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Some Characterizations of the Extended Beta Distribution

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Abstract: In this paper, we establish some new characterization results of the extended beta distribution introduced by Chaudhry et al. [10] [M. A. Chaudhry, A. Qadir, M. Rafique and S. M. Zubair, Journal of Computational and Applied Mathematics, 78(1), 19-32 (1997)] by truncated moment, order statistics and upper record values.

Keywords: Characterizations; Extended Beta Distribution; Order Statistics; Truncated Moment; Upper Record Values.

1 Introduction

The extended beta distribution was introduced by Chaudhry et al. [10]. Further studies continued with the contributions of many authors and researchers at different times. For example, Al-Saleh and Agarwal [7, 8] addressed the extended beta distribution as a mixture of distributions with applications to Bayesian analysis. Nagar et al. [15] investigated several properties of the extended beta distribution. Furthermore, Nagar et al. [16] explored the distribution of the product of two independent extended beta random variables. Also, Nagar et al. [17] derived the Fisher information matrix, and Renyi and Shannon entropies for the extended beta distribution. Pieces of literature manifest that, despite extensive work on the extended beta distribution, no attention has been paid to its characterizations. According to Nagaraja [18], "a characterization is a certain distributional or statistical property of a statistic or statistics that uniquely defines the associated stochastic model". The problems of characterizations of probability distributions have been investigated by several authors and researchers. See, for example, Ahsanullah [2], Ahsanullah et al. [4, 5, 6], Galambos and Kotz [13], Kotz and Shanbhag [14], Nagaraja [18], and references therein. In this paper, we establish some new characterization results by truncated moment, order statistics and upper record values of the extended beta distribution introduced by Chaudhry et al. [10].

The organization of this paper is as follows: In Section Two, some basic distributional properties of the extended beta distribution are presented. Based on these distributional properties, we establish some new characterizations of the extended beta distribution by truncated moment, order statistics and upper record values in Section Three. The concluding remarks are manifested in Section Four.

2 Some Basic Distributional Properties

In this section, following Chaudhry et al. [10], some basic distributional properties of the extended beta distribution are presented.

2.1 PDF and CDF

Chaudhry et al. [10] introduced the following distribution

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$$f_X(x) = \begin{cases} \dfrac{1}{B(p, q; b)} x^{p-1}(1-x)^{q-1} \exp\left[\dfrac{-b}{x(1-x)}\right], & 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{2.1}$$

where $p, q \in (-\infty, \infty)$, $b > 0$, and $B(p, q; b) = \int_0^1 t^{p-1}(1-t)^{q-1} \exp\left[\dfrac{-b}{t(1-t)}\right] dt$ denotes the extended beta function. A random variable X with probability density function (pdf) given by (2.1) has the extended beta distribution with parameters p, q and b. The cumulative distribution function (cdf) corresponding to Eq. (2.1) is given by

$$F(x) = \frac{B_x(p, q; b)}{B(p, q; b)}, \tag{2.2}$$

where $B_x(p, q; b) = \int_0^x t^{p-1}(1-t)^{q-1} \exp\left[\dfrac{-b}{t(1-t)}\right] dt$, $(0 \le x < 1)$, denotes the extended incomplete beta function; see Chaudhry and Zubair [11, Eq. 5.161, p. 240]. According to Chaudhry et al. [10], "this distribution should be useful in extending the statistical results for strictly positive variables to deal with variables that can take arbitrarily large negative values as well". For some selected values of the parameters, the graphs of the pdf and cdf are given in Figures 2.1 (a, b) and 2.2 (a, b) respectively. The effects of the parameters can easily be observed from these graphs. For example, Figure 2.1 (a) reveals that the distribution of the random variable X is continuous, symmetric and bell-shaped for $p = 2.5$, $q = 2.5$, and $b = 0.1, 0.2, 0.3, 0.4, 0.5$. On the other hand, Figure 2.1 (b) shows that, for $p = -2.5$, $q = -2.5$, and $b = 0.1, 0.2, 0.3, 0.4, 0.5$, the distribution of the random variable X defines a continuous probability distribution with two uniform modes (or peaks), resembling a "two-humped" distribution, with a bathtub shape between the two peaks.

