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Quasi Variational Inclusions Involving Three Operators

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Abstract: In this paper, we consider some new classes of the quasi-variational inclusions involving three monotone operators. Some interesting problems such as variational inclusions involving sum of two monotone operators, difference of two monotone operators, system of absolute value equations, hemivariational inequalities and variational inequalities are the special cases of quasi variational inequalities. It is shown that quasi-variational inclusions are equivalent to the implicit fixed point problems. Some new iterative methods for solving quasi-variational inclusions and related optimization problems are suggested by using resolvent methods, resolvent equations and dynamical systems coupled with finite difference technique. Convergence analysis of these methods is investigated under monotonicity. Some special cases are discussed as applications of the main results.

Keywords: Quasi variational inclusions, iterative resolvent methods, resolvent equations, dynamical system, nonexpansive mappings,convergence criteria.
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1 Introduction

Variational inclusion theory contains a wealth of new ideas and techniques. which can be viewed as a novel extension and generalization of the variational inequalities. The origin of the variational inequalities can be traced back to Stampacchia [1] in potential theory. It is amazing that a wide class of unrelated problems, which occur in various fields of pure and applied sciences, can be studied in the general and unified framework of variational inclusions. See [1,2,3,4,5,6,7,8,9,10,11,12, 13,14,15,16,17,18].

Noor and Noor [6,7] proved that the quasi variational inclusions are equivalent to the implicit fixed point problem. This equivalent formulation played an important role in developing numerical methods, sensitivity analysis, dynamical systems and other aspects of quasi variational inclusions. The resolvent equations were introduced and studied by Noor [6,7]. This technique has been used to study the existence of a solution as well as to develop various iterative methods for solving the variational inclusions. Noor and Noor [9] have proved that quasi-variational inclusions are equivalent to the resolvent equations. This equivalence has been used to study the existence, stability and sensitivity analysis of the solution of variational inclusions.

In this paper, we consider a new class of quasi variational inclusions involving involving three monotone operators. It is known that quasi variational inclusions are equivalent to fixed point problem. We used this alternative form to suggest and investigate some new implicit and explicit iterative methods for solving nonlinear quasi variational inclusions. The convergence criteria of the proposed implicit methods is discussed under some mild conditions. Several important special cases are discussed as applications of our results. It is expected the techniques and ideas of this paper may be starting point for further research.

2 Formulations and basic facts

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let $\mathscr{T}, \mathscr{B} : \mathscr{H} \rightarrow \mathscr{H}$ be nonlinear operators and $\mathscr{A}(.,.) : \mathscr{H} \times \mathscr{H} \rightarrow \mathscr{R} \cup \{+\infty\}$ be a maximal monotone operator with respect to the first argument.

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We consider the problem of finding $\mu \in \mathscr{H}$ such that

$$0 \in \mathscr{T}\mu + \mathscr{B}(\mu) + \mathscr{A}(\mu,\mu). \tag{2.1}$$

Inclusion of type (2.1) is called the quasi variational inclusion involving three operators, We would like to emphasize the operator $\mathscr{T}$ is strongly monotone, the operator $\mathscr{B}$ is Lipschitz continuous and the bifunction $\mathscr{A}(.,.)$ is maximal monotone operator with respect to first variable. Several important problems arising in pure and applied sciences can be studied in the frame wrok of the form (2.1).

We now discuss several important and interesting problems, which can be deduced from the problem (2.1).

(I). If $\mathscr{A}(\mu,\mu) = \mathscr{A}(u)$, then problem (2.1) reduces to finding $\mu \in \mathscr{H}$ such that

$$0 \in \mathscr{T}\mu + \mathscr{B}(\mu) + \mathscr{A}(\mu), \tag{2.2}$$

which is known as finding zeros of the sum of three operators.
(II). If $\mathscr{B} = 0$ and $\mathscr{A}(\mu,\mu) = \mathscr{A}(u)$, the problem (2.1) collapse to finding $\mu \in \mathscr{H}$ such that

$$0 \in \mathscr{T}\mu + \mathscr{A}(\mu), \tag{2.3}$$

is known as finding zeros of the sum of two monotone operators and have been studied extensively in recent years.

(III). If $\mathscr{A}(\mu,\mu) = 0, \quad \mathscr{B} = -\mathscr{D}$, the problem (2.1) collapses to finding $\mu \in \mathscr{H}$ such that

$$0 \in \mathscr{T}\mu - \mathscr{D}(\mu), \tag{2.4}$$

which is called the problem of finding zeros of the difference of two operators, introduced and studied by Noor et al.[19]. Problem (2.4) can be interpreted as variational inclusion involving difference of two monotone operators, which is itself a very difficult problem. This problem can be viewed as a problem of finding the the minimum of two difference of convex functions, known DC-problem. Such type of problems have applications in optimization theory and imaging process in medical sciences and earthquake.

(IV). If $\mathscr{A}(.,.) = \partial\varphi(.,.)$, where $\partial\varphi(.,.)$ is the subdifferential of a proper, convex and lower-semicontinuous function

$$\varphi(.,.) : \mathscr{H} \times \mathscr{H} \to \mathscr{R} \cup \{+\infty\}$$

with respect to the first argument, then problem (2.1) is equivalent to finding $\mu \in \mathscr{H}$ such that.

$$\langle \mathscr{T}\mu + \mathscr{B}(\mu), \nu - \mu\rangle + \varphi(\nu,\mu) - \varphi(\mu,\mu) \geq 0, \quad \forall \mu \in \mathscr{H}. \tag{2.5}$$

The problem of the type (2.5) is called the mixed quasi variational inequality problem, which has many important and significant applications in regional, physical, mathematical, pure and applied sciences, see [2,3,6,7,8, 9,20].
(V). If $\varphi(,.,)$ is the indicator function of a closed convex multivalued function $\Omega(\mu)$ in $\mathscr{H}$, then problem (2.5) is equivalent to finding $\mu \in \Omega(\mu)$ such that

$$\langle \mathscr{T}\mu + \mathscr{B}(\mu), \nu - \mu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu), \tag{2.6}$$

which is called the nonlinear quasi variational inequalities, see Noor and Noor [10].
(VI). If $\Omega^*(\mu) = \{\mu \in \mathscr{H} : \langle \mu, \nu\rangle \geq 0, \quad \forall \nu \in \Omega(\mu)\}$ is a polar (dual) cone of a convex-valued cone $\Omega(\mu)$ in $\mathscr{H}$, then problem (2.6) is equivalent to finding μ. such that

$$\mu \in \Omega(\mu), \quad \mathscr{T}\mu + \mathscr{B}(\mu) \in \Omega^*(\mu) \quad \text{and} \quad \langle \mathscr{T}\mu - \mathscr{B}(\mu), \mu\rangle = 0, \tag{2.7}$$

which is known as the strongly nonlinear quasi complementarity problems [10,11]. Obviously strongly complementarity problems include the complementarity problems and their variant forms. For applications and numerical methods, see Cottle et al [26] and Noor[23,24, 25,26] in game theory, management sciences and quadratic programming as special cases.
(VII). If $\Omega(\mu) = \Omega$, where Ω is a convex set in $\mathscr{H}$, then problem (2.6) reduces to finding $\mu \in \Omega$ such that

$$\langle \mathscr{T}\mu + \mathscr{B}(\mu), \nu - \mu\rangle \geq 0, \quad \forall \nu \in \Omega, \tag{2.8}$$

which is known as the mildly nonlinear variational inequalities, introduced and studied by Noor [4,10].
(VIII). The problem (2.8) includes the absolute value equations, which is being investigated extensively in recent years using quite different techniques and ideas. To be more precise, take $\Omega = \mathscr{H}, \quad \mathscr{B}(\mu) = \mathscr{B}|\mu|$, then one easily show that problem (2.8) is equivalent to finding $\mu \in \mathscr{H}$ such that

$$\mathscr{T}u + \mathscr{B}|\mu| = b, \tag{2.9}$$

which is called the absolute value equations, where b is a given data. Clearly, system of absolute value equations is a very important special case of strongly nonlinear variational inequalities, which were introduced by Noor [4].

