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Inferences from Ailamujia Distribution based on Progressive Type-II Censored Data

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Abstract: In this paper, the parameters of Ailamujia distribution are estimated under partially accelerated life test (PALT). The maximum likelihood and Bayesian methods are used. The Bayes estimates of the parameters under squared error (SE) loss functions and linear exponential (LINEX) loss function by using Markov Chain Monte Carlo (MCMC) methods are obtained. Further, the explicit expression for moments and recurrence relations between moments of progressive Type-II censored data are also obtained. Finally, simulation study is carried out to compare the different methods of estimation.

Keywords: Ailamujia distribution; Partially accelerated life test; Progressive type-II censoring; Maximum likelihood estimation; Bayes estimation; Markov chain Monte Carlo method; Moments; Recurrence relations.

1 Introduction

In the reliability analysis and lifetime studies, when the experimenter does not want to test all the experimental units, in that case, the censoring is of great importance in planning and execution of experiments. There are many reasons why we use gradual control when it is required to remove units from the test at different points rather than the terminal point of the experiment or when some of the remaining units can be used in the experiment that was removed earlier in another test.

PALTs are very important approaches that are used to obtain enough failure data of test items in a shorter time. In Accelerated Life Testing (ALT), the test items run only at accelerated condition (e.g., vibration temperature and pressure) to induce early failures and quickly yield information on test items. In PALT some of test items can be run under normal use conditions and the others can be run under accelerated conditions. PALT is the reasonable for estimating the acceleration factor.

The stress can be applied to different methods such as constant-stress and step-stress. In constant-stress PALT (CSPALT), a sample of test items is tested either at normal use condition or at accelerated condition only until either failure occurs or the test is terminated. That is, each item runs at a constant-stress level until the test is terminated. Under step-stress PALT (SSPALT), a sample of test items is first run at normal use condition, and if it does not fail for a determined time, then it will run at accelerated condition until a pre-selected number of failures occur or a pre-determined time has reached, see Nelson [1].

For SSPALT, many studies have been carried out in step-stress models, for example, Gouno *et al*. [2] investigated the optimal step-stress test for the exponential distribution under progressive Type-I censoring. The optimal design for step-stress life test under progressively Type-II censoring data of exponential distribution was studied by Ismail and Sarhan[3]. Ismail [4] obtained the maximum likelihood estimators of the acceleration factor and parameters of Weibull distribution with adaptive type-II progressively hybrid censoring data. Ismail [5] used type-I censoring data and estimated the parameter of Pareto distribution under likelihood and Bayesian methods for step-stress life test model.

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Further, Mohie El-Din *et al*. [6], under progressive first-failure censoring, applied the simple SSALT for Weibull distribution. Mohie El-Din *et al*. [7] provided inferences on SSALT for extension of the exponential distribution.

Recurrence relations between moments of progressively type-II censored order statistics have been studied for different distributions by several authors. For example, see Saran and Pushkarna [8], Aggarwala and Balakrishnan [9], Mahmoud *et al*. [10], Balakrishnan and Saleh [11,12], Athar *et al*. [13], Athar and Akhter [14] and references therein. This paper map is organized as follows:

A model description and test assumptions are presented in Section 2. The MLEs of the unknown parameters are obtained and confidence intervals by using Fisher information matrix are evaluated in Section 3. In Section 4, MCMC are used to obtain the Bayes estimates of parameters . Section 5 deals with the moment properties of progressive Type-II censored order statistics from Ailamujia distribution. In Section 6, simulation study for different methods of estimation is presented. The discussion of this work is concluded in Section 7.

2 Model Description and Assumptions

Ailamujia distribution is introduced by Lv *et al*. [15] and further studied by Pan *et al*. [16]. Long [17] and Li [18] estimated the parameters of Ailamujia distribution. A random variable T said to follows Ailamujia distribution if its probability density function (PDF) and survival function (SF) are given as

$$f(t) = 4t\beta^2 e^{-2\beta t}; t, \beta \geq 0, \tag{1}$$

$$\bar{F}(t) = (1+2t\beta)e^{-2\beta t}, t, \beta \geq 0, \tag{2}$$

where $\bar{F}(t) = 1 - F(t)$.

2.1 Basic Assumptions

1. The n identical and independent items are put on test. The test is terminated at the m^{th} failure, where m is prefixed integer $(m \leq n)$.