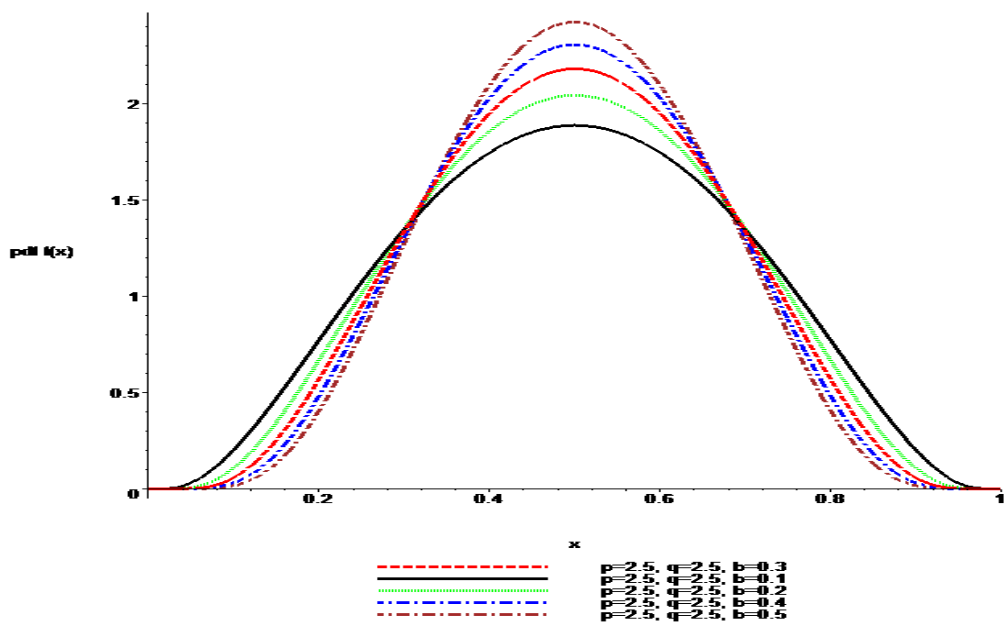

Fig. 2.1 (a): PDF Plot for $p = 2.5$, $q = 2.5$, and $b = 0.1, 0.2, 0.3, 0.4, 0.5$.

2.2 Moments

For some integer $k > 0$, the kth moment is given by

$$E\left(X^k\right)=\frac{B\left(p+k,q;b\right)}{B\left(p,q;b\right)}. \tag{2.3}$$

