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Inferences for Weibull-Gamma Distribution in Presence of Partially Accelerated Life Test

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Abstract: In this paper, the point at issue is to deliberate point and interval estimations for the parameters of Weibull-Gamma distribution (WGD) using progressively Type-II censored (PROG-II-C) sample under step stress partially accelerated life test (SSPALT) model. The maximum likelihood (ML), Bayes, and four parametric bootstrap methods are used to obtain the point estimations for the distribution parameters and the acceleration factor. Furthermore, the approximate confidence intervals (ACIs), four bootstrap confidence intervals and credible intervals of the estimators have been gotten. The results of Bayes estimators are computed under the squared error loss (SEL) function using Markov Chain Monte Carlo (MCMC) method. Gibbs within the Metropolis–Hasting algorithm is applied to generate MCMC samples from the posterior density functions. Simulation results are carried out to explicate the precision of the estimators for the aforementioned parameters.

Keywords: Partially accelerated life test; Maximum likelihood estimation; Bias Corrected Confidence Interval (Boot-BC); Accelerated Bias Corrected Confidence Interval (Boot-BCa); MCMC approach.

1 Introduction

Strong competition among manufacturers and the desire not to lose, leading to testing products under severe conditions (stress), such as high temperatures and high voltages to emphasize product quality and reduce test time, such tests called accelerated life testing. There are three common types of the stresses. These types are step-stress, progressive-stress and constant-stress see Nelson [1]. Such testing conducted under stresses is called accelerated life test (ALT) or partially accelerated life test (PALT) according to the used strategy in designing the test . In a SSPALT unit starts at normal use condition for a specified time then the unit is set under stress unless it fails. Generally, stress is applied until the test unit fails or the test is terminated based on a certain censoring scheme, where the censoring scheme which is used in this paper is PROG-II-C. The PROG-II-C scheme can be described as follows. First, the experimenter places n independent and identical units on the life test. When the first failure occurs, say at time $t_{(1)}$, r_1 units are randomly removed from remaining $n-1$ surviving units. When the second failure occurs at time $t_{(2)}$, r_2 units are randomly removed from remaining $n-r_1-2$ surviving units. This experiment terminates when the m th failure occurs at time t_m, and $r_m = n-m-\sum_{i=1}^{m-1} r_i$ surviving units are removed from the test. For more information on progressive censoring, we refer the reader to Balakrishnan and Aggarwala [2], Balakrishnan [3], Soliman et al. [4], Musleh and Helu [5] and EL-Sagheer [6]. El-Sagheer [7] studied the estimation of WG parameters under normal conditions based on PROG-II-C data. The SSPALT have been studied by several authors based on different schemes of censoring observations for example, see Goel [8], Bhattcharyya and Soejoeti [9], Bai et al. [10], Abdel-Ghaly et al. [11], Abdel-Ghani [12] and El-Sagheer and Ahsanullah [13]. Ismail and Sarhan [14] and El-Sagheer and Ahsanullah [13] discussed SSPALT through PROG-II-C data from exponential distribution and Lomax distribution respectively. In this article SSPALT model appertaining to PROG-II-C data from WG distribution is canvassed. The remainder of this article is organized as follows: Section 2 provides a description of WG distribution and the tampered random variable (TRV) model. In Section 3 the maximum likelihood estimates (MLEs) of the parameters under consideration are estimated in addition to the corresponding ACIs. Section 4 includes concerns with four types of

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bootstrap confidence intervals. Section 5 is devoted to the Bayesian approach that uses the famed MCMC technique. An illustrative example is developed to explain the theoretical results in Section 6. Simulation study is presented in Section 7 to assess the performance of our estimates. Eventually conclusion is inserted in Section 8.

2 Model Description

A brief specification is given in this section about WG distribution. Also, the transformed probability density function (pdf) of WG distribution under the TRV model is presented.

2.1 Weibull-Gamma Distribution

The WG distribution is suitable for the phenomenon of loss of signals in telecommunications which is called fading when multipath is superimposed on shadowing, see Bithas [15]. TheWG distribution is disseminated by Nadarajah and Kotz [16].

A random variable T is said to have WG distribution if its pdf given by:

$$f(t;\alpha,\theta,\beta)=\frac{\theta\beta}{\alpha}t^{\theta-1}\left(1+\frac{1}{\alpha}t^{\theta}\right)^{-(\beta+1)},t>0;\alpha,\theta,\beta>0, \tag{1}$$

the corresponding survival function is

$$S(t)=\left(1+\frac{1}{\alpha}t^{\theta}\right)^{-\beta}, \tag{2}$$

and the corresponding hazard rate function is given by

$$h(t)=\frac{\theta\beta}{\alpha}t^{\theta-1}\left(1+\frac{1}{\alpha}t^{\theta}\right)^{-1},t>0;\alpha,\theta,\beta>0. \tag{3}$$

For more details about WG distribution and its properties see Bithas [15], Molenberghs and Verbeke [17] and Mahmoud et al. [18].

2.2 Test Steps

The following assumptions are used throughout the paper:

(1)n identical and independent units are put on the life test and the life time of individual unit has WG distribution.
(2)At the beginning, each of the units functions under normal use condition. If it does not fail and exceeds a pre-specified time τ, it is put under accelerated condition (stress).
(3)The test is terminated when the mth failure occurs, where m is prefixed before $(m\leq n)$.
(4)At the time of the ith failure, a random number of the surviving items $R_i=1,2,...,m-1$, are randomly selected and removed from the test. Finally, at the time of the mth failure, the remaining surviving items $R_m=n-m-\sum_{i=1}^{m-1}R_i$ are removed from the test and the test is terminated.
(5)The TRV model is applied. It was designed by Degroot and Goel [19]. According to this model the lifetime of a unit under SSPALT can be written as

$$Y=\begin{cases}T, & if\ T\leq\tau,\\ \tau+\frac{1}{\lambda}(T-\tau), & if\ T>\tau,\end{cases} \tag{4}$$

where T is the lifetime of the units under normal condition, τ is the stress change time, and λ is the acceleration factor, where $(\lambda>1)$.
(6)According to the TRV model, the pdf of WG(α,θ,β) distribution under SSPALT is given by

$$f(y)=\begin{cases}f_1(y)=\frac{\alpha\theta}{\beta}y^{\alpha-1}\left(1+\frac{1}{\beta}y^{\alpha}\right)^{-(\theta+1)}, & 0<y\leq\tau,\\ f_2(y)=\frac{\alpha\theta\lambda}{\beta}\left(\psi(\lambda)^{\alpha-1}\right)\left(1+\frac{1}{\beta}(\psi(\lambda))^{\alpha}\right)^{-(\theta+1)}, & y>\tau>0,\end{cases} \tag{5}$$

where $\psi(\lambda)=\tau+\lambda(y-\tau)$.