(IX), If $\varphi(.,\mu) \equiv \varphi(\mu), \quad \forall \mu \in \mathscr{H}$, then problem (2.5) is equivalent to finding $u \in H$ such that

$$\langle \mathscr{T}\mu + \mathscr{B}(\mu), \nu - \mu\rangle + \varphi(\nu) - \varphi(\mu) \geq 0, \quad \forall \mu \in \mathscr{H}, \tag{2.10}$$

which is known as the mixed varaitional inequality. For the applications, sensitivity analysis and numerical methods, see[2,3-7,11-20].
(X). If $\langle \mathscr{B}(\mu), \nu\rangle = -\mathscr{B}(\mu,\nu)$, then (2.8) collapses to finding $\mu \in \Omega(\mu)$ such that

$$\langle \mathscr{T}\mu, \nu - \mu\rangle \geq \mathscr{B}(\mu, \nu - \mu), \quad \forall \nu \in \Omega(\mu),$$

which is called nonlinear quasi hemivariational inequality and appears to be a new one.
If $\Omega(\mu) = \Omega$, then the problem (2.11) becomes the

hemivariational inequality, which was introduced by Panagiotopoulos [36] in structural analysis. For the applications, formulation and other aspects of hemivariational inequalities, see [33,36] and the references therein.

(XI). If φ is the indicator function of a closed convex set Ω in $\mathscr{H}$, then problem (2.10) is equivalent to finding $u \in \Omega$ such that

$$\langle \mathscr{T}\mu + \mathscr{B}(\mu), \nu - \mu >\geq 0, \quad \forall \nu \in \Omega, \tag{2.11}$$

which is called the mildly nonlinear variational inequality introduced and studied by Noor [4] in 1975.

(XII). If $\mathscr{B} = 0$, then problem (2.11) is equivalent to finding $u \in \Omega$ such that

$$\langle \mathscr{T}\mu, \nu - \mu >\geq 0, \quad \forall \nu \in \Omega, \tag{2.12}$$

which is called the classical variational inequality introduced and studied by Stampacchia [24] in 1964. For recent applications, numerical methods, sensitivity analysis and physical formulations, see [1-28] and the references therein.

*Remark.*It is worth mentioning that for appropriate and suitable choices of the operators $\mathscr{T}$, $\mathscr{B}$, $\mathscr{A}$, set-valued convex set $\Omega(\mu)$ and the spaces, one can obtain several classes of variational inequalities, complementarity problems and optimization problems as special cases of the nonlinear quasi variational inclusion (2.1). This shows that the problem (2.1) is quite general and unifying one. It is interesting problem to develop efficient and implementable numerical methods for solving the nonlinear quasi-variational inclusions and their variant forms.

It is well known that the resolvent operator $\mathscr{J}_{\mathscr{A}(\mu)}$ is required to satisfy the following assumption, which plays an important part in the derivation of the results..

Assumption 1

$$\|\mathscr{J}_{\mathscr{A}(\mu)}\omega - \mathscr{J}_{\mathscr{A}(\mu)}\omega\| \leq \nu\|\mu - \nu\|, \forall \mu, \nu, \omega \in \mathscr{H} \tag{2.13}$$

Definition 1. *An operator $\mathscr{T} : \mathscr{H} \to \mathscr{H}$ is said to be:*

1. *Strongly monotone, if there exist a constant $\alpha > 0$, such that*
   $$\langle \mathscr{T}\mu - \mathscr{T}\nu, \mu - \nu\rangle \geq \alpha\|\mu - \nu\|^2, \quad \forall \mu, \nu \in \mathscr{H}.$$
2. *Lipschitz continuous, if there exist a constant $\beta > 0$, such that*
   $$\|\mathscr{T}\mu - \mathscr{T}\nu\| \leq \beta\|\mu - \nu\|, \quad \forall \mu, \nu \in \mathscr{H}.$$
3. *Monotone, if*
   $$\langle \mathscr{T}\mu - \mathscr{T}\nu, \mu - \nu\rangle \geq 0, \quad \forall \mu, \nu \in \mathscr{H}.$$
4. *Pseudo monotone, if*
   $$\langle \mathscr{T}\mu, \nu - \mu\rangle \geq 0 \quad \Rightarrow \quad \langle \mathscr{T}\nu, \nu - \mu\rangle \geq 0, \quad \forall \mu, \nu \in .$$

*Remark.*Every strongly monotone operator is a monotone operator and monotone operator is a pseudo monotone operator, but the converse is not true.

3 Iterative resolvent methods

In this section, we prove that the quasi variational inclusions are equivalent to the fixed point problems. This alternative equivalent formulations are used to suggest some new iterative methods for solving the quasi variational inclusions.

Lemma 1.[9] *The function $\mu \in \mathscr{H}$ is a solution of the quasi variational inclusion (2.1), if and only if, $\mu \in \mathscr{H}$ satisfies the relation*

$$\mu = \mathscr{J}_{\mathscr{A}(\mu)}[\mu - \rho(\mathscr{T}\mu + \mathscr{B}(\mu))], \tag{3.14}$$

where $\mathscr{J}_{\mathscr{A}(u)}$ is the resolvent operator and $\rho > 0$ is a constant.

Lemma 1 implies that the quasi variational inclusion (2.1) is equivalent to the fixed point problem (3.14). We consider this alternative equivalent formulation (3.14) to suggest the several implicit iterative methods for solving the problem (2.1).

Algorithm 1 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_n)}[\mu_n - \rho(\mathscr{T}\mu_n + \mathscr{B}(\mu_n))],$$

which is known as the projection method and has been studied extensively.

Algorithm 2 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_{n+1})}[\mu_n - \rho(\mathscr{T}\mu_{n+1} + \mathscr{B}(\mu_{n+1}))].$$

which is known as the implicit resolvent method and is equivalent to the following two-step method.

Algorithm 3 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}\omega_n &= \mathscr{J}_{\mathscr{A}(\mu_n)}[\mu_n - \rho(\mathscr{T}\mu_n + \mathscr{B}(\mu_n))]\\ \mu_{n+1} &= \mathscr{J}_{\mathscr{A}(\omega_n)}[\mu_n - \rho(\mathscr{T}\omega_n + \mathscr{B}(\omega_n))].\end{aligned}$$

Algorithm 4 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_{n+1})}[\mu_{n+1} - \rho(\mathscr{T}\mu_{n+1} + \mathscr{B}(\mu_{n+1}))],$$

which is known as the modified resolvent method and is equivalent to the iterative method.

Algorithm 5 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned}\omega_n &= \mathscr{J}_{\mathscr{A}(\mu_n)}[\mu_n - \rho(\mathscr{T}u_n + \mathscr{B}(\mu_n))]\\ \mu_{n+1} &= \mathscr{J}_{\mathscr{A}(\omega_n)}[\omega_n - \rho(\mathscr{T}\omega_n\\ &\quad + \mathscr{B}(\omega_n))],\end{aligned}$$

which is two-step predictor-corrector method for solving the problem (2.1).

We can rewrite the equation (3.14) as:

$$\mu = \mathscr{J}_{\mathscr{A}(\mu)}[\frac{\mu+\mu}{2} - \rho\mathscr{T}\mu - \rho\mathscr{B}(\mu)].$$

This fixed point formulation is used to suggest the following implicit method.