When $k=1$ in Eq. (2.3), the 1st moment (mean) is given by

$$E\left(X\right)=\frac{B\left(p+1,q;b\right)}{B\left(p,q;b\right)}. \tag{2.4}$$

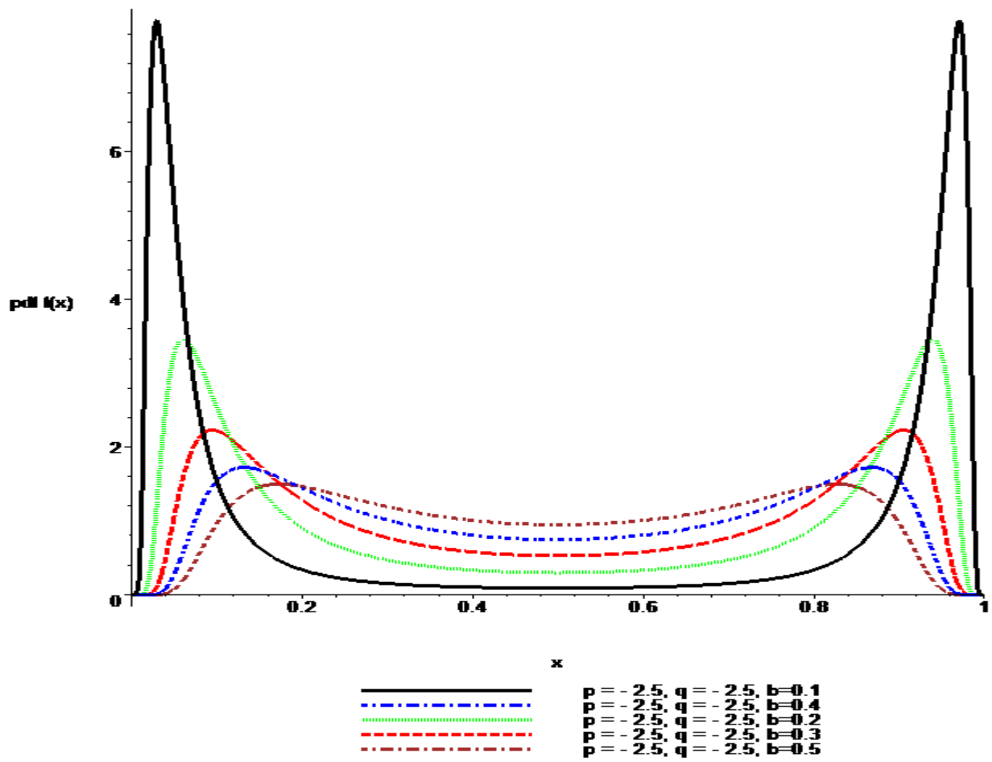

Fig. 2.1 (b): PDF Plot for $p=-2.5$, $q=-2.5$, and $b=0.1, 0.2, 0.3, 0.4, 0.5$

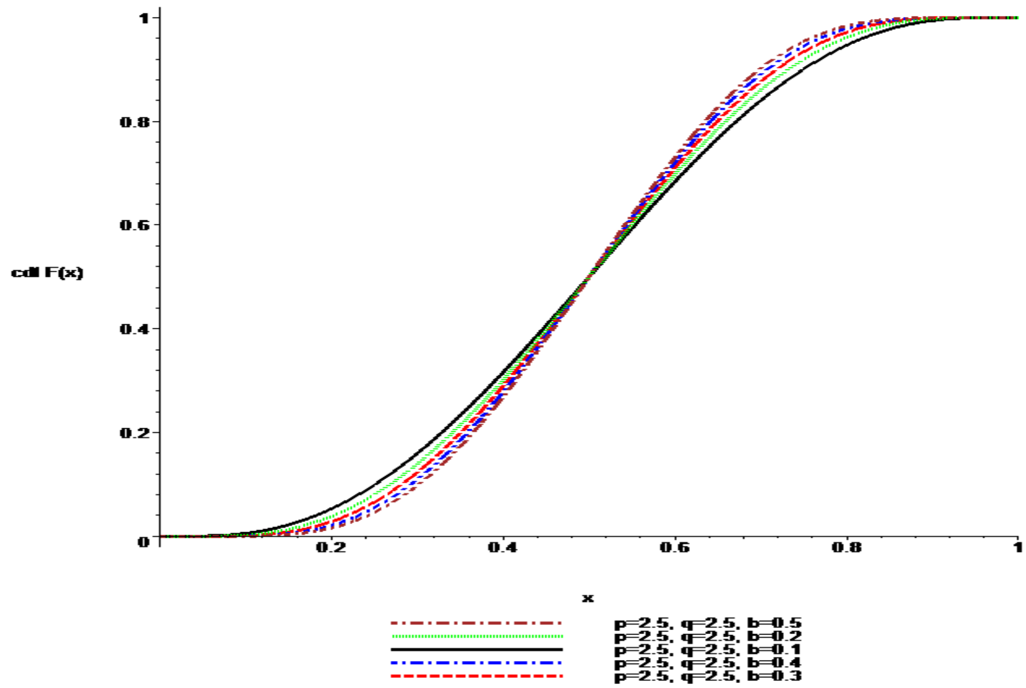

Fig. 2.2 (a): CDF Plot for $p = 2.5$, $q = 2.5$, and $b = 0.1, 0.2, 0.3, 0.4, 0.5$.