2. Under normal use condition each of the n units are tested. By pre-specified time τ, if it does not fail, it is put under accelerated condition.

Let the total lifetime X of an item be:

$$X = \begin{cases} T, & \text{if } T \leq \tau \\ \tau + \lambda^{-1}(T - \tau), & \text{if } T > \tau \end{cases} \tag{3}$$

where T is the lifetime of an item at normal use condition and $\lambda > 1$ is the accelerated factor.

Then, the PDF and the SF of X can be written as

$$f(x) = \begin{cases} f_1(x) = 4\beta^2 x e^{-2\beta x}, 0 < x < \tau, \\ f_2(x) = 4\beta^2\lambda(\tau + \lambda(x-\tau))e^{-2\beta(\tau+\lambda(x-\tau))}, \ x \geq \tau \end{cases} \tag{4}$$

$$\bar{F}(x) = \begin{cases} \bar{F}_1(x) = (1+2\beta x)e^{-2\beta x}, 0 < x < \tau \\ \bar{F}_2(x) = [1 + 2\beta(\tau + \lambda(x-\tau))]e^{-2\beta(\tau+\lambda(x-\tau))}, \ x \geq \tau \end{cases} \tag{5}$$

3 Maximum Likelihood Estimation

Suppose that m_ℓ represents the number of units failed at normal use condition and m_c represents the number of units failed at accelerated use condition where the total number of units failed m is such that $m = m_\ell + m_c$.

Let $X^{\tilde{R}}_{1:m:n} < X^{\tilde{R}}_{2:m:n} < ... < X^{\tilde{R}}_{m_\ell:m:n} < \tau < X^{\tilde{R}}_{m_{\ell+1}:m:n} < ... < X^{\tilde{R}}_{m:m:n}$ denote progressively Type-II censored samples from a sample of size n with progressive censoring scheme $\tilde{R} = (R_1, R_2, ..., R_m)$, $m \le n$. Let $X^{\tilde{R}}_{i:m:n} \equiv X_i$, $i = 1, 2, ..., m$ be the observed values of lifetime X obtained from progressive censoring scheme under SSPALT. The likelihood function under the progressively type-II censored samples when the lifetimes of the units under test follow the Ailamujia distribution with PDF and SF are given in (4) and (5) with unknown parameter vector $\varphi = (\beta, \lambda)$ is

$$\begin{aligned} L(\varphi;x) &= A\prod_{i=1}^{m_\ell} f_1(x_i)\,[\bar{F}_1(x_i)]^{R_i} \prod_{i=m_{\ell+1}}^{m} f_2(x_i)\,[\bar{F}_2(x_i)]^{R_i} \\ &= A\prod_{i=1}^{m_\ell}\left\{4x_i\beta^2 e^{-2\beta x_i}\left[(1+2x_i\beta)e^{-2\beta x_i}\right]^{R_i}\right\} \prod_{i=m_{\ell+1}}^{m}\left\{4z_i\beta^2\lambda e^{-2\beta z_i}\left[[1+2\beta z_i)]e^{-2\beta z_i)}\right]^{R_i}\right\} \\ &= A(2\beta)^{2m}\lambda^{m-m_\ell}\prod_{i=1}^{m_\ell}\left\{x_i(1+2x_i\beta)^{R_i}e^{-2\beta(R_i+1)x_i}\right\} \prod_{i=m_{\ell+1}}^{m}\left\{z_i(1+2z_i\beta)^{R_i}e^{-2\beta(R_i+1)z_i}\right\}, \end{aligned} \tag{6}$$

where

$$\begin{aligned} A &= n(n-1-R_1)(n-2-R_1-R_2)...(n-m_\ell-R_1-R_2-...-R_{m_\ell})(n-m_{\ell+1}-R_1-R_2-...-R_{m_{\ell+1}}) \\ &\quad ...(n-m-R_1-R_2-...-R_m) \text{ and } z = \tau+\lambda(x-\tau). \end{aligned}$$

Then the log-likelihood function can be written as

$$\begin{aligned} \ln L(\varphi;x) &= \ln A + 2m\ln(2\beta) + (m-m_\ell)\ln\lambda + \sum_{i=1}^{m_\ell}\ln x_i + \sum_{i=1}^{m_\ell} R_i\ln(1+2\beta x_i) \\ &\quad -2\beta\sum_{i=1}^{m_\ell} x_i(R_i+1) + \sum_{i=m_{\ell+1}}^{m}\ln z_i + \sum_{i=m_{\ell+1}}^{m} R_i\ln(1+2\beta z_i) - 2\beta\sum_{i=m_{\ell+1}}^{m} z_i(R_i+1). \end{aligned} \tag{7}$$