Algorithm 6 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_{n+1})}[\frac{\mu_n+\mu_{n+1}}{2} - \rho\mathscr{T}\mu_{n+1} - \rho\mathscr{B}(\mu_{n+1})],$$

Algorithm 7 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\omega_n = \mathscr{J}_{\mathscr{A}(\mu_n)}[\mu_n - \rho\mathscr{T}\mu_n - \rho\mathscr{B}(\mu_n)]$$
$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\omega_n)}[\frac{\omega_n+\mu_n}{2} - \rho\mathscr{T}\omega_n - \rho\mathscr{B}(\omega_n)], \quad \lambda$$

From equation (3.14), we have

$$\mu = \mathscr{J}_{\mathscr{A}(\mu)}[\mu - \rho\mathscr{T}(\frac{\mu+\mu}{2}) - \rho\mathscr{B}(\frac{\mu+\mu}{2})]. \tag{3.15}$$

This fixed point formulation (3.15) is used to suggest the implicit method for solving the problem (2.1) as

Algorithm 8 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_{n+1})}[\mu_n - \rho\mathscr{T}(\frac{\mu_n+\mu_{n+1}}{2}) - \rho\mathscr{B}(\frac{\mu_n+\mu_{n+1}}{2})],$$

We can use the predictor-corrector technique to rewrite Algorithm 8 as:

Algorithm 9 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\omega_n = \mathscr{J}_{\mathscr{A}(\mu_n)}[\mu_n - \rho\mathscr{T}\mu_n - \rho\mathscr{B}(\mu_n)],$$
$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\omega_n)}[\mu_n - \rho\mathscr{T}(\frac{\mu_n+\omega_n}{2}) - \rho\mathscr{B}(\frac{\mu_n+\omega_n}{2})],$$

is known as the mid-point implicit method for solving the problem (2.1).

We again use the above fixed formulation to suggest the following implicit iterative method.

Algorithm 10 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_{n+1})}[\mu_{n+1} - \rho\mathscr{T}(\frac{\mu_n+\mu_{n+1}}{2}) - \rho\mathscr{B}(\frac{\mu_n+\mu_{n+1}}{2})].$$

Using the predictor-corrector technique, Algorithm 9 can be written as:

Algorithm 11 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\omega_n = \mathscr{J}_{\mathscr{A}(\mu_n)}[\mu_n - \rho\mathscr{T}\mu_n - \rho\mathscr{B}(\mu_n)],$$
$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\omega_n)}[\omega_n - \rho\mathscr{T}(\frac{\mu_n+\omega_n}{2}) - \rho\mathscr{B}(\frac{\mu_n+\omega_n}{2})].$$

which appears to be new one.

It is obvious that Algorithm 3 and Algorithm 4 have been suggested using different variant of the fixed point formulations (3.14). It is natural to combine these fixed point formulation to suggest a hybrid implicit method for solving the problem (2.1) and related optimization problems, which is the main motivation of this paper.

One can rewrite (3.14) as

$$\mu = \mathscr{J}_{\mathscr{A}(\mu)}[\frac{\mu+\mu}{2} - \rho\mathscr{T}(\frac{\mu+\mu}{2}) - \rho\mathscr{B}(\frac{\mu+\mu}{2})]. \tag{3.16}$$

This equivalent fixed point formulation enables us to suggest the following implicit method for solving the problem (2.1).

Algorithm 12 *For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_{n+1})}\left[\frac{\mu_n+\mu_{n+1}}{2} - \rho\mathscr{T}(\frac{\mu_n+\mu_{n+1}}{2}) - \rho\mathscr{B}(\frac{\mu_n+\mu_{n+1}}{2})\right].$$

To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 4 as the predictor and Algorithm 12 as corrector. Thus, we obtain a new two-step method for solving the the problem (2.1).

Algorithm 13 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\omega_n = \mathscr{J}_{\mathscr{A}(\mu_n)}[\mu_n - \rho\mathscr{T}\mu_n - \rho\mathscr{B}(\mu_n)]$$
$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\omega_n)}[(\frac{\omega_n+\mu_n}{2}) - \rho\mathscr{T}(\frac{\omega_n+\mu_n}{2}) - \rho\mathscr{B}(\frac{\omega_n+\mu_n}{2})].$$

which is a new predictor-corrector two-step method.

For a parameter ξ, one can rewrite the (3.14) as

$$\mu = \mathscr{J}_{\mathscr{A}(\mu)}(1-\xi)\mu + \xi\mu - \rho\mathscr{T}\mu - \rho\mathscr{B}(\mu)].$$

This equivalent fixed point formulation enables to suggest the following inertial method for solving the the problem (2.1).

Algorithm 14 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_n)}[(1-\xi)\mu_n + \xi\mu_{n-1} - \rho\mathscr{T}\mu_n - \rho\mathscr{B}(\mu_n)].$$

It is noted that Algorithm 14 is equivalent to the following two-step method.

Algorithm 15 *For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme*

$$\omega_n = (1-\xi)u_n + \xi u_{n-1}$$
$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_n)}[\omega_n - \rho\mathscr{T}\mu_n - \rho\mathscr{B}(u_n)].$$

Using this idea, we can suggest the following iterative methods for solving nonlinear quasi variational inequalities.

Algorithm 16 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\omega_n = (1-\xi)u_n + \xi u_{n-1}$$
$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_n)}[\omega_n - \rho\mathscr{T}\omega_n - \rho\mathscr{B}(\omega_n)], \quad n = 0,1,2,\ldots.$$

Algorithm 17 *For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme*

$$\omega_n = (1-\alpha)u_n + \alpha u_{n-1}$$
$$u_{n+1} = \mathscr{J}_{\mathscr{A}(\omega_n)}[y_n - \rho\mathscr{T}\omega_n - \rho\mathscr{B}(\omega_n)].$$

Using the technique of Noor et al. [12] and Jabeen et al [30], one can investigate the convergence analysis of these inertial resolvent methods.

4 Resolvent equations technique

In this section, we discuss the resolvent equations associated with the quasi variational inclusions (2.1). It is worth mentioning that the resolvent equations associated with variational inclusions were introduced and studied by Noor [6] and proved that the quasi variational inclusions are equivalent to the implicit resolvent equations to study the sensitivity analysis.

Related to the quasi variational inclusion (2.1), we consider the problem of finding $z, \mu \in \mathscr{H}$ such that

$$\mathscr{T}\mathscr{J}_{\mathscr{A}(\mu)}z + \rho^{-1}\mathscr{R}_{\mathscr{A}(\mu)}z = \mathscr{B}\mathscr{J}_{\mathscr{A}(\mu)}z, \tag{4.17}$$

where $\rho > 0$ is a constant and $\mathscr{R}_{\mathscr{A}(\mu)} = I - \mathscr{J}_{\mathscr{A}(\mu)}$. Here I is the identity operator and $\mathscr{J}_\varphi = (1+\rho\mathscr{A}(\mu))^{-1}$ is the resolvent operator. The equation of the type (4.17) are called the implicit resolvent equations. For the formulation and applications of the resolvent equations, see Noor [6,7] and Noor et al [9,10].

Lemma 2. *The quasi variational inclusion (2.1) has a solution $\mu \in \mathscr{H}$, if and only if, the resolvent equations (4.17) have a solution $z, \mu \in \mathscr{H}$, where*

$$\mu = \mathscr{J}_{\mathscr{A}(\mu)}z \tag{4.18}$$

and

$$z = \mu - \rho(\mathscr{T}\mu - \mathscr{B}(\mu)). \tag{4.19}$$

Proof:

*Proof.*Let $\mu \in \mathscr{H}$ be a solution of (2.1), then, for a constant $\rho > 0$,

$$\rho\mathscr{T}\mu - \rho\mathscr{B}(\mu) + \rho\mathscr{A}(\mu,\mu) \ni 0,$$
$$\Longleftrightarrow$$
$$-\mu + \rho\mathscr{T}\mu - \rho\mathscr{B}(\mu) \;+\; \mu + \rho\mathscr{A}(\mu,\mu) \ni 0$$
$$\Longleftrightarrow$$
$$\mu = \mathscr{J}_{\mathscr{A}(\mu)}[\mu - \rho\mathscr{T}\mu + \rho\mathscr{B}(\mu)].$$

Take $z = \mu - \rho\mathscr{T}\mu + \rho\mathscr{B}(\mu)$, then $z = \mathscr{J}_{\mathscr{A}(\mu)}z$. Thus

$$z = \mathscr{J}_{\mathscr{A}(u)}z - \rho\mathscr{T}_{J_{A(u)}}z + \rho\mathscr{B}_{J_{A(u)}}z,$$

that is

$$\mathscr{T}\mathscr{J}_{\mathscr{A}(\mu)}z + \rho^{-1}\mathscr{R}_{\mathscr{A}(\mu)}z = \mathscr{B}\mathscr{J}_{\mathscr{A}(\mu)}z,$$

the required (4.17).