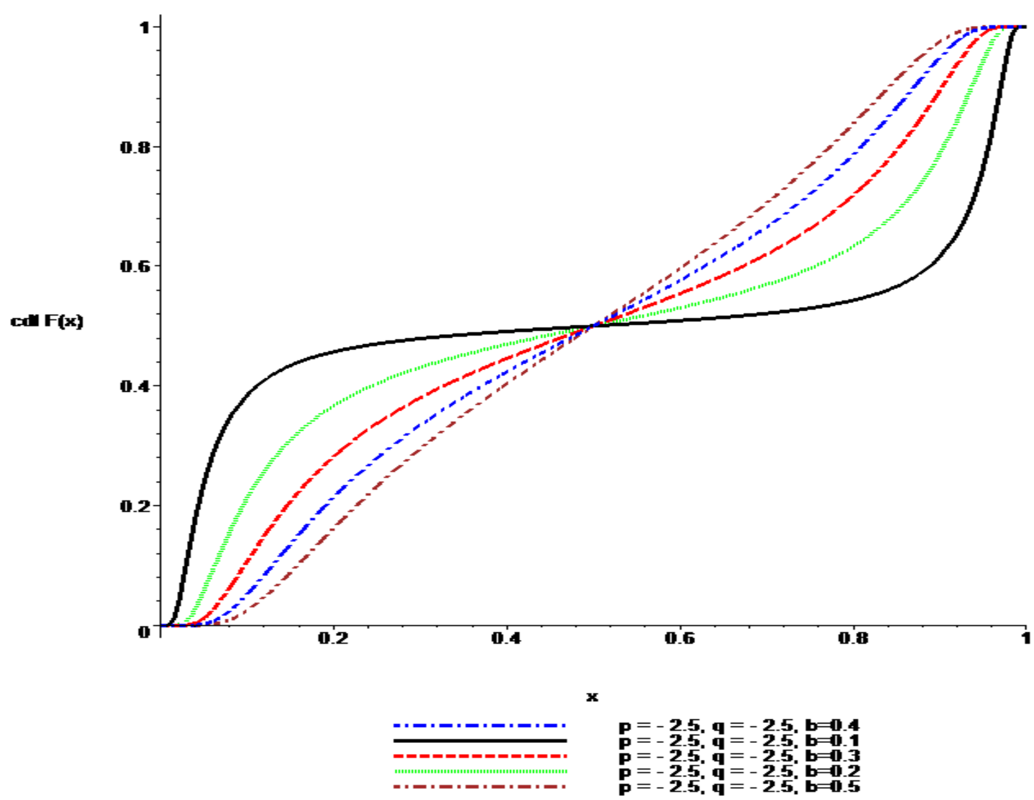

Fig. 2.2 (b): CDF Plot for $p = -2.5$, $q = -2.5$, and $b = 0.1, 0.2, 0.3, 0.4, 0.5$

2.3 Variance

Variance is given by

$$Var(X) = E\left(X^2\right) - \left(E\left(X\right)\right)^2$$

$$= \frac{B\left(p,q;b\right)B\left(p+2,q;b\right) - B^2\left(p+1,q;b\right)}{B^2\left(p,q;b\right)}. \tag{2.5}$$

2.4. Entropy

For a detailed discussion and derivation of the Renyi and Shannon entropies, see Nagar et al. [17].

2.5 Survival and Hazard Functions

The survival and hazard functions are respectively given by

$$S(x) = 1 - F_X(x) = 1 - \frac{B_x\left(p,q;b\right)}{B\left(p,q;b\right)}, \tag{2.6}$$

and

$$h(x) = \frac{f_X(x)}{1 - F_X(x)} = \frac{x^{p-1}\left(1-x\right)^{q-1}\exp\left[\dfrac{-b}{x\left(1-x\right)}\right]}{B\left(p,q;b\right) - B_x\left(p,q;b\right)}. \tag{2.7}$$

For some selected values of the parameters, the graphs of the hazard function (2.7) are given in Figures 2.3 (a, b). The effects of the parameters can easily be seen from these graphs.