(7)Let δ_{1i} and δ_{2i} be indicator functions such that $\delta_{1i} \equiv I(y_i \le \tau), \delta_{2i} \equiv I(y_i > \tau)$, so the number of failures before time τ under normal conditions of the experiment, $n_1 = \sum_{i=1}^{m} \delta_{1i}$ and $m - n_1 = \sum_{i=1}^{m} \delta_{2i}$ is the number of failures after time τ at stress conditions, then the observed progressive censored data are

$$y_{1;m,n}^{\mathbf{R}} < ... < y_{n_1;m,n}^{\mathbf{R}} < \tau < y_{n_1+1;m,n}^{\mathbf{R}} < ... < y_{m;m,n}^{\mathbf{R}}, \tag{6}$$

where $R = (R_1, R_2, ..., R_m)$ and $\sum_{i=1}^{m} R_i = n - m$.

3 Maximum Likelihood Estimation

In this section, the MLEs of the model parameters are obtained. Let $y_i = y_{i;m,n}^{\mathbf{R}}$, $i = 1, 2, ..., m$, be the observed values of the lifetime Y obtained from a PROG-II-C scheme under SSPALT, with censored scheme $R = (R_1, R_2, ..., R_m)$. The likelihood function of the observations $y_1 < ... < y_{n_1} < \tau < y_{n_1+1} < ... < y_m$ can be written in the following form:

$$L(\alpha, \theta, \beta, \lambda) = c\prod_{i=1}^{m}\left\{\left[f_1(y_i)(S_1(y_i))^{R_i}\right]^{\delta_{1i}} \cdot \left[f_2(y_i)(S_2(y_i))^{R_i}\right]^{\delta_{2i}}\right\}, \tag{7}$$

where

$$c = n(n-1-R_1)(n-1-R_1-R_2)...\left(n-m+1-\sum_{i=1}^{m-1} R_i\right). \tag{8}$$

So $L(\alpha, \theta, \beta, \lambda)$ can be written as follows:

$$L(\alpha, \theta, \beta, \lambda) = c\prod_{i=1}^{m}\left\{\left[\frac{\alpha\theta}{\beta}y_i^{\alpha-1}\left(1+\frac{1}{\beta}y_i^{\alpha}\right)^{-\phi_i(\theta)}\right]^{\delta_{1i}} \times \left[\frac{\alpha\theta\lambda}{\beta}(\psi_i(\lambda))^{\alpha-1}\left(1+\frac{1}{\beta}(\psi_i(\lambda))^{\alpha}\right)^{-\phi_i(\theta)}\right]^{\delta_{2i}}\right\}, \tag{9}$$

where

$$\phi_i(\theta) = \theta R_i + \theta + 1 \text{ and } \psi_i(\lambda) = \tau + \lambda(y_i - \tau). \tag{10}$$

The log-likelihood function may then be written as

$$\begin{aligned} \ln L(\alpha, \theta, \beta, \lambda) &= \ln c + m\ln\alpha + m\ln\theta - m\ln\beta + \sum_{i=1}^{m}\delta_{2i}\ln\lambda \\ &+ (\alpha-1)\sum_{i=1}^{m}\delta_{1i}\ln y_i - \sum_{i=1}^{m}\delta_{1i}\,\phi_i(\theta)\ln\left(1+\frac{1}{\beta}y_i^{\alpha}\right) \\ &+ (\alpha-1)\sum_{i=1}^{m}\delta_{2i}\ln\psi_i(\lambda) - \sum_{i=1}^{m}\delta_{2i}\,\phi_i(\theta)\ln\left(1+\frac{1}{\beta}(\psi_i(\lambda))^{\alpha}\right), \end{aligned} \tag{11}$$

and thus we have the likelihood equations for α, θ, β and λ respectively, as

$$\frac{\partial \ln L}{\partial\alpha} = \frac{m}{\alpha} + \sum_{i=1}^{m}\delta_{1i}\ln y_i - \frac{\alpha}{\beta}\sum_{i=1}^{m}\frac{\delta_{1i}\,\phi_i(\theta)\,y_i^{\alpha-1}}{\left(1+\frac{1}{\beta}y_i^{\alpha}\right)} + \sum_{i=1}^{m}\delta_{2i}\ln\psi_i(\lambda) - \frac{\alpha}{\beta}\sum_{i=1}^{m}\frac{\delta_{2i}\,\phi_i(\theta)\,\psi_i(\lambda)^{\alpha-1}}{\left(1+\frac{1}{\beta}\psi_i(\lambda)^{\alpha}\right)} = 0, \tag{12}$$

$$\frac{\partial \ln L}{\partial\theta} = \frac{m}{\theta} - \sum_{i=1}^{m}\delta_{1i}(R_i+1)\ln\left(1+\frac{1}{\beta}y_i^{\alpha}\right) - \sum_{i=1}^{m}\delta_{2i}(R_i+1)\ln\left(1+\frac{1}{\beta}(\psi_i(\lambda))^{\alpha}\right) = 0, \tag{13}$$

$$\frac{\partial \ln L}{\partial\beta} = -\frac{m}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{m}\frac{\delta_{1i}\,\phi_i(\theta)\,y_i^{\alpha}}{\left(1+\frac{1}{\beta}y_i^{\alpha}\right)} + \frac{1}{\beta^2}\sum_{i=1}^{m}\frac{\delta_{1i}\,\phi_i(\theta)\,(\psi_i(\lambda))^{\alpha}}{\left(1+\frac{1}{\beta}(\psi_i(\lambda))^{\alpha}\right)} = 0, \tag{14}$$

and

$$\frac{\partial \ln L}{\partial\lambda} = \frac{1}{\lambda}\sum_{i=1}^{m}\delta_{2i} + (\alpha-1)\sum_{i=1}^{m}\frac{\delta_{2i}(y_i-\tau)}{\psi_i(\lambda)} - \frac{\alpha}{\beta}\sum_{i=1}^{m}\frac{\delta_{2i}\,\phi_i(\theta)(y_i-\tau)(\psi_i(\lambda))^{\alpha-1}}{\left(1+\frac{1}{\beta}(\psi_i(\lambda))^{\alpha}\right)} = 0\,. \tag{15}$$

A system of nonlinear simultaneous equations in four unknowns vaiables α, θ, β and λ is resulted. It is obvious that an exact solution is not easy to get. Therefore, a numerical method such as Newton Raphson can be used to find approximate solution of the above nonlinear system.