Then

$$\frac{\partial \ln L(\varphi;x)}{\partial\beta} = \frac{2m}{\beta} + \sum_{i=1}^{m_\ell}\left[\frac{2R_i x_i}{1+2\beta x_i} - 2x_i(R_i+1)\right] + \sum_{i=m_{\ell+1}}^{m}\left[\frac{2R_i z_i}{1+2\beta z_i} - 2z_i(R_i+1)\right]. \tag{8}$$

$$\frac{\partial \ln L(\varphi;x)}{\partial\lambda} = \frac{(m-m_\ell)}{\lambda} + \sum_{i=m_{\ell+1}}^{m}\left[\frac{x_i-\tau}{z_i} + \frac{2\beta R_i(x_i-\tau)}{1+2\beta z_i} - 2\beta(x_i-\tau)(R_i+1)\right]. \tag{9}$$

Now equating $\frac{\partial \ln L(\varphi;x)}{\partial\beta} = 0$ and $\frac{\partial \ln L(\varphi;x)}{\partial\lambda} = 0$ in (8) and (9) yields a system of two nonlinear equations in two parameters β and λ. It may be noted that the closed form solution is not possible. Therefore, Newton-Raphson method is used to find a numerical solution for these equations.

3.1 Observed Fisher Information Matrix

In this subsection, the approximate confidence intervals of the parameters β and λ, by using the asymptotic distributions of the MLE, are constructed. The asymptotic distribution of the MLEs of β and λ can be written as

$$\left[(\hat{\beta}-\beta), (\hat{\lambda}-\lambda)\right] \to N(0, I^{-1}(\beta,\lambda)),$$

where $I^{-1}(\beta,\lambda)$ is the variance-covariance matrix of the parameters $\varphi = (\beta, \lambda)$. This can be approximated by the elements of observed Fisher information matrix $I_{ij}(\beta,\lambda)$, $i, j = 1, 2$, by $I_{ij}(\hat{\beta},\hat{\lambda})$, where

$$I_{ij}(\hat{\varphi}) = \frac{\partial^2 \ln L(\varphi)}{\partial\varphi_i\partial\varphi_j}\,|_{\varphi=\hat{\varphi}}\,. \tag{10}$$

Now from (8) and (9), we get

$$\frac{\partial^2 \ln L(\varphi;x)}{\partial \beta^2} = -\frac{m}{\beta} - \sum_{i=1}^{m_\ell}\left[\frac{4R_i x_i^2}{(1+2\beta x_i)^2}\right] + \sum_{i=m_{\ell+1}}^{m}\left[\frac{4R_i z_i^2}{(1+2\beta z_i)^2}\right]. \tag{11}$$

$$\frac{\partial^2 \ln L(\varphi;x)}{\partial \beta \partial \lambda} = \sum_{i=m_{\ell+1}}^{m}\left[\frac{2R_i(x_i-\tau)}{(1+2\beta z_i)^2} - 2(x_i-\tau)(R_i+1)\right]. \tag{12}$$

$$\frac{\partial^2 \ln L(\varphi;x)}{\partial \lambda^2} = -\frac{(m-m_\ell)}{\lambda^2} + \sum_{i=m_{\ell+1}}^{m}\left[-\frac{(x_i-\tau)^2}{z_i^2} - \frac{4\beta R_i(x_i-\tau)^2}{(1+2\beta z_i)^2}\right]. \tag{13}$$

The approximate $100(1-\delta)\%$ two-sided confidence intervals for β and λ can be written as

$$\left[(\hat{\beta}_{ML} \pm z_{\delta/2}\sqrt{I_{11}^{-1}(\hat{\beta}_{ML})}, (\hat{\lambda}_{ML} \pm z_{\delta/2}\sqrt{I_{22}^{-1}(\hat{\lambda}_{ML})}\right], \tag{14}$$

where $z_{\delta/2}$ is the upper $100(\delta/2)^{th}$ percentile of the standard normal distribution.