From Lemma 2, it follows that the quasi variational inclusion (2.1) and the resolvent equations (4.17) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving the quasi variational inclusions and related optimization problems. We use the resolvent equations (4.17) to suggest some new iterative methods for solving the quasi variational inclusions. From (4.18) and (4.19), we have

$$\begin{aligned} z &= \mathscr{J}_{\mathscr{A}(\mu)}z - \rho\mathscr{T}\mathscr{J}_{\mathscr{A}(\mu)}z + \rho\mathscr{B}(\mathscr{J}_{\mathscr{A}(\mu)}z) \\ &= \mathscr{J}_{\mathscr{A}(\mu)}[\mu - \rho\mathscr{T}\mu + \rho\mathscr{B}(\mu)] \\ &\quad -\rho\mathscr{T}\mathscr{J}_{\mathscr{A}(\mu)}[\mu - \rho\mathscr{T}\mu + \rho\mathscr{J}_{\mathscr{A}(\mu)}(\mu)]. \end{aligned}$$

Thus, we have

$$\begin{aligned} \mu &= \rho\mathscr{T}\mu - \rho\mathscr{B}(\mu) + [\mathscr{J}_{\mathscr{A}(\mu)}[\mu - \rho\mathscr{T}\mu \\ &\quad + \rho\mathscr{B}(\mu)] - \rho\mathscr{T}\mathscr{J}_{\mathscr{A}(\mu)}[\mu - \rho\mathscr{T}\mu + \rho\mathscr{B}(\mu)]. \end{aligned}$$

Consequently, for a constant $\alpha_n > 0$, we have

$$\begin{aligned} \mu &= (1-\alpha_n)\mu \\ &\quad + \alpha_n\mathscr{J}_{\mathscr{A}(\mu)}\{\Pi_{\Omega(\mu)}[\mu - \rho\mathscr{T}u + \rho\mathscr{B}(\mu)] \\ &\quad + \rho\mathscr{T}\mu - \rho\mathscr{B}(\mu) \\ &\quad - \rho\mathscr{T}\mathscr{J}_{\mathscr{A}(\mu)}[\mu - \rho\mathscr{T}\mu + \rho\mathscr{B}(\mu)]\} \\ &= (1-\alpha_n)\mu \\ &\quad + \alpha_n\mathscr{J}_{\mathscr{A}(\mu)}\{\omega - \rho\mathscr{T}\omega + \rho\mathscr{T}\mu - \rho\mathscr{B}(\mu)\}, \end{aligned} \tag{4.20}$$

where

$$\omega = \mathscr{J}_{\mathscr{A}(\mu)}[\mu - \rho\mathscr{T}\mu + \rho\mathscr{B}(\mu)]. \tag{4.21}$$

Using (4.20) and (4.21), we can suggest the following new predictor-corrector method for solving the quasi variational inclusion (2.1).

Algorithm 18 *For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mathcal{J}_{\mathcal{A}(\mu)}[\mu_n - \rho \mathcal{T}\mu_n + \rho \mathcal{B}(\mu_n)] \\ \mu_{n+1} &= (1-\alpha_n)\mu_n \\ &\quad + \alpha_n \mathcal{J}_{\mathcal{A}(\mu)}\left\{\omega_n - \rho \mathcal{T}\omega_n - \rho \mathcal{B}(\mu_n) + \rho \mathcal{T}\mu_n\right\}. \end{aligned}$$

If $\alpha_n = 1$, then Algorithm **??** reduces to

Algorithm 19 *For a given $\mu_0 \in \mathcal{H}$, compute μ_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mathcal{J}_{\mathcal{A}(\mu)}[\mu_n - \rho \mathcal{T}\mu_n + \rho \mathcal{B}(\mu)] \\ \mu_{n+1} &= \mathcal{J}_{\mathcal{A}(\mu)}[\omega_n - \rho \mathcal{T}\omega_n + \rho \mathcal{T}\mu_n - \rho \mathcal{B}(\mu_n)], \end{aligned}$$

which appears to be a new one.
In a similar way, we can suggest and analyse the predictor-corrector inertial method for solving the quasi variational inclusion(2.1), which involve only one resolvent.

Algorithm 20 *For given $u_0, u_1 \in H$, compute u_{n+1} by the iterative scheme*

$$\begin{aligned} \omega_n &= \mu_n - \xi(\mu_n - \mu_{n-1}) \\ \mu_{n+1} &= \mathcal{J}_{\mathcal{A}(\mu)}[\omega_n - \rho \mathcal{T}\omega_n + \rho \mathcal{T}\mu_n - \rho \mathcal{B}(\mu_n)]. \end{aligned}$$

One can study the convergence of the Algorithm 20 using the technique of Noor et al [12] and Jabeen et al [30].

*Remark.*We have only given some glimpse of the technique of the resolvent equations for solving the quasi variational inclusions. One can explore the applications of the resolvent equations in developing efficient numerical methods for variational inclusions and related nonlinear optimization problems.

5 Dynamical Systems Technique

In this section, we consider the dynamical systems technique for solving quasi variational inclusions. Dupuis and Nagurney [29] introduced and studied dynamical systems associated with variational inequalities using the fixed point problems. Thus it is clear that the variational inequalities are equivalent to a first order initial value problem. Consequently, equilibrium and nonlinear problems arising in various branches in pure and applied sciences can now be studied in the setting of dynamical systems. It has been shown that the dynamical systems are useful in developing some efficient numerical techniques for solving variational inequalities and related optimization problems. We consider some iterative methods for solving the quasi variational inclusions. We investigate the convergence analysis of these new methods involving only the monotonicity of the operator.

We now define the residue vector $R(\mu)$ by the relation

$$R(\mu) = \mu - \mathcal{J}_{\mathcal{A}(\mu)}[\mu - \rho \mathcal{T}\mu + \rho \mathcal{B}(\mu)]. \tag{5.22}$$

Invoking Lemma 1, one can easily conclude that $\mu \in \mathcal{H}$ is a solution of the problem (2.1), if and only if, $\mu \in \mathcal{H}$ is a zero of the equation

$$R(\mu) = 0. \tag{5.23}$$

We now consider a dynamical system associated with the quasi variational inequalities. Using the equivalent formulation (1), we suggest a class of resolvent dynamical systems as

$$\begin{aligned} \frac{d\mu}{dt} &= \lambda\{\mathcal{J}_{\mathcal{A}(\mu)}[\mu - \rho \mathcal{T}u + \rho \mathcal{B}(\mu)] - \mu\}, \\ \mu(t_0) &= \alpha, \end{aligned} \tag{5.24}$$

where λ is a parameter. The system of type (5.22) is called the resolvent dynamical system associated with the problem (2.1). Here the right hand is related to the projection and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in $\mathcal{H}$. This implies that the qualitative results such as the existence, uniqueness and continuous dependence of the solution of (5.22) can be studied.

We use the resolvent dynamical system (5.22) to suggest some iterative for solving the quasi variational inclusion (2.1). These methods can be viewed in the sense of Noor [14] involving the double resolvent.