The algorithm has been implemented using the following steps:

(1)Use the method of moments or some other proper estimates of the parameters as initial points of iteration, denote the initials as $(\alpha_\circ, \theta_\circ, \beta_\circ, \lambda_\circ)$ for the parameters $(\alpha, \theta, \beta, \lambda)$.
(2)Calculate $\left(\frac{\partial \ln L}{\partial \alpha}, \frac{\partial \ln L}{\partial \theta}, \frac{\partial \ln L}{\partial \beta}, \frac{\partial \ln L}{\partial \lambda}\right)_{(\alpha_k,\theta_k,\beta_k,\lambda_k)}$ and the observed Fisher Information matrix $I^{-1}(\alpha, \theta, \beta, \lambda)$.
(3)Update $(\alpha, \theta, \beta, \lambda)$ as

$$(\alpha_{k+1}, \theta_{k+1}, \beta_{k+1}, \lambda_{k+1}) = (\alpha_k, \theta_k, \beta_k, \lambda_k) + \left(\frac{\partial \ln L}{\partial \alpha}, \frac{\partial \ln L}{\partial \theta}, \frac{\partial \ln L}{\partial \beta}, \frac{\partial \ln L}{\partial \lambda}\right)_{(\alpha_k,\theta_k,\beta_k,\lambda_k)} \times I^{-1}(\alpha, \theta, \beta, \lambda).$$

(4)Put $k = k+1$, and then return to step 1.
(5)Continue the consecutive steps until $|(\alpha_{k+1}, \theta_{k+1}, \beta_{k+1}, \lambda_{k+1}) - (\alpha_k, \theta_k, \beta_k, \lambda_k)| \le \varepsilon \to 0$.The final estimates of $(\alpha, \theta, \beta, \lambda)$ are the MLEs of the parameters, denoted as$\left(\hat{\alpha}, \hat{\theta}, \hat{\beta}, \hat{\lambda}\right)$.

To set up $(1-\zeta)100\%$ approximate confidence intervals for the parameters α, θ, β and λ, on the form

$$\left.\begin{array}{ll} (\hat{\alpha}_L, \hat{\alpha}_U) = \hat{\alpha} \pm z_{1-\frac{\zeta}{2}} \sqrt{var(\hat{\alpha})} & (\hat{\theta}_L, \hat{\theta}_U) = \hat{\theta} \pm z_{1-\frac{\zeta}{2}} \sqrt{var(\hat{\theta})} \\ (\hat{\beta}_L, \hat{\beta}_U) = \hat{\beta} \pm z_{1-\frac{\zeta}{2}} \sqrt{var(\hat{\beta})} & (\hat{\lambda}_L, \hat{\lambda}_U) = \hat{\lambda} \pm z_{1-\frac{\zeta}{2}} \sqrt{var(\hat{\lambda})} \end{array}\right\}, \tag{16}$$

where $z_{1-\frac{\zeta}{2}}$is the percentile of the standard normal distribution with left-tail probability $1-\frac{\zeta}{2}$ and $var(\hat{\alpha}), var(\hat{\theta}), var(\hat{\beta}), var(\hat{\lambda})$ represent the asymptotic variances of maximum likelihood estimates which can be calculated using the inverse of the Fisher information matrix, for more details see Cohen [20]. The asymptotic variance–covariance matrix for the maximum likelihood estimates can be put as follows

$$F^{-1} = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \alpha^2} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 \ln L}{\partial \theta \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \theta^2} & -\frac{\partial^2 \ln L}{\partial \theta \partial \beta} & -\frac{\partial^2 \ln L}{\partial \theta \partial \lambda} \\ -\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \beta \partial \theta} & -\frac{\partial^2 \ln L}{\partial \beta^2} & -\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \lambda \partial \theta} & -\frac{\partial^2 \ln L}{\partial \lambda \partial \beta} & -\frac{\partial^2 \ln L}{\partial \lambda^2} \end{bmatrix}^{-1}_{\downarrow(\hat{\alpha},\hat{\theta},\hat{\beta},\hat{\lambda})} = \begin{bmatrix} var(\hat{\alpha}) & Cov(\hat{\alpha}\hat{\theta}) & Cov(\hat{\alpha}\hat{\beta}) & Cov(\hat{\alpha}\hat{\lambda}) \\ Cov(\hat{\theta}\hat{\alpha}) & var(\hat{\theta}) & Cov(\hat{\theta}\hat{\beta}) & Cov(\hat{\theta}\hat{\lambda}) \\ Cov(\hat{\beta}\hat{\alpha}) & Cov(\hat{\beta}\hat{\theta}) & var(\hat{\beta}) & Cov(\hat{\beta}\hat{\lambda}) \\ Cov(\hat{\lambda}\hat{\alpha}) & Cov(\hat{\lambda}\hat{\theta}) & Cov(\hat{\lambda}\hat{\beta}) & var(\hat{\lambda}) \end{bmatrix}^{-1}, \tag{17}$$

where

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \alpha^2} = \frac{-m}{\alpha^2} - \beta \sum_{i=1}^{m} \frac{\delta_{1i}\, \phi_i(\theta)\, y_i^{\alpha-1}\left(1 + \frac{y_i^\alpha}{\beta} + \alpha \ln y_i\right)}{(\beta + y_i^\alpha)^2} \\ -\beta \sum_{i=1}^{m} \frac{\delta_{2i}\, \phi_i(\theta)\, (\psi_i(\lambda))^{\alpha-1}\left(1 + \frac{(\psi_i(\lambda))^\alpha}{\beta} + \alpha \ln(\psi_i(\lambda))\right)}{(\beta + (\psi_i(\lambda))^\alpha)^2}, \end{aligned} \tag{18}$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \theta} = -\alpha \sum_{i=1}^{m} \frac{\delta_{1i}(R_i+1) y_i^{\alpha-1}}{(\beta + y_i^\alpha)} - \alpha \sum_{i=1}^{m} \frac{\delta_{2i}\,(R_i+1)\,(\psi_i(\lambda))^{\alpha-1}}{(\beta + (\psi_i(\lambda))^\alpha)}, \tag{19}$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \alpha \sum_{i=1}^{m} \frac{\delta_{1i}\, \phi_i(\theta)\, y_i^{\alpha-1}}{(\beta + y_i^\alpha)^2} + \alpha \sum_{i=1}^{m} \frac{\delta_{2i}\, \phi_i(\theta)\, (\psi_i(\lambda))^{\alpha-1}}{(\beta + (\psi_i(\lambda))^\alpha)^2}, \tag{20}$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = \sum_{i=1}^{m} \frac{\delta_{2i}\,(y_i - \tau)}{\psi_i(\lambda)} - \alpha\beta \sum_{i=1}^{m} \frac{\delta_{2i}\, \phi_i(\theta)\, (y_i - \tau)\left(\alpha - 1 - \frac{(\psi_i(\lambda))^\alpha}{\beta}\right)(\psi_i(\lambda))^{\alpha-2}}{(\beta + (\psi_i(\lambda))^\alpha)^2}, \tag{21}$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = -\frac{m}{\theta^2}, \tag{22}$$

$$\frac{\partial^2 \ln L}{\partial \theta \partial \beta} = \frac{1}{\beta} \sum_{i=1}^{m} \frac{\delta_{1i}\,(R_i+1) y_i^\alpha}{(\beta + y_i^\alpha)} + \frac{1}{\beta} \sum_{i=1}^{m} \frac{\delta_{2i}\,(R_i+1)\,(\psi_i(\lambda))^\alpha}{(\beta + (\psi_i(\lambda))^\alpha)}, \tag{23}$$

$$\frac{\partial^2 \ln L}{\partial\theta\partial\lambda} = -\alpha \sum_{i=1}^{m} \frac{\delta_{2i}\,(R_i+1)\,(y_i-\tau)\,(\psi_i(\lambda))^{\alpha-1}}{(\beta+(\psi_i(\lambda))^{\alpha})}, \tag{24}$$