4 Bayes Estimation

In this section, Bayes estimates (BE) of the model parameters β and λ under progressive Type-II censoring are obtained using SE and LINEX loss functions. We shall use the informative priors as

$$\pi_1(\beta) \propto \beta^{\alpha_1-1}e^{-\beta/\gamma_1}, \quad \alpha_1,\gamma_1 > 0, \tag{15}$$

$$\pi_2(\lambda) \propto \lambda^{\alpha_2-1}e^{-\lambda/\gamma_2}, \quad \alpha_2,\gamma_2 > 0. \tag{16}$$

The gamma distribution is used as prior distribution because it covers all prior information of the experimenter. For more details one can refer to Kim *et al*. [19] and Singh *et al*. [20].

The joint prior of the parameters β and λ is given by

$$\pi(\beta,\lambda) \propto \beta^{\alpha_1-1}\lambda^{\alpha_2-1}e^{-\beta/\gamma_1}e^{-\lambda/\gamma_2}. \tag{17}$$

From (17) and (6), the joint posterior density function is

$$\pi^*(\beta,\lambda;x) \propto L(\beta,\lambda;x)\pi(\beta,\lambda) \propto \beta^{2m+\alpha_1-1}\lambda^{m-m_\ell+\alpha_2-1}e^{-\beta/\gamma_1}e^{-\lambda/\gamma_2}$$

$$\times \prod_{i=1}^{m_\ell}\left\{x_i(1+2x_i\beta)^{R_i}e^{-2\beta(R_i+1)x_i}\right\}\prod_{i=m_{\ell+1}}^{m}\left\{z_i(1+2z_i\beta)^{R_i}e^{-2\beta(R_i+1)z_i}\right\}. \tag{18}$$

Now, based on SE and LINEX loss functions, the BE of the function $\Psi(\varphi)$ of the model parameters β and λ can be written as

$$\hat{\Psi}_{SE}(\varphi) = E[\Psi(\varphi)|x] = \int_{\varphi}\Psi(\varphi)\pi^*(\varphi|x)d\varphi, \tag{19}$$

$$\hat{\Psi}_{LINEX}(\varphi) = -\frac{1}{c}\ln[E(\exp(-c\Psi(\varphi)|x)] = -\frac{1}{c}\ln\left[\int_{\varphi}\exp(-c\Psi(\varphi)|x)\pi^*(\varphi|x)d\varphi\right], \tag{20}$$

where $\Psi(\varphi) \equiv \Psi(\beta,\lambda)$ and $E(\cdot)$ is the expected value.

Since we cannot compute the integrals given in equations (19) and (20) explicitly. Thus, it can be approximated by using MCMC technique.

4.1 Bayesian Estimation by MCMC Method

In this subsection, a Gibbs sampling which is MCMC algorithm for obtaining a sequence of random samples from posterior function to compute the BE and credible intervals under the SE and LINEX loss functions is studied. Now, from (18), the conditional posterior density functions of the parameters β and λ respectively can be written as

$$\pi_1^*(\beta|\lambda;x) \propto \beta^{2m+\alpha_1-1} e^{-\beta/\gamma_1} \prod_{i=1}^{m_\ell} \left\{(1+2x_i\beta)^{R_i} e^{-2\beta(R_i+1)x_i}\right\} \times \prod_{i=m_{\ell+1}}^{m} \left\{(1+2z_i\beta)^{R_i} e^{-2\beta(R_i+1)z_i}\right\}, \tag{21}$$

$$\pi_2^*(\lambda|\beta;x) \propto \lambda^{m-m_\ell+\alpha_2-1} e^{-\lambda/\gamma_2} \prod_{i=m_{\ell+1}}^{m} \left\{z_i(1+2z_i\beta)^{R_i} e^{-2\beta(R_i+1)z_i}\right\}. \tag{22}$$

The conditional posterior distributions of β and λ in (21) and (22) cannot be obtained analytically to well known distribution. Thus, to solve this problem, we use the Metropolis-Hastings algorithm to generate random samples by using normal proposal distribution, see Upadhyay and Gupta [21]. To compute the Bayes estimators we used the following Algorithm.

Algorithm 4.1.