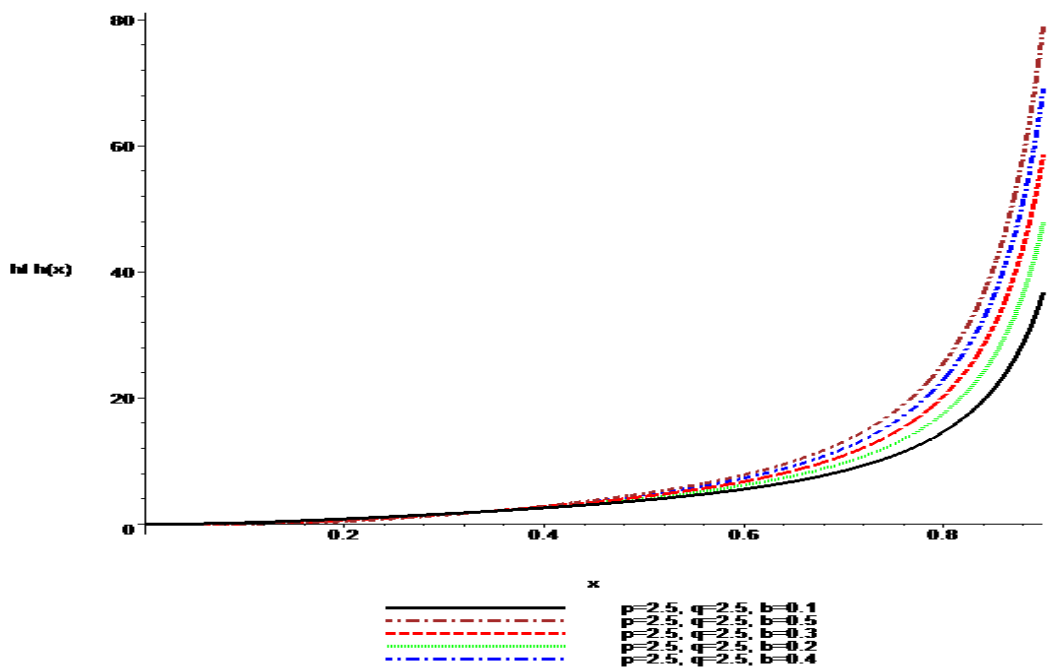

Fig. 2.3 (a): HF $h(x)$ Plot for $p = 2.5$, $q = 2.5$, and $b = 0.1, 0.2, 0.3, 0.4, 0.5$.

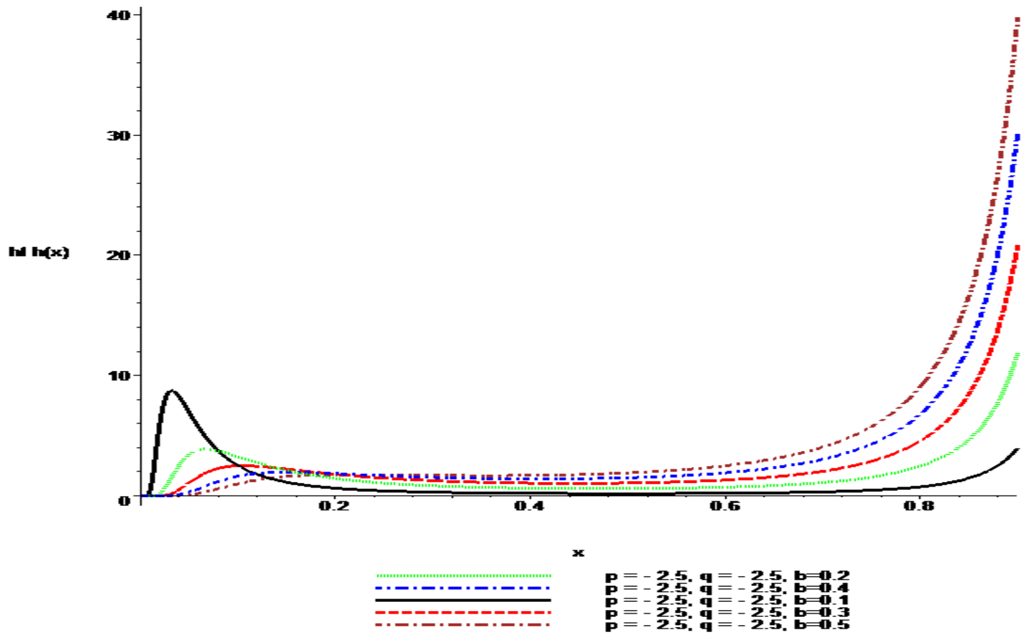

Fig. 2.3 (b): HF $h(x)$ Plot for $p=-2.5$, $q=-2.5$, and $b=0.1, 0.2, 0.3, 0.4, 0.5$.

3 Characterization Results

In this section, we establish our proposed characterization results of the extended beta distribution introduced by Chaudhry et al. [10] by truncated moment, order statistics and upper record values. They require the following assumption and lemmas:

Assumption 3.1. Suppose the random variable X is absolutely continuous with the cumulative distribution function $F(x)$ and the probability density function $f(x)$. We assume that $\omega = \inf\{x \mid F(x) > 0\}$, and $\delta = \sup\{x \mid F(x) < 1\}$. We also assume that $f(x)$ is differentiable for all x, and $E(X)$ exists.

Lemma 3.1. If the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = 1$, and if $E(X|X \le x) = g(x)\tau(x)$, where $\tau(x) = \frac{f(x)}{F(x)}$ and $g(x)$ is a continuous differentiable function of x with the condition that $\int_0^x \frac{u - g^{/}(u)}{g(u)} du$ is finite for $x > 0$, $f(x) = c e^{\int_0^x \frac{u - g^{/}(u)}{g(u)} du}$, where c is a constant defined by the condition $\int_0^\infty f(x)dx = 1$.

