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Solitary Waves, Shock Waves and Singular Solitons of the Generalized Ostrovsky-Benjamin-Bona-Mahoney Equation

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Abstract: This paper obtains the solitary wave, shock wave as well as singular soliton solutions to the generalized Ostrovsky-Benjamin-Bona-Mahoney (gO-BBM) equation. The ansatz method is applied to obtain the solutions. Several constraint conditions for the parameters are derived that establish the existence of the soliton solutions.

1 Introduction

The theory of solitons plays a dominant role in many nonlinear dynamical systems [1,2,3,4,5]. Of particular interest is in the field of nonlinear optical comminications systems. Although the perfect material is currently not available, optical nonlinear media with fast response times and refractive indices that depend on intensity would be ideally suited to modulate the power and phase of optical waves for all-optical devices. These all-optical devices include fibre optic sensors, routers and large-scale optical integrated circuits [6,7].

For other example, they also appear in shallow water wave flow, plasma physics, nuclear physics, oceanography, theoretical physics and among others [8,9,10,11,12,13].

The dynamics of solitons are governed by several nonlinear evolution equations (NLEEs) that are studied in applied mathematics. This paper addresses one such NLEE that descibes the soliton dynamics due to ocean currents. It is the generalized Ostrovsky-Benjamin-Bona-Mahoney (gO-BBM) equation. The Ostrovsky equation arises in the study of geophysical fluids while the Benjamin-Bona-Mahoney (BBM)

equation is studied in the context of shallow water wave dynamics [3,14,15,16,17]. This equation models the dynamics that is much closer to realistic situations.

The aim of this paper is to look for solitons and shock wave solutions to this equation. This is accomplished by using the ansatz method.

2 Governing Equation

The gO-BBM equation is given by [18,19]

$$\left[u_t + u_x - \alpha \left(u^2 \right)_x - \beta u_{xxt} \right]_x = \gamma \left(u + u^2 \right), \qquad (1)$$

where α, β and γ are constants. This is a model equation for the motion of ocean currents. In this paper the gO-BBM equation with full nonlinearity is studied, namely

$$\left[\left(u^l \right)_t + k^2 \left(u^n \right)_x + a \left(u^{2n} \right)_x + b \left(u^n \right)_{xxt} \right]_x = c_1 u^n + c_2 u^{2n} \qquad (2)$$

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where k, a, b, c_1 and c_2 are all real constants, while l, $n \in Z^+$. For the special case when $l = n = 1$ the gO-BBM equation collapses to the regular O-BBM (1).

Generally (2) is not integrable by the classical method of Inverse Scattering Transform [9]. It is always useful and desirable to construct exact analytical solutions (in particular soliton solutions) for the understanding of nonlinear physical phenomena. Based on the exact solutions, we can directly and accurately analyze the properties of propagating waves in nonlinear physical systems. In view of the enormous success of the solitary wave ansatz method for solving many NLEEs with constant and variable coefficients, we use it in to find the exact solitary waves, shock waves and singular soliton solutions to (2), for any positive integers n and l.

3 Solitons and Shock Waves

This section will focus on extraction the soliton solutions to the gO-BBM equation. There are three types of solitons that will be studied in this section. They are solitary waves, shock waves and singular solitons.

3.1 Solitary Waves

In order to solve (2), the starting hypothesis is given by [18,19]

$$u(x,t) = \frac{A}{\cosh^p \tau}, \quad p > 0, \tag{3}$$

where

$$\tau = B(x - vt). \tag{4}$$

Here, A is the amplitude of the soliton while v is the speed and B is the inverse width. The exponent p is unknown at this point and its value will be determined during the process of deriving the solution of this equation. Substituting (3) into (2) leads to

$$\frac{A^l p^2 l^2 B^2 v}{\cosh^{pl} \tau} - \frac{A^l B^2 v p l (pl+1)}{\cosh^{pl+2} \tau} + \frac{bA^n p^4 n^4 B^4 v}{\cosh^{pn} \tau}$$
$$- \frac{k^2 p^2 n^2 A^n B^2}{\cosh^{pn} \tau} + \frac{k^2 pn(pn+1)A^n B^2}{\cosh^{pn+2} \tau}$$
$$- \frac{4an^2 p^2 A^{2n} B^2}{\cosh^{2np} \tau} + \frac{2anp(2np+1)A^{2n} B^2}{\cosh^{2np+2} \tau}$$
$$- \frac{bA^n B^4 pn(pn+1)v \left\{ p^2 n^2 + (pn+2)^2 \right\}}{\cosh^{pn+2} \tau}$$
$$+ \frac{bA^n B^4 vpn(pn+1)(pn+2)(pn+3)}{\cosh^{pn+4} \tau}$$
$$+ \frac{c_1 A^n}{\cosh^{pn} \tau} + \frac{c_2 A^{2n}}{\cosh^{2np} \tau} = 0. \tag{5}$$