1. Let $\hat{\beta}$ and $\hat{\lambda}$ be the MLEs with starting values$(\beta^{(0)}$ and $\lambda^{(0)})$ of β and λ.
2. Set h=1.
3. Using MH algorithm to generate $\beta^{(h)}$from $\pi_1^*(\beta^{(h-1)}|\lambda^{(h-1)};x)$ with the proposed distribution $N\left(\beta^{(h-1)}, \sqrt{var(\hat{\beta}_{ML})}\right)$ and then generate $\lambda^{(h)}$ from $\pi_2^*(\lambda^{(h-1)}|\beta^{(h)};x)$ with the proposed distribution $N\left(\lambda^{(h-1)}, \sqrt{var(\hat{\lambda}_{ML})}\right)$.
4. Set h=h+1.
5. Repeat steps 2 to 4 N times.
6. Obtain $\beta^{(h)}$ and $\lambda^{(h)}$, h=M+1,.....,N , and now, the approximate means of $\varphi(\beta,\lambda)$ and $\exp[-c\,\varphi(\beta,\lambda)]$ are given, respectively, by

$$E\left[\Psi(\beta,\,\lambda)|x\right] = \frac{1}{N-M}\sum_{h=M+1}^{N} \Psi(\beta^{(h)},\,\lambda^{(h)}), \tag{23}$$

$$E\left[\exp(-c\Psi(\beta,\,\lambda)|x)\right] = \frac{1}{N-M}\sum_{h=M+1}^{N} \exp(-c\Psi(\beta^{(h)},\,\lambda^{(h)})), \tag{24}$$

where M is the burn-in period.

Then, the Bayes MCMC point estimate of $\Psi(\beta,\,\lambda)$ under SE and LINEX loss are given by

$$\hat{\Psi}_{SE}(\beta,\,\lambda) = E\left[\Psi(\beta,\,\lambda)|x\right]. \tag{25}$$

$$\hat{\Psi}_{LINEX}(\beta,\,\lambda) = -\frac{1}{c}\ln[E(\exp(-c\Psi(\beta,\,\lambda)|x)]. \tag{26}$$

To find the credible intervals of β and λ, repeated steps 1 to 5, Q times and arrange the values in ascending order as $\hat{\beta}_{SE}^{[1]}, ..., \hat{\beta}_{SE}^{[Q]}$ and $\hat{\lambda}_{SE}^{[1]}, ..., \hat{\lambda}_{SE}^{[Q]}$. Then the credible intervals β and λ can be written as $(\hat{\beta}_{SE}^{[(\frac{\delta}{2})Q]}, \hat{\beta}_{SE}^{[(1-\frac{\delta}{2})Q]})$ and $(\hat{\lambda}_{SE}^{[(\frac{\delta}{2})Q]}, \hat{\lambda}_{SE}^{[(1-\frac{\delta}{2})Q]})$, respectively.

5 Moments and recursive relation

Let $X^{\tilde{R}}_{1:m:n} < X^{\tilde{R}}_{2:m:n} < ... < X^{\tilde{R}}_{m:m:n}$ be the m ordered observed failure times of items under the progressive censoring scheme $\tilde{R}$ from a sample of size n . The joint probability density function of $X^{\tilde{R}}_{1:m:n}, X^{\tilde{R}}_{2:m:n}, ..., X^{\tilde{R}}_{m:m:n}$ with common PDF $f(x)$and CDF $F(x)$ is given by

$$f^{X^{\tilde{R}}_{1:m:n},\ldots,X^{\tilde{R}}_{m:m:n}}(x_1,\ldots,x_m) = c(n,m-1)\prod_{j=1}^{m} f(x_j)[1-F(x_j)]^{R_j}$$

$$-\infty \le x_1 < x_2 < \cdots < x_m \le \infty \tag{27}$$

where,

$$n = m+\sum_{j=1}^{m} R_j,\ n,m \in \mathrm{N}, R_j = \mathrm{N}_0,\ 1 \le j \le m,$$

and $$A = c(n,m-1) = \prod_{j=1}^{m}\left(n-\sum_{i=1}^{j-1} R_i - j+1\right) = \prod_{j=1}^{m} R_j^*, \text{ with } R_j^* = \sum_{k=j}^{m}(R_k+1).$$

Observe that $R_1^* = n$.