For simplicity, we take $\lambda = 1$. Thus the dynamical system (5.22) becomes

$$\begin{aligned} \frac{d\mu}{dt} + \mu &= \mathcal{J}_{\mathcal{A}(\mu)}[\mu - \rho \mathcal{T}u + \rho \mathcal{B}(\mu)], \\ \mu(t_0) &= \alpha. \end{aligned} \tag{5.25}$$

The forward difference scheme is used to construct the implicit iterative method.
Discretizing (5.25), we have

$$\frac{\mu_{n+1} - \mu_n}{h} + \mu_{n+1} = \mathcal{J}_{\mathcal{A}(\mu_n)}[\mu_n - \rho \mathcal{T}\mu_{n+1} + \rho \mathcal{B}(\mu_{n+1})],$$

where h is the step size.
Now, we can suggest the following implicit iterative method for solving the quasi variational inclusion (2.1).

Algorithm 21 *For a given μ_0,, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mathcal{J}_{\mathcal{A}(\mu_{n+1})}\left[\mu_n - \rho \mathcal{T}\mu_{n+1} + \rho \mathcal{B}(\mu_{n+1}) - \frac{\mu_{n+1} - \mu_n}{h}\right],$$

This is an implicit method, which is quite different from the implicit method of [4].
Algorithm 21 is equivalent to the following two-step method.

Algorithm 22 *For a given μ_0,, compute μ_{n+1} by the iterative scheme*

$$\omega_n = \mathscr{J}_{\mathscr{A}(\mu_n)}[\mu_n - \rho\mathscr{T}\mu_n + \rho\mathscr{B}(\mu_n)]$$
$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\omega_n)}\left[\mu_n - \rho\mathscr{T}\omega_n + \rho\mathscr{B}(\omega_n) - \frac{\omega_n - \mu_n}{h}\right].$$

Discretizing (5.26), we now suggest an other implicit iterative method for solving (2.1).

$$\frac{\mu_{n+1} - \mu_n}{h} + \mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_{n+1})}[\mu_{n+1} - \rho\mathscr{T}\mu_{n+1} + \rho\mathscr{B}(\mu_{n+1})],$$

where h is the step size.
This formulation enables us to suggest the two-step iterative method.

Algorithm 23 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\omega_n = \mathscr{J}_{\mathscr{A}(\mu_n)}[\mu_n - \rho\mathscr{T}\mu_n + \rho\mathscr{B}(\mu_n)]$$
$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\omega_n)}\left[\omega_n - \rho\mathscr{T}\omega_n + \rho\mathscr{B}(\omega_n) - \frac{\omega_n - \mu_n}{h}\right].$$

Again using the project dynamical systems, we can suggested some iterative methods for solving the quasi variational inclusions and related optimization problems.

Algorithm 24 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_{n+1})}\left[\frac{(h+1)\mu_n - \mu_{n+1}}{h} - \rho\mathscr{T}\mu_n + \rho\mathscr{B}(\mu_n)\right].$$

or equivalently

Algorithm 25 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\omega_n = \mathscr{J}_{\mathscr{A}(\mu_n)}[\mu_n - \rho\mathscr{T}\mu_n + \rho\mathscr{A}(\mu_n)]$$
$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\omega_n)}\left[\frac{(h+1)\mu_n - \omega_n}{h} - \rho\mathscr{T}u_n + \rho\mathscr{A}(\mu_n)\right].$$

Discretizing (5.24), we have

$$\frac{\mu_n - \mu_{n-1}}{h} + \mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_{n+1})}[\mu_n - \rho\mathscr{T}\mu_{n+1} + \rho\mathscr{B}(\mu_{n+1})],$$

where h is the step size.
This helps us to suggest the following implicit iterative method for solving the problem (2.1).

Algorithm 26 *For a given $\mu_0 \in \mathscr{H}$, compute μ_{n+1} by the iterative scheme*

$$\omega_n = \mathscr{J}_{\mathscr{A}(\mu_n)}[\mu_n - \rho\mathscr{T}\mu_n + \rho\mathscr{B}(\mu_n)]$$
$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\omega_n)}\left[\frac{(h+1)\mu_n - \omega_n}{h} - \rho\mathscr{T}\mu_n + \rho\mathscr{B}(\mu_n)\right].$$

Discretizing (5.24), we propose another implicit iterative method.

$$\frac{\mu_{n+1} - \mu_n}{h} + \mu_n = \mathscr{J}_{\mathscr{A}(\mu_{n+1})}[\mu_n - \rho\mathscr{T}\mu_{n+1} + \rho\mathscr{B}(\mu_{n+1})],$$

where h is the step size.
For $h = 1$, we can suggest an implicit iterative method for solving the problem (2.1).

Algorithm 27 *For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative scheme*

$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\mu_{n+1})}[\mu_n - \rho\mathscr{T}\mu_{n+1} + \rho\mathscr{B}(\mu_{n+1})],$$

Algorithm 27 is an implicit iterative method in the sense of Koperlevich, see Noor [14].

Using (5.23), we have

$$\frac{d\mu}{dt} + \mu = \mathscr{J}_{\mathscr{A}((1-\alpha)\mu+\alpha\mu)}[(1-\alpha)\mu + \alpha\mu - \rho\mathscr{T}((1-\alpha)\mu + \alpha\mu) + \rho\mathscr{A}((1-\alpha)\mu + \alpha\mu))], \tag{5.26}$$

where $\alpha \in [0,1]$ is a constant.

Discretization (5.26)and taking $h = 1$, we have

$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}((1-\alpha)\mu_n+\alpha\mu_{n-1})}\left[(1-\alpha)\mu_n + \alpha\mu_{n-1} - \rho\mathscr{T}((1-\alpha)\mu_n + \alpha\mu_{n-1}) + \rho\mathscr{A}((1-\alpha)\mu_n + \alpha\mu_{n-1})\right], \tag{5.27}$$

which is an inertial type iterative method for solving the quasi variational inclusion (2.1). Using the predictor-corrector techniques, we have

Algorithm 28 *For a given $\mu_0 \in H$, compute μ_{n+1} by the iterative schemes*

$$\omega_n = (1-\alpha)\mu_n + \alpha\mu_{n-1}$$
$$\mu_{n+1} = \mathscr{J}_{\mathscr{A}(\omega_n)}\left[\omega_n - \rho\mathscr{T}(\omega_n) + \rho\mathscr{A}(\omega_n)\right],$$

which is known as the inertial two-step iterative method.

*Remark.*For appropriate and suitable choice of the operators $\mathscr{T}, \mathscr{A}$, parameter α and the spaces, one can propose a wide class of implicit, explicit and inertial type methods for solving quasi variational inclusions and related nonlinear optimization problems. Using the techniques and ideas of Noor et al [12], one can discuss the convergence analysis of the proposed methods.

6 Nonexpansive mappings

In this section, we consider the non-expansive mapping technique to suggest some iterative methods for solving quasi variational inclusions (2.1). First of all, we recall the following fact.

Let S be a nonexpansive mapping. We denote the set of the fixed points of S by $\mathcal{F}(S)$ and the set of the solutions of the quasi variational inclusion (2.1) by $QRI(H,T,B)$. If $\mu^* \in \mathcal{F}(S) \cap QRVI(K,T)$, then $x^* \in F(S)$ and $\mu^* \in VI(K,T)$. Thus from Lemma 1, it follows that

$$\begin{aligned} \mu^* = S\mu^* &= \mathcal{J}_{\mathcal{A}(\mu)}[\mu^* - \rho \mathcal{T}\mu^* + \rho \mathcal{B}(\mu^*] \\ &= S\mathcal{J}_{\mathcal{A}(\mu)}[\mu^* - \rho \mathcal{T}\mu^* + \rho \mathcal{B}(\mu^*)], \end{aligned}$$

where $\rho > 0$ is a constant.

This fixed point formulation is used to suggest the following iterative method for finding a common element of two different sets of solutions of the fixed points of the nonexpansive mappings and the variational inclusions.