Proof. For proof, see Shakil et al. [20].

Lemma 3.2. If the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = 1$, and if $E(X \mid X \ge x) = \tilde{g}(x) r(x)$, where $r(x) = \frac{f(x)}{1 - F(x)}$ and $\tilde{g}(x)$ is a continuous differentiable function of x with the condition that $\int_x^\infty \frac{u+\left[\tilde{g}(u)\right]'}{\tilde{g}(u)}du$ is finite for $x>0$, then $f(x)=c\,e^{-\int_0^x \frac{u+\left[\tilde{g}(u)\right]'}{\tilde{g}(u)}du}$, where c is a constant determined by the condition $\int_0^\infty f(x)dx=1$.

Proof. For proof, see Shakil et al. [20].

3.1 Characterization by Truncated Moments

Theorem 3.1. If the random variable X satisfies the Assumption 3.1 with $\omega=0$ and $\delta=1$, $E(X|X\le x)=g(x)\frac{f(x)}{F(x)}$, where

$$g(x)=\frac{B_x(p+1,q;b)}{x^{p-1}(1-x)^{q-1}\exp\left[\frac{-b}{x(1-x)}\right]}, \tag{3.1}$$

if and only if X has the pdf $f_X(x)=\frac{1}{B(p,q;b)}x^{p-1}(1-x)^{q-1}\exp\left[\frac{-b}{x(1-x)}\right]$.

Proof. Suppose that $E(X|X\le x)=g(x)\frac{f(x)}{F(x)}$. Then, since $E(X|X\le x)=\frac{\int_0^x u f(u)du}{F(x)}$, we have $g(x)=\frac{\int_0^x u f(u)du}{f(x)}$. Now, if the random variable X satisfies the Assumption 3.1 and has the distribution with the pdf (2.1), we have

$$g(x)=\frac{\int_0^x u f(u)du}{f(x)}=\frac{\int_0^x u^p(1-u)^{q-1}\exp\left[\frac{-b}{u(1-u)}\right]du}{x^{p-1}(1-x)^{q-1}\exp\left[\frac{-b}{x(1-x)}\right]}.$$

$$=\frac{B_x(p+1,q;b)}{x^{p-1}(1-x)^{q-1}\exp\left[\frac{-b}{x(1-x)}\right]},$$

using the definition of the extended incomplete beta function.

Conversely, suppose that

$$g(x)=\frac{B_x(p+1,q;b)}{x^{p-1}(1-x)^{q-1}\exp\left[\frac{-b}{x(1-x)}\right]}.$$

Since, by Lemma 3.1, $g'(x)=x-g(x)\frac{f'(x)}{f(x)}$, (see Shakil et al. [20]), differentiating $g(x)$ with respect to x, we have

$$g'(x)=x-g(x)\left(\frac{p-1}{x}-\frac{q-1}{1-x}+\frac{b}{x^2}\frac{1-2x}{(1-x)^2}\right),$$

from which we obtain

$$\frac{x-g'(x)}{g(x)}=\frac{p-1}{x}-\frac{q-1}{1-x}+\frac{b}{x^2}\frac{1-2x}{(1-x)^2}.$$

Now, since, by Lemma 3.1, we have

$$\frac{x-g'(x)}{g(x)}=\frac{f'(x)}{f(x)},$$

it follows that

$$\frac{f'(x)}{f(x)}=\frac{p-1}{x}-\frac{q-1}{1-x}+\frac{b}{x^2}\frac{1-2x}{(1-x)^2}.$$

Integrating the above-mentioned expression with respect to x and simplifying, we obtain

$$\ln f(x)=\ln\left(c\,x^{p-1}(1-x)^{q-1}\exp\left[\frac{-b}{x(1-x)}\right]\right)$$

or,

$$f(x)=c\,x^{p-1}(1-x)^{q-1}\exp\left[\frac{-b}{x(1-x)}\right],$$

where c is the normalizing constant to be defined. Thus, integrating the above-mentioned equation with respect to x from $x=0$ to $x=1$, and using the condition $\int_0^1 f(x)\,dx=1$, as well as the definition of the extended beta function, see Chaudhry and Zubair [11, Eq. 5.60, Page 221], we obtain $c=\frac{1}{B(p,q;b)}$. This completes the proof of Theorem 3.1.

