2 = 0.186$ $\widehat{\alpha}_3 = 0.406$
BGGD $\left(\widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\gamma}_3, \widehat{\lambda}, 0.1\right)$	$\widehat{\gamma}_1 = 0.024, \widehat{\gamma}_2 = 0.150$ $\widehat{\gamma}_3 = 0.310, \widehat{\lambda} = 0.0044$
BEMWED $\left(\widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\gamma}_3, \widehat{\lambda}, 0.1, 0.42\right)$	$\widehat{\gamma}_1 = 0.212, \widehat{\gamma}_2 = 1.315$ $\widehat{\gamma}_3 = 2.645, \widehat{\lambda} = 0.096$
BEMWD $\left(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_3, \widehat{\theta}, \widehat{\beta}, \widehat{\gamma}\right)$	$\widehat{\alpha}_1 = 0.314, \widehat{\alpha}_2 = 1.945$ $\widehat{\alpha}_3 = 3.923, \widehat{\theta} = 0.061$ $\widehat{\beta} = 0.386, \widehat{\gamma} = 0.594$

Table 3: The statistics L, AIC and CAIC for American Football League (NFL) data

Model	L	AIC	CAIC
BEGWGD $(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_3, 0.1, 0.2, 0.2, 0.5)$	-354.03	714.06	714.69
BGGD $\left(\widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\gamma}_3, \widehat{\lambda}, 0.1\right)$	-260.5	529.0	530.03
BEMWED $\left(\widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\gamma}_3, \widehat{\lambda}, 0.1, 0.42\right)$	-239.86	487.36	488.39
BEMWD $\left(\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_3, \widehat{\theta}, \widehat{\beta}, \widehat{\gamma}\right)$	-239.442	490.88	493.284

Since the value of $(-L)$ (see Table 3) is smaller for the BEMW distribution compared with those values of the other models and the values ofAIC and $CAIC$ (see Table 3) are smaller for the BEMW distribution compared with those values of the other models except BEMWED , then the introduced distribution seems to be a very competitive model to these data.

The profiles of the log-likelihood function of α_1, α_2, α_3, θ, β and γ for American Football League (NFL) data are plotted in Fig. 1, Fig. 2, Fig. 3, Fig. 4, Fig. 5. and Fig. 6. respectively. From the plots of the profiles of the log-likelihood function of α_1, α_2, α_3, θ, β and γ ,we observe that the likelihood equations have a unique solution.

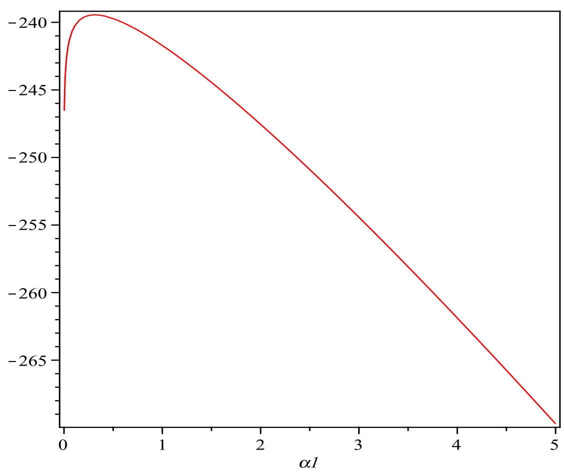

Fig. 1: The profile of the log-likelihood function of α_1 .American Football League (NFL) data.

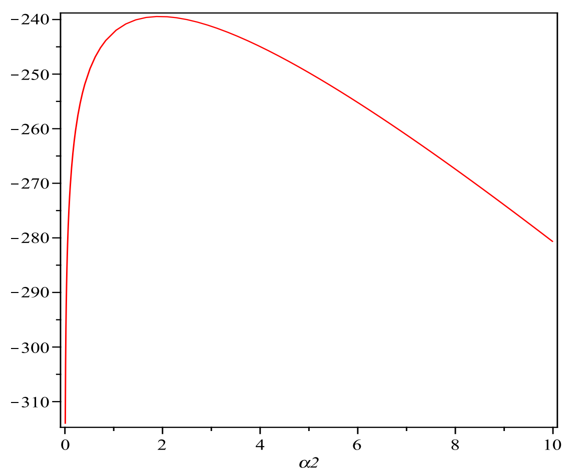

Fig. 2: The profile of the log-likelihood function of α_2 .American Football League (NFL) data.

7 Conclusions

In this paper, we proposed a new bivariate exponentiated modified Weibull (BEMW) distribution, whose marginals are EMW distributions. Some statistical properties of this distribution have been studied and discussed. The maximum likelihood estimates of the parameters are derived. A real data set is analyzed using the new distribution, Bivariate Generalized Gompertz distribution, Bivariate Exponentiated Modified Weibull Extension distribution, Bivariate Exponentiated Generalized Weibull Gompertz distribution. Based on the comparisons between all these models, we conclude that, the introduced model is highly competitive in the sense of fitting this real data set.