Algorithm 29 *For a given $u_0 \in \mathcal{H}$, compute the approximate solution x_n by the iterative schemes*

$$u_{n+1} = (1-a_n)u_n + a_n S\mathcal{J}_{\mathcal{A}(\mu)}[\mu_n - \rho \mathcal{T}\mu_n + \rho \mathcal{B}(\mu_n)],$$

where $a_n \in [0,1]$ for all $n \geq 0$ and S is the nonexpansive operator.

Algorithm 29 is also known as a Mann iteration.

Related to the variational inequalities, we have the problem of solving the resolvent equations (4.17) involving the non-expansive mapping S. To be more precise, let $\mathcal{R}_{\mathcal{A}(\mu)} = I - S\mathcal{J}_{\mathcal{A}(\mu)}$, where $\mathcal{J}_{\mathcal{A}(\mu)}$ is the resolvent, I is the identity operator and S is the nonexpansive operator. We consider the problem of finding $z \in \mathcal{H}$ such that

$$\mathcal{T}S\mathcal{J}_{\mathcal{A}(\mu)}z + \rho^{-1}\mathcal{R}_{\mathcal{A}(\mu)}z = \mathcal{B}S\mathcal{J}_{\mathcal{A}(\mu)}z, \tag{6.28}$$

which is called the implicit resolvent equation involving the nonexpansive operator S. For $S = I$, the identity operator, we obtain the implicit resolvent equation (4.17). Using essentially the technique of the resolvent operator, one can establish the equivalence between the resolvent equations and variational inclusions. This alternative equivalence has played a fundamental and basic role in developing some efficient and robust methods for solving variational inclusions and related optimization problems. The resolvent equation technique has been used to study the sensitivity analysis and asymptotical stability of the variational inclusions. It has been shown that the resolvent equation technique is more flexible and general than the resolvent method and its variant form, see Noor [37].

Definition 2. *An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is called ζ-Lipschitzian if, there exists a constant $\mu > 0$, such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \zeta\|x-y\|, \quad \forall x,y \in \mathcal{H}.$$

Definition 3. *An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is called α_1-inverse strongly monotone (or co-coercive)if, there exists a constant $\alpha > 0$, such that*

$$\langle \mathcal{T}x - \mathcal{T}y, x-y\rangle \geq \alpha\|\mathcal{T}x - \mathcal{T}y\|^2, \quad \forall x,y \in \mathcal{H}.$$

Definition 4. *An operator $\mathcal{T} : \mathcal{H} \to$ is called r-strongly monotone if, there exists a constant $r > 0$ such that*

$$\langle Tx - Ty, x-y\rangle \geq r\|x-y\|^2, \quad \forall x,y \in \mathcal{H}.$$

Definition 5. *An operator $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ is called relaxed (γ,r)-cocoercive if, there exists constants $\gamma > 0, r > 0$, such that*

$$\langle \mathcal{T}x - \mathcal{T}y, x-y\rangle \geq -\gamma\|\mathcal{T}x - \mathcal{T}y\|^2 + r\|x-y\|^2, \quad \forall x,y \in \mathcal{H}.$$

Remark. Clearly a r-strongly monotone operator or a γ-inverse strongly monotone operator must be a relaxed (γ,r)-cocoercive operator, but the converse is not true. Therefore the class of the relaxed (γ,r)-cocoercive operators is the most general class, and hence definition 2.4 includes both the definition 2.2 and the definition 2.3 as special cases.

Remark. From Definition 3, it follows that if $\mathcal{T}$ is α-inverse strongly monotone (or co-coercive), than $\mathcal{T}$ is also Lipschitz continuous with constant $\frac{1}{\alpha}$.

Lemma 3. *Suppose $\{\delta_k\}_{k=0}^{\infty}$ is a nonnegative sequence satisfying the following inequality:*

$$\delta_{k+1} \leq (1-\lambda_k)\delta_k + \sigma_k, \; k \geq 0,$$

with $\lambda_k \in [0,1]$, $\sum_{k=0}^{\infty}\lambda_k = \infty$, and $\sigma_k = o(\lambda_k)$. Then $\lim_{k\to\infty}\delta_k = 0$.

In this section, we use the resolvent equations to suggest and analyze an iterative method for finding the common element of the nonexpansive mappings and the variational inclusion $QRVI(T,K)$. For this purpose, we need the following result.

Lemma 4. *The element $\mu \in \mathcal{H}$ is a solution of quasi variational inclusion (2.1), if and only if, $z \in \mathcal{H}$ satisfies the implicit resolvent equation (4.17), where*

$$\mu = S\mathcal{J}_{\mathcal{A}(\mu)}z, \tag{6.29}$$

$$z = u - \rho \mathcal{T}\mu + \rho \mathcal{B}(\mu), \tag{6.30}$$

where $\rho > 0$ is a constant.

From Lemma 4, it follows that the quasi variational inclusion (2.1) and the implicit resolvent equation (6.28) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving variational inclusion and related optimization problems, see [37] and the references therein. We denote the set of the solutions of the resolvent equations by IRE(H,T,S).

Using Lemma 4 and Remark 6, we now suggest and analyze a new iterative algorithm for finding the common element of the solution sets of the quasi variational inclusions and nonexpansive mappings S and this is the main motivation of this paper.

Algorithm 30 *For a given* $z_0 \in H$, *compute the approximate solution* z_{n+1} *by the iterative schemes*

$$\mu_n = S\mathcal{J}_{\mathcal{A}(\mu)}z_n \quad (6.31)$$
$$z_{n+1} = (1-a_n)z_n + a_n\{u_n - \rho\mathcal{T}\mu_n + \rho\mathcal{B}(\mu_n)\}, \quad (6.32)$$

where $a_n \in [0,1]$ *for all* $n \geq 0$ *and* S *is a nonexpansive operator.*

For $S = I$, the identity operator, Algorithm 30 reduces to the following iterative method for solving quasi variational inclusion(2.1) and appears to be a new one.

Algorithm 31 *For a given* $z_0 \in H$, *compute the approximate solution* z_{n+1} *by the iterative schemes*

$$\mu_n = S\mathcal{J}_{\mathcal{A}(\mu)}z_n$$
$$z_{n+1} = (1-a_n)z_n + a_n\{u_n - \rho\mathcal{T}\mu_n + \rho\mathcal{B}(\mu_n)\}$$

If $\mathcal{J}_{\mathcal{A}(\mu)} = \mathcal{J}_{\mathcal{A}}$, the convex set in H, then Algorithms 30 and 31 reduce to the following algorithms for solving variational inclusion.

Algorithm 32 *For a given* $z_0 \in H$, *compute the approximate solution* z_{n+1} *by the iterative schemes*

$$\mu_n = S\mathcal{J}_{\mathcal{A}}z_n$$
$$z_{n+1} = (1-a_n)z_n + a_n\{\mu_n - \rho\mathcal{T}\mu_n + \rho\mathcal{B}(\mu_n)\}$$

where $a_n \in [0,1]$ for all $n \geq 0$ and S is a nonexpansive operator.

We now study the convergence of Algorithm 30.

Theorem 1. *Let* $\mathcal{T}$ *be a relaxed* (γ, r)*-cocoercive and* μ*-Lipschitzian mapping and* S *be a nonexpansive mapping such that* $F(S) \cap IRE(H,T,S) \neq \emptyset$. *Let* $\{z_n\}$ *be a sequence defined by Algorithm 30, for any initial point* $z_0 \in H$. *If Assumption 2.1 holds and*

$$\theta = \frac{\sqrt{1+2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2} + \rho\xi}{1-\nu} < 1,$$
$$\rho < \frac{1-\nu}{\xi}, \quad \nu < 1, \quad a_n \in [0,1] \quad (6.33)$$

and

$$\sum_{n=0}^{\infty} a_n = \infty,$$

then z_n *converges strongly to* $z^* \in F(S) \cap IRE(H,T,S)$.