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial\beta^2} &= \frac{m}{\beta^2} + \frac{1}{\beta}\sum_{i=1}^{m} \frac{\delta_{1i}\,\phi_i(\theta)\,y_i^{\alpha}\left(-2-\frac{2}{\beta}y_i^{\alpha}+\frac{1}{\beta}\right)}{(\beta+y_i^{\alpha})^2} \\ &+\frac{1}{\beta}\sum_{i=1}^{m} \frac{\delta_{2i}\,\phi_i(\theta)\,(\psi_i(\lambda))^{\alpha}\left(-2-\frac{2}{\beta}(\psi_i(\lambda))^{\alpha}+\frac{1}{\beta}\right)}{(\beta+(\psi_i(\lambda))^{\alpha})^2},\end{aligned} \tag{25}$$

$$\frac{\partial^2 \ln L}{\partial\beta\partial\lambda} = \alpha \sum_{i=1}^{m} \frac{\delta_{2i}\,\phi_i(\theta)\,(y_i-\tau)\,(\psi_i(\lambda))^{\alpha-1}}{(\beta+(\psi_i(\lambda))^{\alpha})^2}, \tag{26}$$

and

$$\begin{aligned}\frac{\partial^2 \ln L}{\partial\lambda^2} &= -\frac{1}{\lambda^2}\sum_{i=1}^{m}\delta_{2i} - (\alpha-1)\sum_{i=1}^{m}\frac{\delta_{2i}\,(y_i-\tau)^2}{(\psi_i(\lambda))^2} \\ &-\alpha\beta\sum_{i=1}^{m}\frac{\delta_{2i}\,\phi_i(\theta)\,(y_i-\tau)^2\,(\psi_i(\lambda))^{\alpha-2}}{(\beta+(\psi_i(\lambda))^{\alpha})^2}\left(\alpha-1-\frac{(\psi_i(\lambda))^{\alpha}}{\beta}\right).\end{aligned} \tag{27}$$

4 Bootstrap Confidence Intervals

There are three types of resampling plans, non-parametric, semi-parametric and parametric. Bootstrap methods depend on these three resampling plans. For more details about resampling plans see Efron [21]. Here, confidence intervals are proposed based on the parameteric bootstrap methods where the parametric model for the data is known $f(y;.)$ up to the unknown parameters $(\alpha,\theta,\beta,\lambda)$, so that bootstrap data are sampled from $f\left(y;\hat\alpha,\hat\theta,\hat\beta,\hat\lambda\right)$,where $\left(\hat\alpha,\hat\theta,\hat\beta,\hat\lambda\right)$ are the MLEs from the original data. A lot of papers dealt only with percentile bootstrap method (Boot-p) based on the idea of Efron [21] and bootstrap-t method (Boot-t) based on the idea of Hall [22], such as Soliman et al. [4], El-Sagheer and Ahsanullah [13] and among others. In this paper, we deal with additional two types of Bootstrap CIs: (i) Boot-BC based on the idea of Diciccio and Efron [10]. (ii) Boot-BCa based on the idea of Diciccio and Efron [23]. For more survey of the parametric bootstrap methods, see Davison and Hinkley [24] and a more recently reviewed article by Kreiss and Paparoditis [25]. The following algorithm is followed to obtain bootstrap samples for the four methods:

(1)Based on the original progressively type-II sample, $\underline{y} \equiv y^{\mathbf{R}}_{1;m,n} < \ldots < y^{\mathbf{R}}_{n_1;m,n} < y^{\mathbf{R}}_{n_1+1;m,n} < \ldots < y^{\mathbf{R}}_{m;m,n}$, compute $\hat\alpha$, $\hat\theta$, $\hat\beta$ and $\hat\lambda$.
(2)Use $\hat\alpha$, $\hat\theta$, $\hat\beta$ and $\hat\lambda$ to generate a bootstrap sample $\underline{y}^*$ with the same values of R_i, $i=1,2,\ldots,m$ using algorithm presented in Balakrishnan and Sandhu [26].
(3)As in Step1 based on $\underline{y}^*$, compute the bootstrap sample estimates of $\hat\alpha$, $\hat\theta$, $\hat\beta$ and $\hat\lambda$ say $\hat\alpha^*$, $\hat\theta^*$, $\hat\beta^*$ and $\hat\lambda^*$.
(4)Repeat the previous steps 2 and 3 B times and arrange all $\hat\alpha^*$, $\hat\theta^*$, $\hat\beta^*$ and $\hat\lambda^*$ in ascending order to obtain the bootstrap sample $\left(\hat\Omega_k^{*[1]},\hat\Omega_k^{*[2]},\ldots,\hat\Omega_k^{*[B]}\right)$, $k=1,2,3,4$, where $\hat\Omega_1^*=\hat\alpha^*$, $\hat\Omega_2^*=\hat\theta^*$, $\hat\Omega_3^*=\hat\beta$, $\hat\Omega_4^*=\hat\lambda^*$.

4.1 Bootstrap-p Confidence Interval

Let $\Phi(z)=P(\hat\Omega_k^*\le z)$ be the cumulative distribution function of $\hat\Omega_k^*$. Define $\hat\Omega^*_{kBoot}=\Phi^{-1}(z)$ for given z. The approximate bootstrap-p $100(1-\zeta)\%$ confidence interval of $\hat\Omega_k^*$ is given by

$$\left[\,\hat\Omega^*_{kBoot}(\tfrac{\zeta}{2})\;,\;\hat\Omega^*_{kBoot}(1-\tfrac{\zeta}{2})\,\right]. \tag{28}$$

4.2 Bootstrap-t Confidence Interval

Consider the order statistics $\mu_k^{*[1]} < \mu_k^{*[2]} < \ldots < \mu_k^{*[B]}$ where

$$\mu_k^{*[j]} = \frac{\sqrt{B}(\hat{\Omega}_k^{*[j]} - \hat{\Omega}_k)}{\sqrt{Var\left(\hat{\Omega}_k^{*[j]}\right)}}, \; j = 1,2,...,B; \; k = 1,2,3,4, \tag{29}$$

where $\hat{\Omega}_k = \hat{\alpha}$, $\hat{\Omega}_k = \hat{\theta}$, $\hat{\Omega}_k = \hat{\beta}$ and $\hat{\Omega}_k = \hat{\lambda}$ while $Var\left(\hat{\Omega}_k^{*[j]}\right)$ is obtained using the inverse of the Fisher information matrix as done before in (17). Let $W(z) = P\left(\mu_k^* < z\right), k = 1,2,3,4$ be the cumulative distribution function of μ_k^*. For a given z, define

$$\hat{\Omega}_{kBoot-t}^* = \hat{\Omega}_k + B^{\frac{-1}{2}}\sqrt{Var\left(\hat{\Omega}_k^*\right)}W^{-1}(z). \tag{30}$$