Theorem 3.2. If the random variable X satisfies the Assumption 3.1 with $\omega=0$ and $\delta=1$, $E(X|X\geq x)=\tilde{g}(x)\frac{f(x)}{1-F(x)}$, where $\tilde{g}(x)=\frac{(E(X)-g(x)f(x))B(p,q;b)}{x^{p-1}(1-x)^{q-1}\exp\left[\frac{-b}{x(1-x)}\right]}$, where $g(x)$ is given by Eq.(3.1) and $E(X)$ is given by Eq. (2.4), if and only if

$$f_X(x)=\frac{1}{B(p,q;b)}x^{p-1}(1-x)^{q-1}\exp\left[\frac{-b}{x(1-x)}\right].$$

Proof. Suppose that $E\left(X|X\geq x\right)=\tilde{g}(x)\frac{f(x)}{1-F(x)}$. Then, since $E\left(X|X\geq x\right)=\frac{\int_x^1 u\,f(u)\,du}{1-F(x)}$, we have $\tilde{g}(x)=\frac{\int_x^1 u\,f(u)\,du}{f(x)}$. Now, if the random variable X satisfies the Assumptions 3.1 and has the distribution with the pdf (2.1), we have

$$\tilde{g}(x)=\frac{\int_x^1 u\,f(u)\,du}{f(x)}=\frac{\int_0^1 u\,f(u)\,du-\int_0^x u\,f(u)\,du}{f(x)}$$

$$=\frac{\left(E(X)-g(x)f(x)\right)B(p,q;b)}{x^{p-1}(1-x)^{q-1}\exp\left[\frac{-b}{x(1-x)}\right]}.$$

Conversely, suppose that $\tilde{g}(x)=\frac{\left(E(X)-g(x)f(x)\right)B(p,q;b)}{x^{p-1}(1-x)^{q-1}\exp\left[\frac{-b}{x(1-x)}\right]}$.

Since, by Lemma 3.2, $\left(\tilde{g}(x)\right)^{/}=-x-\tilde{g}(x)\frac{f^{/}(x)}{f(x)}$, (see Shakil et al. [20], differentiating $\tilde{g}(x)$ with respect to x, we have

$$\left(\tilde{g}(x)\right)^{/}=-x-\tilde{g}(x)\left(\frac{p-1}{x}-\frac{q-1}{1-x}+\frac{b}{x^2}\frac{1-2x}{(1-x)^2}\right),$$

From which we obtain

$$\frac{x+\left(\tilde{g}(x)\right)^{/}}{\tilde{g}(x)}=-\left(\frac{p-1}{x}-\frac{q-1}{1-x}+\frac{b}{x^2}\frac{1-2x}{(1-x)^2}\right).$$

Now, since, by Lemma 3.2, we have

$$\frac{f^{/}(x)}{f(x)}=-\frac{x+\left[\tilde{g}(x)\right]^{/}}{\tilde{g}(x)},$$

it follows that

$$\frac{f^{/}(x)}{f(x)}=\frac{p-1}{x}-\frac{q-1}{1-x}+\frac{b}{x^2}\frac{1-2x}{(1-x)^2}.$$

Integrating the above-mentioned expression with respect to x and simplifying, we obtain

$$\ln f(x) = \ln\left(c\, x^{p-1}(1-x)^{q-1} \exp\left[\frac{-b}{x(1-x)} \right] \right),$$

or,

$$f(x) = c\, x^{p-1}(1-x)^{q-1} \exp\left[\frac{-b}{x(1-x)} \right],$$

where c is the normalizing constant to be defined. Thus, integrating the above-mentioned equation with respect to x from $x=0$ to $x=1$, and using the condition $\int_0^1 f(x)\,dx = 1$, as well as the definition of the extended beta function, we obtain $c = \dfrac{1}{B(p,q;b)}$. This completes the proof of Theorem 3.2.