Proof. Let $z^* \in H$ be a solution of $F(S) \cap IRE(H,T,S)$. Then, from Lemma 3.1, we have

$$\mu^* = a_n S\mathcal{J}_{\mathcal{A}(\mu^*)}z^* \quad (6.34)$$
$$z^* = (1-a_n)z^* + a_n\{\mu^* - \rho\mathcal{T}\mu^* + \rho\mathcal{B}(\mu^*), \quad (6.35)$$

where $a_n \in [0,1]$ and $u^* \in K$ is a solution of RVI(K,I). To prove the result, we need first to evaluate $\|z_{n+1} - z^*\|$ for all $n \geq 0$. From (6.32) and (6.35), we have

$$\begin{aligned}\|z_{n+1} - z^*\| &= \|(1-a_n)z_n + a_n\{\mu_n - \rho\mathcal{T}\mu_n + \rho\mathcal{B}(\mu_n)\} \\ &\quad -(1-a_n)z^* \\ &\quad -a_n\{\mu^* - \rho\mathcal{T}\mu^* + \rho\mathcal{B}(\mu_n^*)\}\| \\ &\leq (1-a_n)||z_n - z^*|| \\ &\quad + a_n||u_n - u^* - \rho(\mathcal{T}\mu_n - \mathcal{T}u^*)|| \\ &\quad + \rho\|\mathcal{B}(\mu_n - \mathcal{B}(\mu_n^*)\|. \end{aligned} \quad (6.36)$$

From the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on $\mathcal{T}$, we have

$$\begin{aligned} &\|u_n - u^* - \rho(\mathcal{T}u_n - \mathcal{T}u^*)\|^2 \\ &= \|u_n - u^*\|^2 - 2\rho\langle\mathcal{T}\mu_n - \mathcal{T}u^*, u_n - u^*\rangle \\ &\quad + \rho^2\|\mathcal{T}\mu_n - \mathcal{T}u^*\|^2 \\ &\leq ||u_n - u^*||^2 - 2\rho[-\gamma\|\mathcal{T}\mu_n - \mathcal{T}u^*\|^2 + r||u_n - u^*||^2] \\ &\quad + \rho^2||Tu_n - Tu^*||^2 \\ &\leq ||u_n - u^*||^2 + 2\rho\gamma\mu^2||u_n - u^*||^2 - 2\rho r||u_n - u^*||^2 \\ &\quad + \rho^2\mu^2||u_n - u^*||^2 \\ &= [1 + 2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2]||u_n - u^*||^2 \end{aligned} \quad (6.37)$$

Combining (6.36), (6.37) and using the Lipschitz continuity of the operator $\mathcal{B}$, we have

$$\|z_{n+1} - z^*\| \leq (1-a_n)\|z_n - z^*\| + a_n\theta_1\|\mu_n - \mu^*\|. \quad (6.38)$$

where

$$\theta_1 = \sqrt{1+2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2} + \rho\xi. \quad (6.39)$$

From (6.31), (6.34) and the Assumption 1, we have

$$\begin{aligned}\|\mu_n - \mu^*\| &\leq a_n\|S\mathcal{J}_{\mathcal{A}(\mu_n)}z_n - S\mathcal{J}_{\mathcal{A}(\mu^*)}z^*\| \\ &\leq \|S\mathcal{J}_{\mathcal{A}(\mu_n)}z_n - S\mathcal{J}_{\mathcal{A}(\mu_n)}z^*\| \\ &\quad + \|S\mathcal{J}_{\mathcal{A}(\mu_n)}z^* - S\mathcal{J}_{\mathcal{A}(\mu^*)}z^*\| \\ &\leq \nu\|\mu_n - \mu^*\| + \|z_n - z^*\|, \end{aligned}$$

which implies that

$$\|u_n - u^*\| \leq \frac{1}{1-\nu}\|z_n - z^*\|. \quad (6.40)$$

From (6.38) and (6.40), we obtain that

$$\begin{aligned}\|z_{n+1} - z^*\| &\leq (1-a_n)\|z_n - z^*\| + a_n\theta\|z_n - z^*\| \\ &= [1 - a_n(1-\theta)]\|z_n - z^*\|, \end{aligned} \quad (6.41)$$

where

$$\theta = \frac{\sqrt{1+2\rho\gamma\mu^2 - 2\rho r + \rho^2\mu^2} + \rho\xi}{1-\nu}.$$

Hence by (6.41) and Lemma 6.28, it follows that

$$\lim_{n\to\infty} \|z_n - z^*\| = 0,$$

which complete the proof.

We now prove the strong convergence of Algorithm 30 under the α-inverse strongly monotonicity.

Theorem 2. *Let $\mathscr{T}$ be an α-inverse strongly monotonic mapping with constant $\alpha >$. and S be a nonexpansive mapping such that $F(S)\cap IRE(H,T)\neq\emptyset$. If the operator $\mathscr{B}$ is Lipschitz continuous with constant ξ and*

$$\rho(1+\alpha\xi)\leq\alpha(2-\nu),\quad \nu\in(0,1), \tag{6.42}$$

then the approximate solution obtained from Algorithm 30 converges strongly to $z^*\in F(S)\cap IRE(H,T)$.

Proof. Let $\mathscr{T}$ be α-inverse strongly monotone with the constant $\alpha>0$, then $\mathscr{T}$ is $\frac{1}{\alpha}$–Lipschitzian continuous. Consider

$$\begin{aligned}
&\|\mu_n-\mu^*-\rho[\mathscr{T}\mu_n-\mathscr{T}\mu^*]\|^2\\
&=\|\mu_n-\mu^*\|^2+\rho^2\|\mathscr{T}\mu_n-\mathscr{T}\mu^*\|^2\\
&\quad-2\rho\langle\mathscr{T}\mu_n-\mathscr{T}\mu^*,\mu_n-\mu^*\rangle\\
&\leq\|\mu_n-\mu^*\|^2+\rho^2\|\mathscr{T}\mu_n-\mathscr{T}\mu^*\|^2\\
&\quad-2\rho\alpha\|\mathscr{T}\mu_n-\mathscr{T}\mu^*\|^2\\
&=\|\mu_n-\mu^*\|^2+(\rho^2-2\rho\alpha)|\mathscr{T}\mu_n-\mathscr{T}\mu^*\|^2\\
&\leq\|\mu_n-\mu^*\|^2+(\rho^2-2\rho\alpha)\cdot\frac{1}{\alpha^2}\|\mu_n-\mu^*\|^2\\
&=\left(1+\frac{(\rho^2-2\rho\alpha)}{\alpha^2}\right)\|\mu_n-\mu^*\|^2.
\end{aligned} \tag{6.43}$$

From (6.40) and (6.43), we have

$$\begin{aligned}
\|z_{n+1}-z^*\|\leq&\ (1-a_n)\|z_n-z^*\|\\
&+a_n\|\mu_n-\mu^*-\rho(\mathscr{T}\mu_n-\mathscr{T}\mu^*)\|\\
&+\alpha_n\rho\|\mathscr{B}(\mu_n)-\mathscr{B}(\mu^*)\|\\
\leq&\ (1-a_n)||x_n-x^*||+a_n\theta_2\|\mu_n-\mu^*\|\\
&+\alpha_n\rho\xi\|\mu_n-\mu^*\|\\
=&\ [1-a_n(1-\theta_3)]\|z_n-z^*\|,
\end{aligned}$$

where

$$\theta_2=(1+\frac{(\rho^2-2\rho\alpha)}{\alpha^2})^{1/2}.$$

and

$$\theta_3=\frac{\sqrt{1+\frac{\rho^2-2\rho\alpha}{\alpha^2}}+\rho\xi}{1-\nu}<1.$$

Form (6.42), it follows that $\theta_3<1$. Thus it follows

$$\lim_{n\to\infty}\|z_n-z^*\|=0$$

from Lemma 3. completing the proof.