Thus, the approximate bootstrap-t $100(1-\zeta)\%$ confidence interval of $\hat{\Omega}_k^*$ is given by

$$\left[\hat{\Omega}_{kBoot-t}^*(\tfrac{\zeta}{2}) \,, \hat{\Omega}_{kBoot-t}^*(1-\tfrac{\zeta}{2})\right]. \tag{31}$$

4.3 Bootstrap Bias Corrected Confidence Interval

Let $\Phi(z) = \zeta$ be the standard normal cumulative distribution function, with $z_\zeta = \Phi^{-1}(\zeta)$. Define the bias-correction constant $z_\circ$ from the following probability $P(\hat{\Omega}_k^* \leq \hat{\Omega}_k) = G(z_\circ), k = 1,2,3,4$, where $G(.)$ is cumulative distribution function of the bootstrap distribution and

$$P(\hat{\Omega}_k^* \leq \hat{\Omega}_k) = \frac{\#\left\{\hat{\Omega}_k^{*[j]} < \hat{\Omega}_k\right\}}{B}, \; j = 1,2,...,B; \; k = 1,2,3,4.$$

Thus

$$z_\circ = \Phi^{-1}\left(\frac{\#\left\{\hat{\Omega}_k^{*[j]} < \hat{\Omega}_k\right\}}{B}\right), \; j = 1,2,...,B; \; k = 1,2,3,4. \tag{32}$$

For a given ζ, and the bias-correction constant $z_\circ$,then

$$\hat{\Omega}_{kBoot-BC}^* = G^{-1}\left[\Phi\left(2z_\circ + z_\zeta\right)\right]. \tag{33}$$

Thus, the approximate bootstrap-BC $100(1-\zeta)\%$ confidence interval of $\hat{\Omega}_{kBoot-BC}^*$ is given by

$$\left[\hat{\Omega}_{kBoot-BC}^*(\tfrac{\zeta}{2}) \,, \hat{\Omega}_{kBoot-BC}^*(1-\tfrac{\zeta}{2})\right]. \tag{34}$$

4.4 Bootstrap Bias Corrected Accelerated Confidence Interval

Let $\Phi(z) = \zeta$ be the standard normal cumulative distribution function, with $z_\zeta = \Phi^{-1}(\zeta)$ and the bias-correction constant $z_\circ$ which is defined in (32). Then

$$\hat{\Omega}_{kBoot-BCa}^* = G^{-1}\left[\Phi\left(z_\circ + \frac{z_\circ + z_\zeta}{1 - a(z_\circ + z_\zeta)}\right)\right], k = 1,2,3,4, \tag{35}$$

where a is called the acceleration factor wich is estimated by a simple jack-knife method. Let $\underline{y}_i$ represent the original data with the ith point omitted, say $\underline{y}_2 = y_{1;m,n}^{\mathbf{R}} < y_{3;m,n}^{\mathbf{R}} < \ldots < y_{n_1;m,n}^{\mathbf{R}} < y_{n_1+1;m,n}^{\mathbf{R}} < \ldots < y_{m;m,n}^{\mathbf{R}}$, and $\hat{\Omega}_k^i = \hat{\Omega}_k(\underline{y}_i)$ be the estimate of Ω_k constructed from this data, $\Omega_1 = \alpha$, $\Omega_2 = \theta$, $\Omega_3 = \lambda$ and $\Omega_4 = \lambda$. Let $\bar{\Omega}_k$ be the mean of the $\hat{\Omega}_k^i$'s. Then a is estimated by

$$a = \frac{\sum_{i=1}^m \left(\bar{\Omega}_k - \hat{\Omega}_k^i\right)^3}{6\left[\sum_{i=1}^m \left(\bar{\Omega}_k - \hat{\Omega}_k^i\right)^2\right]^{\frac{3}{2}}}, k = 1,2,3,4. \tag{36}$$

For more details see Efron and Tibshirani [27] and Davison and Hinkley [24]. If $a = 0$, equation (35) reduces to equation (33). Then, the approximate bootstrap-BC $100(1-\zeta)\%$ confidence interval of $\hat{\Omega}_{kBoot-BCa}^*$ is given by

$$\left[\hat{\Omega}_{kBoot-BCa}^*(\tfrac{\zeta}{2}) \,, \hat{\Omega}_{kBoot-BCa}^*(1-\tfrac{\zeta}{2})\right]. \tag{37}$$

5 Bayesian Estimation Using MCMC Technique

Bayesian statistics is interested in fitting a probability model to a set of data and summarizing the result by a probability distribution on the parameters of the model. The given data comes from the likelihood function and the prior distribution function and the resulting distributions called the posterior distributions. If the independent priors for the parameters α, θ, β and λ takes the following forms:

$$\left.\begin{array}{l}\pi(\alpha) \propto \alpha^{-1},\ \alpha>0,\ \pi(\theta) \propto \theta^{-1},\ \theta>0, \\ \pi(\beta) \propto \beta^{-1},\ \beta>0,\ \pi(\lambda) \propto \lambda^{-1},\ \lambda>1\end{array}\right\}. \tag{38}$$

Then, the joint prior of the parameters α, θ, β and λ can be written as

$$\pi(\alpha,\theta,\beta,\lambda) \propto (\alpha\theta\beta\lambda)^{-1},\alpha>0,\theta>0,\ \beta>0,\lambda>1. \tag{39}$$

The joint posterior density function of α, θ, β and λ, denoted by $\pi^*(\alpha,\theta,\beta,\lambda|\underline{y})$ can be written as

$$\pi^*(\alpha,\theta,\beta,\lambda|\underline{y}) = \frac{L(\alpha,\theta,\beta,\lambda)\times\pi(\alpha,\theta,\beta,\lambda)}{\int_1^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\theta,\beta,\lambda)\times\pi(\alpha,\theta,\beta,\lambda)\,d\alpha d\theta d\beta d\lambda} \tag{40}$$

Therefore, the Bayes estimate of any function of the parameters, say $h(\alpha,\theta,\beta,\lambda)$, using squared error loss function (SEL) is

$$\begin{aligned}\hat{h}(\alpha,\theta,\beta,\lambda) &= E_{\alpha,\theta,\beta,\lambda|\underline{y}}[h(\alpha,\theta,\beta,\lambda)] \\ &= \frac{\int_1^\infty\int_0^\infty\int_0^\infty\int_0^\infty h(\alpha,\theta,\beta,\lambda)\times L(\alpha,\theta,\beta,\lambda)\times\pi(\alpha,\theta,\beta,\lambda)\,d\alpha d\theta d\beta d\lambda}{\int_1^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\alpha,\theta,\beta,\lambda)\times\pi(\alpha,\theta,\beta,\lambda)\,d\alpha d\theta d\beta d\lambda}.\end{aligned} \tag{41}$$