From (5), equating the exponents $2np + 2$ and $pn + 4$ gives

$$p = \frac{2}{n}, \tag{6}$$

which is also obtained by equating the exponents $2np$ and $pn + 2$. Again, from (5), equating the exponent of $\cosh^{pl} \tau$ and $\cosh^{pn} \tau$ leads to $l = n$. With this condition in place (2) is modified to

$$\left[(u^n)_t + k^2 (u^n)_x + a (u^{2n})_x + b (u^n)_{xxt} \right]_x = c_1 u^n + c_2 u^{2n} \tag{7}$$

Next, equating the exponents and the coefficients of like powers of cosh function leads to

$$4B^2(k^2 - v - 4bvB^2) - c_1 = 0, \tag{8}$$
$$6B^2(k^2 - v - 20bvB^2) + (c_2 - 16aB^2)A^n = 0, \tag{9}$$
$$aA^n + 6bvB^2 = 0. \tag{10}$$

Solving this system yields

$$A = \left(-\frac{3c_1}{2c_2} \right)^{\frac{1}{n}} \tag{11}$$

$$B = \frac{1}{2} \sqrt{\frac{c_1 (bc_2 + a)}{b (k^2 c_2 - ac_1)}} \tag{12}$$

$$v = \frac{a(k^2 c_2 - ac_1)}{c_2 (bc_2 + a)}. \tag{13}$$

Equation (11) shows that solitons will exist as long as

$$c_1 c_2 < 0, \tag{14}$$

if n is an even integer. However, if n is an odd integer there is no such restriction. Also, from (12) the following restriction is obtained

$$c_1 b (bc_2 + a) (k^2 c_2 - ac_1) > 0. \tag{15}$$

Thus, the solitary wave solution (valid for any positive integer n) to the gO-BBM equation with full nonlinearity (2) is given by

$$u(x,t) = \left(-\frac{3c_1}{2c_2} \right)^{\frac{1}{n}} \times$$
$$\operatorname{sech}^{\frac{2}{n}} \left[\frac{1}{2} \sqrt{\frac{c_1 (bc_2 + a)}{b (k^2 c_2 - ac_1)}} \left(x - \frac{a (k^2 c_2 - ac_1)}{c_2 (bc_2 + a)} t \right) \right].$$

3.2 Shock waves

Let us now calculate the shock wave solution of the gO-BBM equation with full nonlinearity (2). It is assumed that the shock wave takes the form [12]

$$u(x,t) = A \tanh^p \tau, \quad p > 0, \tag{16}$$

where τ is defined above and v is the speed of the shock wave. The only difference is that, in this case, the parameters A and B are non-zero free parameters. For shock waves, A is a dialation parameter (amplitude) while

B is a measure of steepness of the shock waves. Substituting (16) into (2) yields

$$vplA^lB^2\left\{(pl-1)\tanh^{pl-2}\tau+2pl\tanh^{pl}\tau\right\}$$
$$+vplA^lB^2(pl+1)\tanh^{pl+2}\tau-k^2pnA^nB^2(pn+1)\tanh^{pn+2}\tau$$
$$-k^2pnA^nB^2\left\{(pn-1)\tanh^{pn-2}\tau-2pn\tanh^{pn}\tau\right\}$$
$$-2apnA^{2n}B^2\left\{(2pn-1)\tanh^{2pn-2}\tau-4pn\tanh^{2pn}\tau\right\}$$
$$-2apnA^{2n}B^2(2pn+1)\tanh^{2pn+2}\tau$$
$$+bvpnA^nB^4\left\{(pn-1)(pn-2)(pn-3)\tanh^{pn-4}\tau\right.$$
$$-(pn+1)(pn+2)(pn+3)\tanh^{pn+4}\tau$$
$$+2\left\{p^2n^2+(pn-2)^2\right\}\left\{pn-1\right\}\tanh^{pn-2}\tau$$
$$+2\left\{p^2n^2+(pn+2)^2\right\}\left\{pn+1\right\}\tanh^{pn+2}\tau$$
$$-\left\{4p^3n^3+(pn-1)^2(pn-2)\right\}\tanh^{pn}\tau$$
$$\left.-(pn+1)^2(pn+2)\tanh^{pn}\tau\right\}$$
$$+c_1A^n\tanh^{pn}\tau+c_2A^{2n}\tanh^{2pn}\tau=0. \qquad (17)$$

From (17), equating the exponents $2np+2$ and $pn+4$ gives

$$p=\frac{2}{n}.$$

This is also arrrived at by equating the exponents of the pairs $(2np,\,pn+2)$ or $(2np-2,\,pn)$. Further, equating the exponent of $\tanh^{pl}\tau$ and $\tanh^{pn}\tau$ implies $l=n$. Now from (17) the linearly independent functions are $\tanh^{p+j}\tau$ ($j=0,\pm2,\pm4$) and setting their respective coefficients to zero leads to the following set of algebraic equations

$$k^2-v+8bvB^2=0, \qquad (18)$$
$$6B^2(k^2-v+40bvB^2)-(c_2+32aB^2)A^n=0, \qquad (19)$$
$$aA^n-6bvB^2=0, \qquad (20)$$
$$4B^2(3aA^n+2v-2k^2-34bvB^2)-c_1=0. \qquad (21)$$