3.2 Characterizations by Order Statistics

If $X_1, X_2, \ldots, X_n$ are the n independent copies of the random variable X with absolutely continuous distribution function $F(x)$ and pdf $f(x)$, and if $X_{1,n} \le X_{2,n} \le \ldots \le X_{n,n}$ is the corresponding order statistics, it is known from Ahsanullah et al. [3], chapter 5, or Arnold et al. [9], chapter 2, that $X_{j,n} \mid X_{k,n} = x$, for $1 \le k < j \le n$, is distributed as the $(j-k)th$ order statistics from $(n-k)$ independent observations from the random variable V having the pdf $f_V(v \mid x)$ where $f_V(v \mid x) = \dfrac{f(v)}{1-F(x)}$, $0 \le v < x$, and $X_{i,n} \mid X_{k,n} = x, 1 \le i < k \le n$, is distributed as ith order statistics from k independent observations from the random variable W having the pdf $f_W(w \mid x)$ where $f_W(w \mid x) = \dfrac{f(w)}{F(x)}$, $w < x$. Let $S_{k-1} = \frac{1}{k-1}\left(X_{1,n} + X_{2,n} + \ldots + X_{k-1,n}\right)$, and $T_{k,n} = \frac{1}{n-k}\left(X_{k+1,n} + X_{k+2,n} + \ldots + X_{n.n}\right)$.

Theorem 3.3. Suppose the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = 1$, then $E(S_{k-1} \mid X_{k,n} = x) = g(x)\tau(x)$, where $\tau(x) = \dfrac{f(x)}{F(x)}$ and $g(x) = \dfrac{B_x(p+1,q;b)}{x^{p-1}(1-x)^{q-1}\exp\left[\frac{-b}{x(1-x)}\right]}$,

if and only if X has the pdf $f_X(x) = \dfrac{1}{B(p,q;b)} x^{p-1}(1-x)^{q-1} \exp\left[\dfrac{-b}{x(1-x)} \right]$.

Proof. It is known that $E(S_{k-1} \mid X_{k,n} = x) = E(X \mid X \le x)$; see Ahsanullah et al. [3], and David and Nagaraja [12]. Hence, by Theorem 3.1, the result follows.

Theorem 3.4. Suppose the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = 1$, then

$E(T_{k,n} \mid X_{k,n} = x) = \tilde{g}(x)\dfrac{f(x)}{1-F(x)}$, where $\tilde{g}(x) = \dfrac{\left(E(X) - g(x)f(x)\right)B(p,q;b)}{x^{p-1}(1-x)^{q-1}\exp\left[\dfrac{-b}{x(1-x)}\right]}$ if and only if $f_X(x) = \dfrac{1}{B(p,q;b)}x^{p-1}(1-x)^{q-1}\exp\left[\dfrac{-b}{x(1-x)}\right]$.

Proof. Since $E(T_{k,n} \mid X_{k,n} = x) = E(X \mid X \geq x)$, see Ahsanullah et al. [3], and David and Nagaraja [12], the result follows from Theorem 3.2.

3.2 Characterization by Upper Record Values

For further details on record values, see Ahsanullah [1]. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed absolutely continuous random variables with distribution function $F(x)$ and pdf $f(x)$. If $Y_n = \max(X_1, X_2, \ldots, X_n)$ for $n \geq 1$ and $Y_j > Y_{j-1}, j > 1$, X_j is called an upper record value of $\{X_n,\ n \geq 1\}$. The indices at which the upper records occur are given by the record times $\{U(n) > \min(j \mid j > U(n+1), X_j > X_{U(n-1)}, n > 1)\}$ and $U(1) = 1$. Let the nth upper record value be denoted by $X(n) = X_{U(n)}$.

Theorem 3.5. Suppose the random variable X satisfies the Assumption 3.1 with $\omega = 0$ and $\delta = 1$, then $E(X(n+1) \mid X(n) = x) = \tilde{g}(x)\dfrac{f(x)}{1-F(x)}$, where

$$\tilde{g}(x) = \frac{\left(E(X) - g(x)f(x)\right)B(p,q;b)}{x^{p-1}(1-x)^{q-1}\exp\left[\dfrac{-b}{x(1-x)}\right]},$$

if and only if $f_X(x) = \dfrac{1}{B(p,q;b)}x^{p-1}(1-x)^{q-1}\exp\left[\dfrac{-b}{x(1-x)}\right]$.

Proof. It is known from Ahsanullah et al. [3], and Nevzorov [19] that $E(X(n+1) \mid X(n) = x) = E(X \mid X \geq x)$. Then, the result follows from Theorem 3.2.

4 Conclusion

In this paper, we have considered the three-parameter extended beta distribution introduced by Chaudhry et al. [10]. Some basic distributional properties are presented. Based on these distributional properties, we have established some new characterization results of the extended beta distribution of Chaudhry et al. [10] by truncated moment, order statistics and upper record values. We hope the findings of the paper will be beneficial for the practitioners in various fields of sciences.

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