Generally, the ratio of two integrals given by (41) cannot be obtained in a closed form. In this case, the MCMC technique will be used to generate samples from the posterior distributions and then the Bayes estimates of the parameters α, θ, β and λ will be computed. The main theme of the MCMC technique is to compute an approximate value of integrals in (41). An important sub-class of MCMC methods are Gibbs sampling and more general Metropolis within-Gibbs samplers. The Metropolis algorithm is a random walk that uses an acceptance/rejection rule to converge to the target distribution. The Metropolis algorithm was first proposed in Metropolis et al. [28] and It was then generalized by Hastings [29]. Made into mainstream statistics and engineering via the articles Gelfand and Smith [30] and Gelfand et al. [31] which presented the Gibbs sampler as used in Geman and Geman [32]. From (9), (39) and (40), the joint posterior density function of α, θ, β and λ can be written as

$$\begin{aligned}\pi^*(\alpha,\theta,\beta,\lambda|\underline{y}) \propto\ & \alpha^{m-1}\theta^{m-1}\beta^{-(m+1)}\lambda^{\left(\sum_{i=1}^m\delta_{2i}\right)-1}\times \\ & \prod_{i=1}^m\left\{\left[y_i^\alpha\left(1+\tfrac{1}{\beta}y_i^\alpha\right)^{-\phi_i(\theta)}\right]^{\delta_{1i}}\times\left[(\psi_i(\lambda))^{\alpha-1}\left(1+\tfrac{1}{\beta}(\psi_i(\lambda))^\alpha\right)^{-\phi_i(\theta)}\right]^{\delta_{2i}}\right\}.\end{aligned} \tag{42}$$

The conditional posterior densities of α, θ, β and λ can be given as

$$\pi_1^*(\alpha|\theta,\beta,\lambda,\underline{y}) \propto \alpha^{m-1}\prod_{i=1}^m\left\{\left[y_i^\alpha\left(1+\tfrac{1}{\beta}y_i^\alpha\right)^{-\phi_i(\theta)}\right]^{\delta_{1i}}\times\left[(\psi_i(\lambda))^{\alpha}\left(1+\tfrac{1}{\beta}(\psi_i(\lambda))^\alpha\right)^{-\phi_i(\theta)}\right]^{\delta_{2i}}\right\}, \tag{43}$$

$$\pi_2^*(\theta|\alpha,\beta,\lambda,\underline{y}) \equiv gamma\left[m,\sum_{i=1}^m\left\{\delta_{1i}(R_i+1)\ln\left(1+\tfrac{1}{\beta}y_i^\alpha\right)+\delta_{2i}(R_i+1)\ln\left(1+\tfrac{1}{\beta}(\psi_i(\lambda))^\alpha\right)\right\}\right], \tag{44}$$

$$\pi_3^*(\beta|\alpha,\theta,\lambda,\underline{y}) \propto \beta^{-(m+1)}\prod_{i=1}^m\left\{\left[\left(1+\tfrac{1}{\beta}y_i^\alpha\right)^{-\phi_i(\theta)}\right]^{\delta_{1i}}\times\left[\left(1+\tfrac{1}{\beta}(\psi_i(\lambda))^\alpha\right)^{-\phi_i(\theta)}\right]^{\delta_{2i}}\right\}, \tag{45}$$

and

$$\pi_4^*(\lambda|\alpha,\theta,\beta,\underline{y}) \propto \lambda^{\left(\sum_{i=1}^m\delta_{2i}\right)-1}\prod_{i=1}^m\left[(\psi_i(\lambda))^{\alpha-1}\left(1+\frac{1}{\beta}(\psi_i(\lambda))^\alpha\right)^{-\phi_i(\theta)}\right]^{\delta_{2i}}. \tag{46}$$

Figure 1 shows that all the conditional posterior distributions are almost symmetrical and seem to be quite skewed. Now, the following steps illustrate the method of the Metropolis–Hastings algorithm within Gibbs sampling to generate the posterior samples as suggested by Tierney [33], and in turn obtain the Bayes estimates and the corresponding credible intervals:

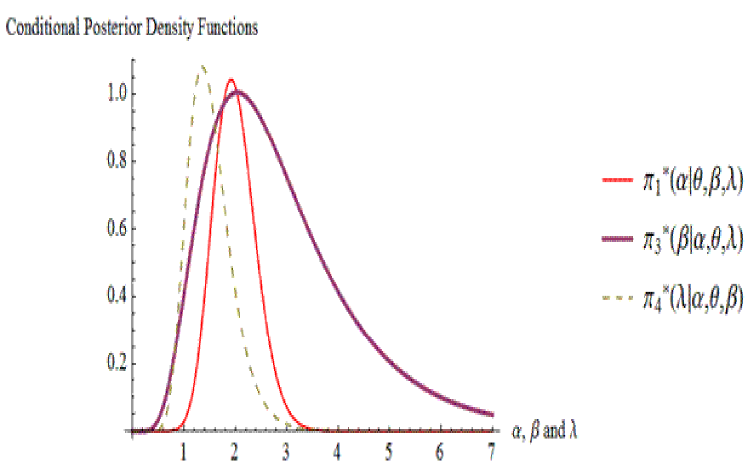

Fig. 1: The Conditional Posterior Density Functions.

(1)Start with an $\left(\alpha^{(0)}=\hat{\alpha},\ \theta^{(0)}=\hat{\theta},\ \beta^{(0)}=\hat{\beta} \text{ and } \lambda^{(0)}=\hat{\lambda}\right)$.
(2)Put $i=1$.
(3)Generate $\theta^{(i)}$ from

$$\text{gamma distribution}\left[m,\sum_{i=1}^{m}\left\{\delta_{1i}\left(R_i+1\right)\ln\left(1+\tfrac{1}{\beta}y_i^{\alpha}\right)+\delta_{2i}\left(R_i+1\right)\ln\left(1+\tfrac{1}{\beta}\left(\psi_i\left(\lambda\right)\right)^{\alpha}\right)\right\}\right].$$

(4)Using the following Metropolis-Hastings method, generate $\alpha^{(i)},\beta^{(i)}$ and $\lambda^{(i)}$ from (43), (45) and (46) with the normal suggested distribution

$$N(\alpha^{(i-1)},var(\alpha)),\ N(\beta^{(i-1)},var(\beta))\text{and } N(\lambda^{(i-1)},var(\lambda)),\text{respectively.}$$

Where $var(\alpha)$, $var(\beta)$ and $var(\lambda)$ can be obtained from the main diagonal in inverse Fisher information matrix (17).
i-Generate a proposal α^* from $N(\alpha^{(i-1)},var(\alpha)),\beta^*$ from $N(\beta^{(i-1)},var(\beta))$ and λ^* from $N(\lambda^{(i-1)},var(\lambda))$.
ii-Evaluate the acceptance probabilities

$$\left.\begin{aligned}\rho_{\alpha}&=\min\left[1,\frac{\pi_1^*(\alpha^*|\theta^{(i)},\beta^{(i-1)},\lambda^{(i-1)},\underline{y})}{\pi_1^*(\alpha^{(i-1)}|\theta^{(i)},\beta^{(i-1)},\lambda^{(i-1)},\underline{y})}\right],\\ \rho_{\beta}&=\min\left[1,\frac{\pi_3^*(\beta^*|\alpha^{(i)},\theta^{(i)},\lambda^{(i-1)},\underline{y})}{\pi_3^*(\beta^{(i-1)}|\alpha^{(i)},\theta^{(i)},\lambda^{(i-1)},\underline{y})}\right],\\ \rho_{\lambda}&=\min\left[1,\frac{\pi_4^*(\lambda^*|\alpha^{(i)},\theta^{(i)},\beta^{(i)},\underline{y})}{\pi_4^*(\lambda^{(i-1)}|\alpha^{(i)},\theta^{(i)},\beta^{(i)},\underline{y})}\right].\end{aligned}\right\}. \tag{47}$$