The marginal density of the r^{th} progressive Type- II censored data for *pdf* $f(x)$ and *cdf* $F(x)$ is given by Balakrishnan and Cramer [22], as

$$f_{X_{r:m:n}}(x) = f(x)\left(\prod_{i=1}^{r}\gamma_i\right)\sum_{j=1}^{r} a_{j,r}[1-F(x)]^{\gamma_j-1},\ x \in \Re, \tag{28}$$

where, $$\gamma_i = \sum_{l=i}^{m}(R_l+1) \text{ and } a_{j,r} = \prod_{\substack{k=1\\k\neq j}}^{r}\frac{1}{\gamma_k-\gamma_j}.$$

Theorem 5.1. Let $X^{\tilde{R}}_{1:m:n}, \ldots, X^{\tilde{R}}_{m:m:n}$ be the progressive Type-II censored data from the Ailamujia distribution as given in (1). Then for $1 \le m \le n$

$$E(X^{\tilde{R}}_{r:m:n})^k = \left(\prod_{i=1}^{r}\gamma_i\right)\sum_{j=1}^{r}\sum_{l=0}^{\gamma_j-1} a_{j,r}\binom{\gamma_j-1}{l}\frac{\Gamma(k+l+2)}{(2\beta)^k\gamma_j^{k+l+2}}, \tag{29}$$

where $\gamma_i = \sum_{t=i}^{m}(R_t+1)$, $a_{j,r} = \prod_{\substack{p=1\\k\neq j}}^{r}\frac{1}{\gamma_p-\gamma_j}$ and $\tilde{R} = (R_1, R_2, ..., R_m)$.

Proof: In view of (28), we have

$$E(X^{\tilde{R}}_{r:m:n})^k = \left(\prod_{i=1}^{r}\gamma_i\right)\sum_{j=1}^{r} a_{j,r}\int_0^\infty x^k[(1+2\beta x)e^{-2\beta x}]^{\gamma_j-1}4x\beta^2 e^{-2\beta x}dx$$

$$= (2\beta)^2\left(\prod_{i=1}^{r}\gamma_i\right)\sum_{j=1}^{r} a_{j,r}\int_0^\infty x^{k+1}(1+2\beta x)^{\gamma_j-1}e^{-2\beta\gamma_j x}dx$$

$$= (2\beta)^{2+l}\left(\prod_{i=1}^{r}\gamma_i\right)\sum_{j=1}^{r} a_{j,r}\sum_{l=0}^{\gamma_j-1}\binom{\gamma_j-1}{l}\int_0^\infty x^{k+l+1}e^{-2\beta\gamma_j x}dx$$

$$= \left(\prod_{i=1}^{r}\gamma_i\right)\sum_{j=1}^{r}\sum_{l=0}^{\gamma_j-1} a_{j,r}\binom{\gamma_j-1}{l}\frac{\Gamma(k+l+2)}{(2\beta)^k\gamma_j^{k+l+2}}. \tag{30}$$

Hence the theorem.

Theorem 5.2. Let $X^{\tilde{R}}_{1:m:n}, \ldots, X^{\tilde{R}}_{m:m:n}$ be the progressive Type -II censored data from the Ailamujia distribution as given in (1), where $1 \le m \le n$. Then

$$E(X^{\tilde{R}}_{r:m:n})^k - E(X^{\tilde{R}}_{r-1:m:n})^k = \frac{k}{4\beta^2\gamma_r}\left[E(X^{\tilde{R}}_{r:m:n})^{k-2} + 2\beta E(X^{\tilde{R}}_{r:m:n})^{k-1}\right]. \tag{31}$$

Proof: In view of Athar and Islam [23] with $\xi(x) = x^k$ and $\gamma_i = n - i + 1 + \sum_{l=i}^{n} R_l$, we have

$$E(X_{r:m:n}^{\tilde{R}})^k - E(X_{r-1:m:n}^{\tilde{R}})^k = k\left(\prod_{i=1}^{r-1}\gamma_i\right)\int_0^{\infty} x^{k-1}\sum_{j=1}^{r} a_{j,r}[1-F(x)]^{\gamma_j}dx \tag{32}$$

For the Ailamujia distribution as given in (1), we have the relation between CDF and PDF as

$$1 - F(x) = \frac{1+2x\beta}{4x\beta^2} f(x) \tag{33}$$

On application of (33) in (32), we get the required result.

6 Simulation study

In this section, for different sample sizes and censoring schemes, the simulation study is carried out by using the following Algorithm (6.1). Progressively Type-II censored samples are generated by Balakrishnan and Sandhu [24].

Algorithm 6.1.