7 Computational Aspects

In this paper, we have introduced and studied a new class of quasi variational inclusions involving three monotone operators. It have been shown that the variational inclusions are equivalent to a new class of resolvent equations. This alternative equivalent formulations are used to investigate several iterative and inertial iterative methods for solving quasi variational inclusions associated with resolvent methods, resolvent equations, dynamical systems. Also this equivalence is used to suggest and analyze an iterative method for finding the common element of set of the solutions of the variational inclusions and the set of the fixed-points of the nonexpansive operator. For more details, see [6,7,8,9,10, 12,13,17,18,25,30,31,32,35,37,41,42] and the references therein. The results are encouraging and perform better than other methods. It is interesting problem to explore the techniques and ideas of this paper to develop other new iterative methods for solving the quasi variational inclusions. This is another direction for future work.

8 Conclusion

In this paper, we have considered and analyzed some classes of quasi variational inclusions involving three operators. We have established between quasi variational inclusions and fixed point problems using the resolvent methods. We have used this alternative equivalent formulation to suggest some new iterative methods for solving the quasi variational inclusions. These new methods include extraresolvent method, modified double resolvent methods and inertial type methods, which are suggested using the techniques of resolvent method, resolvent equations, dynamical systems and nonexpansive mappings. Convergence analysis of the proposed method is discussed for monotone operators. It is an open problem to compare these proposed methods with other methods. Using the ideas and techniques of this paper, one can suggest and investigate several new implicit methods for solving various classes of variational inclusions and related optimization problems.

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References

[1] G. Stampacchia, Formes bilieaires coercives sur les ensembles convexes, *Comput. Rend.l'Acad. Sciences, Paris,* **258**, 4413-4416(1964).
[2] R. Glowinski, J. L. Lions and R. Tremileres, Numerical Analysis of Variational Inequalities, NorthHolland, Amsterdam, Holland, 1981.
[3] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications,* SIAM, Philadelphia, 2000
[4] M. A. Noor, *On Variational Inequalities,* PhD Thesis, Brunel University, London, UK, 1975.
[5] M. A. Noor, General variational inequalities, *Appl. Math. Letters,* **1**, 119-121(1988).
[6] M. A. Noor, Generalized set-valued variational inclusions and resolvent equations, *J. Math. Anal. Appl.,* **228**,206-220(1998).
[7] M. A. Noor, Three-step iterative algorithms for multivalued quasi variational inclusions, *J. Math. Anal. Appl.,* **255**, 589-604(2001).
[8] M. A. Noor, Implicit dynamical systems and quasi variational inequalities, *Appl. Math. Comput.,* **134**, 69-81(2003).
[9] M. A. Noor and K. I. Noor, Sensitivity analysis of quasi variational inclusions, *J. Math. Anal. Appl.,* **236**, 290-299(1999).
[10] M. A. Noor and K. I. Noor, Iterative algorithms for solving nonlinear quasi-variational inequalities, *Montes Taurus J. Pure Appl. Math.,* **4(1)**, 44-58(2022).
[11] M. A. Noor and W. Oettli, On general nonlinear complementarity problems and quasi equilibria, *Le Mathemath.,* **49**, 313-331(1994).
[12] M. A. Noor. K.I. Noor and B. B. Mohsen, Some new classes of general quasi variational inequalities, *AIMS Math.,* **6(6)**, 6406-6421(2021).
[13] M. A. Noor, New approximation schemses for general variational inequalities, *J. Math. Anal. Appl.,* **251**, 217-229(2000).
[14] M. A. Noor, Some developments in general variational inequalities, *Appl. Math. Comput.,* **152**, 199-277(2004),
[15] J. L. Lions and G. Stampacchia, Variational inequalities, *Commun. Pure Appl. Math.,* **20**, 493-512(1967).
[16] M. A. Noor and K. I. Noor, Higher order strongly general convex functions and variational inequalities, *AIMS Math.,* **5(4)**, 3646-3663(2020)
[17] M. A. Noor, K. I. Noor and Th. M. Rassias, New trends in general variational inequalities, *Acta Appl. Mathematica.,* **170(1)**, 981-1064(2020).
[18] M. A. Noor, K. I. Noor and Th. M. Rassias, Some aspects of variational inequalities, *J. Comput. Appl. Math.,* **47**, 493-512(1993)
[19] M. A. Noor, K. I. Noor, A. Hamdi and E. H. El-Shemas, On difference of two monotone operators, *Optim. Letts.,* **3**, 329-335(2009).
[20] N. Kikuchi and J. T. Oden, Contact Problems in Elasticity, SAIM Publishing Co., Philadelphia, (1988).
[21] A. Bensoussan, J. L. Lions, *Application des Inequalities Variationnelles en Control Stochastique,* Paris: Bordas(Dunod), 1978.
[22] B. T.Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, *Soviet Math. Dokl.,* **7**, 2-75(1966).
[23] S. Karamardian, The nonlinear complementarity problems with applications, Part 2, *J. Optim. Theory Appl.,* **4(3)**, 167-181(1969).
[24] M. A. Noor, The quasi-complementarity problem, *J. Math. Anal. Appl.,* **130**, 344-353(1988).
[25] M. A. Noor, Fixed point approach for complementaritty problems, *J. Math. Anal. Appl.,* **133**, 437-448(1988).
[26] R. W. Cottle , J,-S. Pang and R, E. Stone, *The Linear Complementarity Problem,* SIAM Publ. 2009.
[27] G. Cristescu and L. Lupsa, *Non-Connected Convexities and Applications,* Kluwer Academic Publisher, Dordrechet, (2002)
[28] C. P.Niculescu and L. E. Persson, *Convex Functions and Their Applications,* Springer-Verlag, New York, (2018).
[29] P. Dupuis and A. Nagurney, Dynamical systems and variational inequalities, *Annals Oper. Research.,* **44**, 7-42(1993).
[30] S. Jabeen, B. B.Mohsin , M. A. Noor and K. I. Noor. Inertial projection methods for solving general quasi-variational inequalities, *AIMS Math.,* **6(2)**, 1075-1086(2021).
[31] H. Moosaei, S. Ketabchi, M.A. Noor, J. Iqbal and V. Hooshyarbakhsh, Some techniques for solving absolute value equations, *Appl. Math. Comput.,* **268**, 696-705(2015).
[32] M. A. Noor, J. Iqbal, K.I. Noor and E. Al-Said, On an iterative method for solving absolute value equations, *Optim. Lett.,* **6**, 1027-1033(2021).
[33] M. A. Noor, Hemivariational inequalities, *J. Appl. Math. Comput.,* **17(1-2)**, 59 -72(2005).
[34] P. D. Panagiotopoulos, Nonconvex energy functions, hemivariational inequalities and substationary principles, *Acta Mech.,* **42**, 160-183(1983).
[35] F. Alvarez, Weak convergence of a relaxed and inertial hybrid projection-proximal point algorithm for maximal monotone operators in Hilbert space, *SIAM J. Optim.,* **14**, 773-782(2003).
[36] A. Nagurney and D. Zhang, *Projected dynamical systems and variational inequalities with applications,* Kluwer Academic Publishers, Boston, Dordrecht, London 1996.
[37] M. A. Noor, General variational inequalities and nonexpansive mappings, *J. Math. Anal. Appl.,* **331**, 810-824(2007)
[38] M. Patriksson, *Nonlinear Programming and Variational Inequalities: A Unified Approach,* Kluwer Acadamic publishers, Drodrecht, 1998.
[39] S.M. Robinson, Normal maps induced by linear transformations, *Math. Oper. Res.,* **17**, 691-714(1992).
[40] P. Shi, Equivalence of variational inequalities with WienerHopf equations, *Proc. Amer.Math. Soc.,* **111**, 339-346(1991).
[41] Y. Shehu, A. Gibali and S. Sagratella, Inertial projection-type method for solving quasi variational inequalities in real Hilbert space, *J. Optim. Theory Appl.,* **184**, 877-894(2020).
[42] M. A. Noor and K. I. Noor, Absolute value variational inclusions, *Earth. J. Math. Sci.,* **8(1)**, 121-153(2022).