iii-Generate u_1,u_2 and u_3 from a Uniform $(0,1)$ distribution.
iv-If $u_1\leq\rho_{\alpha}$ accept the proposal and set $\alpha^{(i)}=\alpha^*$, else set $\alpha^{(i)}=\alpha^{(i-1)}$.
v-If $u_2\leq\rho_{\beta}$ accept the proposal and set $\beta^{(i)}=\beta^*$, else set $\beta^{(i)}=\beta^{(i-1)}$.
vi-If $u_3\leq\rho_{\lambda}$ accept the proposal and set $\lambda^{(i)}=\lambda^*$, else set $\lambda^{(i)}=\lambda^{(i-1)}$.
(5)Compute $\alpha^{(i)},\beta^{(i)}$ and $\lambda^{(i)}$.
(6)Put $i=i+1$.

(7)Repeat steps $(3-6)$ N-times

(8)In order to guarantee the convergence and to remove the influence of the selection of initial values, the first M simulated varieties are ignored. Then the selected samples are $\alpha^{(i)}, \beta^{(i)}$ and $\lambda^{(i)}$, $i = M+1, \ldots, N$, for sufficiently large N, forms an approximate posterior samples which can be used to obtain the Bayes MCMC point estimates of α, θ, β and λ as

$$\left.\begin{array}{ll} \alpha_{MCMC} = \frac{1}{N-M} \sum_{i=M+1}^{N} \alpha^{(i)}, & \theta_{MCMC} = \frac{1}{N-M} \sum_{i=M+1}^{N} \theta^{(i)}, \\ \beta_{MCMC} = \frac{1}{N-M} \sum_{i=M+1}^{N} \beta^{(i)}, & \lambda_{MCMC} = \frac{1}{N-M} \sum_{i=M+1}^{N} \lambda^{(i)} \end{array}\right\}. \tag{48}$$

(9)To calculate the credible intervals (CRIs) of Ω_k where $\Omega_1 = \alpha$, $\Omega_2 = \theta$, $\Omega_3 = \lambda$ and $\Omega_4 = \lambda$, we take the quantiles of the sample as the endpoints of the intervals. Sort $\{\Omega_k^{M+1}, \Omega_k^{M+2}, \ldots, \Omega_k^{N}\}$ as $\left\{\Omega_k^{(1)}, \Omega_k^{(2)}, \ldots, \Omega_k^{(N-M)}\right\}$. Hence the $100\ (1-\gamma)\%$ symmetric credible interval of Ω_k is

$$\left[\Omega_k\left(\tfrac{\gamma}{2}(N-M)\right),\ \Omega_k\left(\left(1-\tfrac{\gamma}{2}\right)(N-M)\right)\right]. \tag{49}$$

6 Explanatory Example

In this section, a simulation example is presented to assess the estimation procedures. In this example, a PROG-II-C sample from WG distribution under SSPALT model is generated. The algorithm of generation is performed according to the algorithm described in Balakrishnan and Sandhu [26] as the following:

(1)Specify the values of n, m and $R_i, i = 1, 2, \ldots, m$.

(2)Specify the values of the parameters α, θ, β and λ.

(3)Specify the values of the stress change time τ.

(4)Generate a random sample with size n and censoring size m from the random variable Y given by (4), the set of data can be considered as:

$$y_{1;m,n}^{\mathbf{R}} < \ldots < y_{n_1;m,n}^{\mathbf{R}} < y_{n_{1+1};m,n}^{\mathbf{R}} < \ldots < y_{m;m,n}^{\mathbf{R}},$$

where $R = (R_1, R_2, \ldots, R_m)$ and $\sum_{i=1}^{m} R_i = n - m$.

(5)Use the PROG-II-C sample to compute the MLEs of the model parameters. The Newton–Raphson method is applied for solving the nonlinear system to obtain the MLEs of the parameters.

(6)Compute the 95% bootstrap conidence intervals for the model parameters, using the steps described in Section 4.

(7)Compute the Bayes estimates of the model parameters based on MCMC algorithm described in Section 5.

A simulation data for progressive type-II censored sample under SSPALT model from Weibull-Gamma distribution with true values $\alpha = 2.5$, $\theta = 0.4$, $\beta = 1.5$ and acceleration factor $\lambda = 2$, and $\tau = 0.7$, using progressive censoring schemes $n = 30$, $m = 15$ and $R = (3,0,3,0,2,0,2,0,3,0,1,0,1,0,0)$ has been approximated to four decimal places and it has been presented in Table 1.

Table 1. SSPALT simulation data with true values for α, θ, β and λ

Failure times under normal conditions	Failure times under accelerated conditions						
0.4145	0.7424	0.9136	0.9501	1.3307	1.6883	3.0142	9.8213
0.4948	0.7987	0.9441	1.1193	1.3850	2.1915	4.1789	

In the MCMC approach, we run the chain for 12000 times and discard the first 2000 values as 'burn-in'.

Table 2. Different point estimates for $(\alpha, \theta, \beta, \lambda) = (2.5, 0.4, 1.5, 2)$.

Parameters	$(.)_{ML}$	$(.)_{Boot-p}$	$(.)_{Boot-t}$	$(.)_{MCMC}$
α	2.7195	3.3618	2.1797	2.6711
θ	0.3469	0.3529	0.3111	0.343
β	1.4094	1.4287	1.0968	1.375
λ	1.7878	1.8384	1.5715	1.8493

Table 3. 95% confidence intervals for α, θ, β and λ.

Method	α	Length	θ	Length
ACI	$[-1.738, 7.1767]$	8.9145	$[-0.370, 1.0638]$	1.43378
Boot -p CI	$[1.6457, 4.9103]$	3.26454	$[0.1470, 0.7764]$	0.629425
Boot -t CI	$[1.3668, 2.6364]$	1.26961	$[0.1960, 0.3403]$	0.144343
Boot-BC CI	$[1.2999, 4.1972]$	2.8973	$[0.1365, 0.8522]$	0.715704
Boot-BCa CI	$[1.3749, 4.1836]$	2.80869	$[0.0925, 0.7303]$	0.637793
CRI	$[2.6347, 2.6956]$	0.06084	$[0.1927, 0.5379]$	0.34513
Method	β	Length	λ	Length
ACI	$[-4.571, 7.3893]$	11.9599	$[-2.077, 5.6526]$	7.72960
Boot -p CI	$[0.2224, 2.9128]$	2.69035	$[0.7167, 3.9532]$	3.23659
Boot -t CI	$[0.1345, 1.4016]$	1.26705	$[0.7263, 1.7554]$	1.02909
Boot-BC CI	$[0.2842, 2.8691]$	2.58489	$[0.7891, 5.3758]$	4.58672
Boot-BCa CI	$[0.0402, 2.5562]$	2.51595	$[0.8091, 4.1715]$	3.36237
CRI	$[1.3156, 1.3990]$	0.08348	$[1.8192, 1.8706]$	0.05132

Table 4. MCMC results for α, θ, β and λ.