1. Set values for n, m, τ and R_i, $i = 1, 2, \ldots, m$
2. Generate β and λ from $\pi_1(\beta)$ and $\pi_2(\lambda)$ respectively, for the given values of the prior parameters $(\beta, \lambda, \alpha_1, \alpha_2, \gamma_1, \gamma_2)$.
3. For the given values of n and m, $1 \le m \le n$ and CDF in (2), we generate the following progressively Type-II censored samples using the algorithm given in Balakrishnan and Sandhu [24], the set of data can be considered as
$$X_{1:m:n}^{\tilde{R}} < X_{2:m:n}^{\tilde{R}} < \ldots < X_{m_\ell:m:n}^{\tilde{R}} < \tau < X_{m_{\ell+1}:m:n}^{\tilde{R}} < \ldots < X_{m:m:n}^{\tilde{R}},$$
where $\tilde{R} = (R_1, R_2, \ldots, R_m)$ and $\sum_{i=1}^{m} R_i = n - m$.
4. Follow the same steps 1 to 3, to compute the MLEs of the parameters β and λ by solving the nonlinear equations (8) and (9).
5. Calculate the BE of the parameters β and λ relative to SE and LINEX loss functions under MCMC with N = 11000.
6. Approximate CI and credible CIs with confidence level 95% for the parameters β and λ are computed.
7. Replicate the Steps 3−7, 1000 times.
8. MSEs and BEs of the parameters β and λ are computed.
9. Repeat steps 1−8 with different values of n, m and R_i, $i = 1, 2, \ldots, m$.

6.1 Simulation Procedure

The MLEs, MSEs and 95% approximately of CIs of the model parameters are calculated by Monte Carlo simulation method, based on r =1000 Monte Carlo simulations and run the chain for 11000 times 'burn-in' M =1000. The Bayes MSE and 95% credible interval for β and λ are computed. **Table 1** presents different censoring schemes, for different choices of sample sizes. The MSE for β and λ using the ML and MCMC methods under progressive using different censoring schemes are shown in **Table 2**. **Table 3** presents the lengths of the CIs for ML method and MCMC.

Table 1: Progressive censoring schemes for different n and m.

Sc	*n*	*m*	*R*	*Sc*	*n*	*m*	*R*	*Sc*	*n*	*m*	*R*
[1]	50	35	$R_1 = 15$	[4]	90	80	$R_1 = 15$	[7]	120	100	$R_1 = 20$
[2]			$R_{18} = 15$	[5]			$R_{40} = 15$	[8]			$R_{50} = 20$
[3]			$R_{35} = 15$	[6]			$R_{80} = 15$	[9]			$R_{100} = 20$

Table 2: MLE, BE and with their *$MSEs$ for the parameters ($\beta = 0.5$ and $\lambda = 1.7$) and values of the prior ($\alpha_1 = 0.52,\ \gamma_1 = .073$, $\alpha_2 = 0.31$, $\gamma_2 = 0.052$) and $\tau = 4$.