Solving this system gives

$$v=\frac{k^2}{1-8bB^2} \qquad (22)$$

$$A=\left\{\frac{6bB^2k^2}{a(1-8bB^2)}\right\}^{\frac{1}{n}} \qquad (23)$$

$$c_1=0,\quad c_2=0. \qquad (24)$$

It is clear that $8bB^2\neq1$ and $a\neq0$. On account that $c_1=c_2=0$ then (2) collapses, after simplification, to the reduced gO-BBM equation with full nonlinearity

$$(u^n)_t+k^2(u^n)_x+a(u^{2n})_x+b(u^n)_{xxt}=0, \qquad (25)$$

where the integration constant is taken to be zero without any loss of generality. The shock wave solution is

$$u(x,t)=\left\{\frac{6bB^2k^2}{a(1-8bB^2)}\right\}^{\frac{1}{n}}\tanh^{\frac{2}{n}}\left[B\left(x-\frac{k^2}{1-8bB^2}t\right)\right].$$

3.3 Singular Solitons

In this section, the search is for a singular soliton solution to the gO-BBM equation with full nonlinearity (2). For this case, the ansatz is

$$u(x,t)=A\operatorname{csch}^p\tau,\quad p>0. \qquad (26)$$

For the case of singular solitons too, the parameters A and B are free non-zero parameters with v being the speed of the soliton. Substituting (26) into (2) gives

$$vA^lp^2l^2B^2\operatorname{csch}^{pl}\tau+pl(pl+1)vA^lB^2\operatorname{csch}^{pl+2}\tau$$
$$-(k^2p^2n^2A^nB^2-bvA^np^4n^4B^4-c_1A^n)\operatorname{csch}^{pn}\tau$$
$$-k^2pn(pn+1)A^nB^2\operatorname{csch}^{pn+2}\tau-4ap^2n^2A^{2n}B^2\operatorname{csch}^{2pn}\tau$$
$$-2apn(2pn+1)A^{2n}B^2\operatorname{csch}^{2pn+2}\tau$$
$$+bvA^nB^4pn(pn+1)\left\{p^2n^2+(pn+2)^2\right\}\operatorname{csch}^{pn+2}\tau$$
$$+bvA^nB^4pn(pn+1)(pn+2)(pn+3)\operatorname{csch}^{pn+4}\tau$$
$$+c_2A^{2n}\operatorname{csch}^{2pn}\tau=0. \qquad (27)$$

From (27), equating the exponent pairs $(2np+2,\,pn+4)$ or $(2np,\,pn+2)$ give

$$p=\frac{2}{n}.$$

Equating the exponents of $\operatorname{csch}^{pl}\tau$ and $\operatorname{csch}^{pn}\tau$ yields the requirement that $l=n$. Then, setting the coefficients of $\operatorname{csch}^{p+j}\tau$ ($j=0,2,4$) to zero, as these are linearly independent functions, gives

$$4B^2(k^2-v-4bvB^2)-c_1=0, \qquad (28)$$
$$6B^2(k^2-v-20bvB^2)-A^n(c_2-16aB^2)=0, \qquad (29)$$
$$aA^n-6bvB^2=0. \qquad (30)$$

Solving this system yields

$$A=\left(\frac{3c_1}{2c_2}\right)^{\frac{1}{n}}, \qquad (31)$$

$$B=\frac{1}{2}\sqrt{\frac{c_1(bc_2+a)}{b(k^2c_2-ac_1)}}, \qquad (32)$$

$$v=\frac{a(k^2c_2-ac_1)}{c_2(bc_2+a)}. \qquad (33)$$

Equation (31) shows that solitons will exist if $c_1c_2>0$. Also, from (32), the following restriction is obtained

$$bc_1(bc_2+a)(k^2c_2-ac_1)>0. \qquad (34)$$

This, in turn, gives the singular soliton solution which is valid for any positive integer n

$$u(x,t)=\left(\frac{3c_1}{2c_2}\right)^{\frac{1}{n}}\times$$

$$\operatorname{csch}^{\frac{2}{n}}\left[\frac{1}{2}\sqrt{\frac{c_1(bc_2+a)}{b(k^2c_2-ac_1)}}\left(x-\frac{a(k^2c_2-ac_1)}{c_2(bc_2+a)}t\right)\right].$$

4 Conclusion

In this paper, exact soliton solutions to the gO-BBM equation are derived. There are three types of soliton

solutions – solitary waves, shock waves and singular solitons. These results will be immensely useful in the area of geophysical fluids and/or rotating fluids within the shallow water wave approximation.

These results may be extended in many different ways. For example, the conservation laws can be be studied that will yield the corresponding conserved densities from which conserved quantities can be obtained. Further, perturbation terms will be added to study the dynamics of the gO-BBM equation in the adiabatic approximation.

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