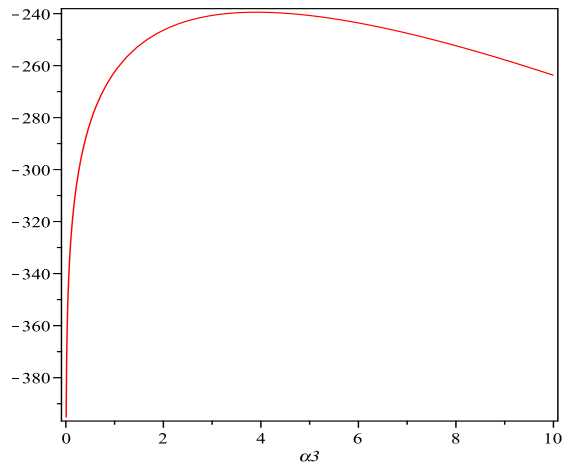

Fig. 3: The profile of the log-likelihood function of α_3 .American Football League (NFL) data.

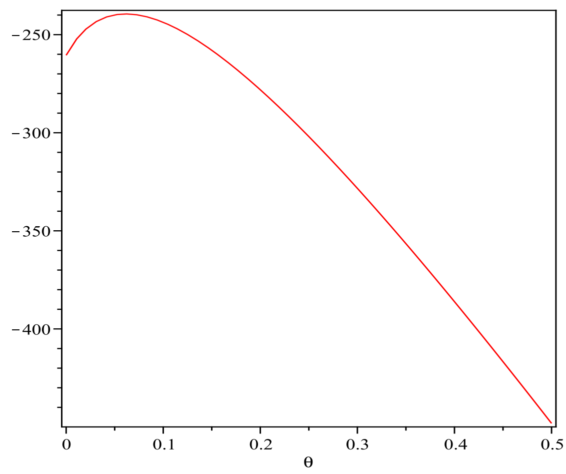

Fig. 4: The profile of the log-likelihood function of θ American Football League (NFL) data.

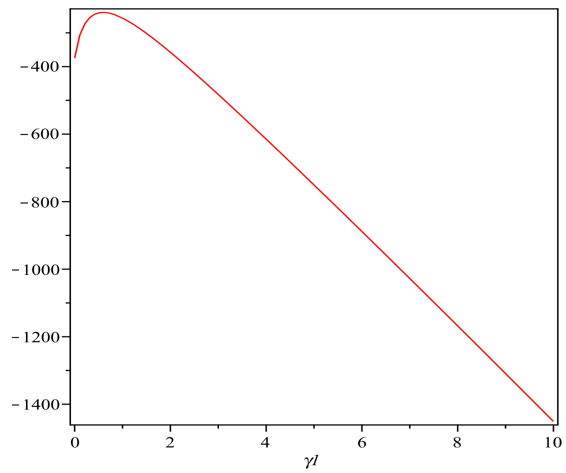

Fig. 5: The profile of the log-likelihood function of γ .American Football League (NFL) data.

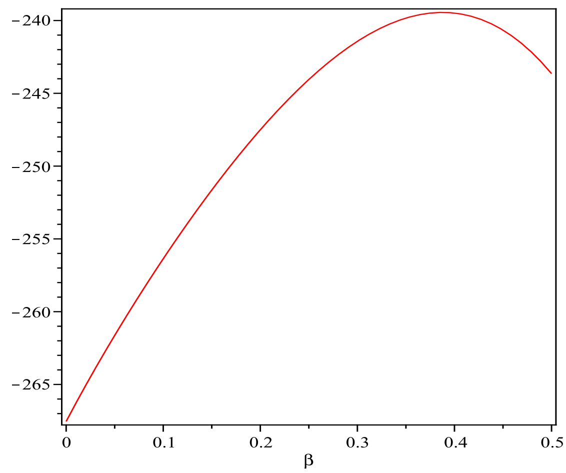

Fig. 6: The profile of the log-likelihood function of β.American Football League (NFL) data.

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H. S. Shahen is a PhD. student at Department of Mathematics, Faculty of Science, Mansoura University, Egypt. She received his master’s degree in Statistics and Computer Science from Mansoura University, Egypt in 2015. She worked as a teaching assistant and an assistant lecturer for eight years at Mansoura University. Her research interests include probability distributions and their application in data analysis, reliability and mathematical statistics.

A. H. El-Bassiouny is a Professor at the Department of Mathematics, Faculty of Science, Mansoura University , Mansoura, Egypt. He received his Bachelor’s degree and master’s from Mansoura University , Mansoura, in 1977 and 1981, respectively, and PhD degrees from University of , Russia, in 1986. Prof. El-Bassiouny was the head of the Department of Mathematics from 2007 to 2011 and dean of the Faculty of Science from 2011 to 2015 in Mansoura University , Mansoura, Egypt. Prof. El-Bassiouny’s main research interests are in probability distributions and their application in data analysis. He is awarded with numerous recognitions for his pioneering studies on distributions.

Mohamed Abouhawwash is a postdoctoral research associate for Department of Electrical and Computer Engineering, College of Engineering, Michigan State University, East Lansing, MI 48824, USA. He is working as assistant professor at Department of Mathematics, Faculty of Science, Mansoura University, Egypt, from where he received his masters and PhD degrees in Statistics and Computer Science in 2011, 2015. He is bestowed with the best masters and PhD thesis awards from Mansoura University in 2012, 2018 respectively. His research interests include evolutionary algorithms, Mathematical optimization, machine learning, probability distributions and their application in data analysis, reliability and mathematical statistics. He is referee and editor of mathematical and evolutionary algorithms journals.