(n,m)	**Sc**		**MLE**		**BE**	
				SE	**LINEX**	**LINEX**
					$c=2$	$c=0.001$
(50, 35)	[1]	β	0.9347 (0.4245)	0.5672 (0.0969)	0.5644 (0.0963)	0.5672 (0.0969)
		λ	2.5212 (0.9211)	1.8812 (0.1832)	1.8823 (0.1839)	1.8812 (0.1832)
	[2]	β	0.9312 (0.4156)	0.5664 (0.0915)	0.5665 (0.0916)	0.5664 (0.0915)
		λ	2.2292 (0.6291)	1.8692 (0.1791)	1.8694 (0.1792)	1.8692 (0.1791)
	[3]	β	0.9154 (0.3995)	0.5592 (0.0866)	0.5599 (0.0869)	0.5592 (0.0866)
		λ	2.1862 (0.5155)	1.8662 (0.1755)	1.8668 (0.1756)	1.8662 (0.1755)
(50, 50)		β	0.9112 (0.3971)	0.5566 (0.0863)	0.5561 (0.0858)	0.5566 (0.0863)
		λ	2.1743 (0.5122)	1.8743 (0.1788)	1.8747 (0.1793)	1.8743 (0.1788)
(90, 80)	[4]	β	0.9035 (0.3876)	0.5464 (0.0816)	0.5461 (0.0812)	0.5464 (0.0816)
		λ	2.0772 (0.5072)	1.8472 (0.1272)	1.8478 (0.1276)	1.8472 (0.1272)
	[5]	β	0.8974 (0.3865)	0.5421 (0.0784)	0.5427 (0.0746)	0.5421 (0.0784)
		λ	2.0976 (0.5190)	1.8562 (0.1290)	1.8558 (0.1286)	1.8562 (0.1290)
	[6]	β	0.8884 (0.3765)	0.5372 (0.0776)	0.5370 (0.0773)	0.5372 (0.0776)
		λ	2.0192 (0.4925)	1.8092 (0.0994)	1.8088 (0.0989)	1.8092 (0.0994)
(90, 90)		β	0.8821 (0.3720)	0.5327 (0.0757)	0.5331 (0.0790)	0.5327 (0.0757)
		λ	2.0079 (0.4784)	1.8019 (0.0912)	1.8022 (0.0922)	1.8019 (0.0912)
(120, 100)	[7]	β	0.8777 (0.3611)	0.5118 (0.0594)	0.5113 (0.0591)	0.5118 (0.0594)
		λ	1.9805 (0.4505)	1.7805 (0.0811)	1.7809 (0.0814)	1.7805 (0.0811)
	[8]	β	0.8591 (0.3562)	0.5348 (0.0689)	0.5344 (0.0685)	0.5348 (0.0689)
		λ	1.9678 (0.4497)	1.7678 (0.0621)	1.7672 (0.0613)	1.7678 (0.0621)
	[9]	β	0.8195 (0.3103)	0.5119 (0.0597)	0.5111 (0.0592)	0.5119 (0.0597)
		λ	1.9737 (0.4519)	1.7437 (0.0339)	1.7440 (0.0345)	1.7437 (0.0339)
(120, 120)		β	0.7819 (0.2614)	0.5214 (0.0655)	0.5219 (0.0659)	0.5214 (0.0655)
		λ	1.9558 (0.4442)	1.7403 (0.0326)	1.7422 (0.0329)	1.7403 (0.0326)

∗MSE are given in bracket.

Table 3: Lengths and coverage probabilities (CPs) of 95% CIs for estimates of the parameters ($\beta = 0.5$ and $\lambda = 1.7$) when values of the prior parameters ($\alpha_1 = 0.52,\ \gamma_1 = .073,\ \alpha_2 = 0.31,\ \gamma_2 = 0.052$) and $\tau = 4$

(n, m)	**Sc**		**MLE**		**BE**	
			Length	**CP**	**Length**	**CP**
(50, 35)	[1]	β	1.0492	0.774	0.2332	0.897
		λ	3.5249	0.872	1.2954	0.876
	[2]	β	1.1637	0.947	0.2633	1.000
		λ	3.3210	0.770	1.1873	0.843
	[3]	β	1.0722	0.782	0.2754	0.976
		λ	3.1344	0.940	1.1709	1.000
(50,50)		β	1.2872	0.820	0.3613	0.888
		λ	3.0621	1.000	0.9473	0.945
(90,80)	[4]	β	0.9823	0.928	0.1976	0.840
		λ	2.9282	0.773	1.0848	0.920
	[5]	β	1.1622	0.888	0.2153	1.000
		λ	2.8594	0.912	1.0145	1.000
	[6]	β	1.1442	0.790	0.2172	0.786
		λ	2.8865	0.860	1.0198	0.921
(90,90)		β	1.2855	0.942	0.2357	0.966
		λ	2.6324	0.971	0.9507	1.000
(120,100)	[7]	β	0.9765	0.750	0.1705	0.873
		λ	2.7438	1.000	1.1085	0.850
	[8]	β	0.9825	0.992	0.1765	0.970
		λ	2.8438	0.789	1.0824	1.000
	[9]	β	0.9941	0.970	0.1854	0.876
		λ	2.5611	0.850	1.0496	0.941
(120,120)		β	0.9732	0.820	0.1861	0.958
		λ	2.3523	0.942	0.9765	0.947

7 Conclusion

We can notice from the simulation study that:

1. In Table 2, for the sample size n, the MSE of the MCMC method is smaller than MSE of the MLEs, thus the MCMC method is better than the MLEs.
2. In Table 3, the lengths of MCMC CI method is shorter than lengths of MLE CI in all samples and the CPs of MCMC method is better than CPs of the MLEs, except for a few cases. This may be due to fluctuations in data.
3. The BEs of β, λ have the smallest MSEs under LINEX loss function ($c = 2$) as compared with estimates under MLEs and SE loss function.
4. For the sample size n, the MSE estimates decrease with increasing the censoring sizes m, so we can say that large sample sizes with large censored samples give better estimates.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article

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