Parameters	Mean	Median	Mode	Variance	S.D	Skewness
α	2.6711	2.6764	2.6871	0.00031	0.01771	-0.7293
θ	0.343	0.3349	0.3188	0.00789	0.08884	0.52349
β	1.375	1.3806	1.3919	0.00048	0.02199	-1.6770
λ	1.8493	1.8503	1.8523	0.00018	0.01331	-0.66505

7 Simulation Study

This section provides some results based on Monte Carlo simulations to assess the performance of the different methods. All computations were computerized using (MATHEMATICA program version 9.0). PROG-II-C Weibull-Gamma samples are generated according to SSPALT model using the algorithm proposed by Balakrishnan and Aggarwala [2]. The comparison between the different methods of the resulting estimators of α, θ, β, and λ has been considered in their mean square error (MSE) which is computed, for$k = 1, 2, 3$ and ($\Omega_1 = \alpha,\ \Omega_2 = \theta,\ \Omega_3 = \beta,\ \Omega_4 = \lambda$),as

$$MSE(\Omega_k) = \frac{1}{M}\sum_{i=1}^{M}\left(\hat{\Omega}_k^{(i)} - \Omega_k\right)^2,$$

where $M = 1000$ is the number of simulated samples. Another criterion is used to compare (CIs) obtained by using asymptotic distributions of the MLEs and MCMC credible intervals (CRIs). The comparison of them is made in terms of the average confidence interval lengths (ACLs) and coverage probability (CP). The CP of a confidence interval is the proportion of the time that the interval contains the true value of interest. In this study, the following censoring schemes (CSs) are taken into consideration:

Scheme A : $R_1 = n - m,\ R_i = 0$ for $i \neq 1$.

Scheme B : $R_{\frac{m}{2}} = R_{\frac{m}{2}+1} = \frac{n-m}{2},\ R_i = 0$ for $i \neq \frac{m}{2}$ and $i \neq \frac{m}{2} + 1$.

Scheme C : $R_m = n - m,\ R_i = 0$ for $i \neq m$.

Table 5. MSE of ML and Bayes MCMC estimates for the parameters with $(\alpha,\theta,\beta,\lambda) = (2.5,0.4,1.5,2)$ at $\tau = 0.5$.

		MLE				*MCMC*			
(n,m)	CS	α	θ	β	λ	α	θ	β	λ
(30,20)	A	3.1948 (1.3837)	0.3611 (0.0249)	1.1239 (0.7984)	1.7313 (0.7697)	3.1971 (1.3910)	0.3608 (0.0251)	1.1159 (0.7916)	1.7319 (0.7639)
	B	3.2044 (1.3893)	0.3613 (0.0297)	1.1302 (0.7668)	1.7071 (0.6612)	3.2012 (1.3822)	0.3629 (0.0302)	1.1314 (0.7642)	1.7066 (0.6646)
	C	3.2668 (1.4570)	0.3763 (0.0341)	1.1233 (0.7684)	1.6439 (0.7761)	3.2642 (1.4537)	0.3779 (0.0350)	1.1191 (0.7595)	1.6419 (0.7758)
(40,20)	A	3.2225 (1.3181)	0.3593 (0.0280)	1.0905 (0.7969)	1.6888 (0.7191)	3.2224 (1.3184)	0.3592 (0.0279)	1.0858 (0.7904)	1.6905 (0.7252)
	B	3.1231 (1.2790)	0.3660 (0.0285)	1.1317 (0.7214)	1.7564 (0.6572)	3.1181 (1.2677)	0.3679 (0.0295)	1.1370 (0.7181)	1.7571 (0.6588)
	C	3.2774 (1.4747)	0.3687 (0.0345)	1.0530 (0.7957)	1.6467 (0.8291)	3.2734 (1.4651)	0.3717 (0.0358)	1.0602 (0.7966)	1.6462 (0.8283)
(40,30)	A	3.1866 (1.2653)	0.3562 (0.0222)	1.1459 (0.7339)	1.7403 (0.6213)	3.1859 (1.2650)	0.3575 (0.0228)	1.1479 (0.7307)	1.7388 (0.6192)
	B	3.1429 (1.2216)	0.3641 (0.0252)	1.1861 (0.7385)	1.7562 (0.6026)	3.1405 (1.2202)	0.3663 (0.0261)	1.1942 (0.7348)	1.7570 (0.6057)
	C	3.1798 (1.2970)	0.3783 (0.0293)	1.2008 (0.7378)	1.7078 (0.7092)	3.1769 (1.2938)	0.3817 (0.0309)	1.2098 (0.7342)	1.7067 (0.7086)
(60,40)	A	3.1916 (1.2482)	0.3539 (0.0187)	1.1433 (0.7773)	1.6940 (0.6146)	3.1860 (1.2392)	0.3568 (0.0190)	1.1606 (0.7796)	1.6928 (0.6125)
	B	3.1117 (1.1502)	0.3587 (0.0189)	1.1982 (0.6934)	1.7993 (0.5557)	3.1047 (1.1422)	0.3635 (0.0195)	1.2264 (0.6978)	1.8003 (0.5574)
	C	3.1844 (1.2524)	0.3778 (0.0239)	1.1634 (0.6929)	1.6771 (0.6386)	3.1761 (1.2370)	0.3854 (0.0260)	1.1994 (0.6971)	1.6768 (0.6396)

Table 6. MSE of ML and Bayes MCMC estimates for the parameters with $(\alpha,\theta,\beta,\lambda) = (2.5,0.4,1.5,2)$ at $\tau = 0.7$.

		MLE				*MCMC*			
(n,m)	CS	α	θ	β	λ	α	θ	β	λ
(30,20)	A	3.1681 (1.3055)	0.3796 (0.0305)	1.3455 (0.5651)	1.9207 (0.8491)	3.1658 (1.3018)	0.3806 (0.0310)	1.3488 (0.5606)	1.9222 (0.8539)
	B	3.1570 (1.2736)	0.3879 (0.0326)	1.3585 (0.5611)	1.8882 (0.8209)	3.1535 (1.2684)	0.3896 (0.0329)	1.3687 (0.5599)	1.8890 (0.8219)
	C	3.2224 (1.3821)	0.3809 (0.0353)	1.3104 (0.6013)	1.8599 (1.0186)	3.2176 (1.3725)	0.3846 (0.0370)	1.3260 (0.6036)	1.8601 (1.0167)
(40,20)	A	3.2204 (1.3908)	0.3754 (0.0289)	1.2800 (0.5912)	1.8660 (0.8714)	3.2186 (1.3853)	0.3765 (0.0293)	1.2869 (0.5905)	1.8675 (0.8767)
	B	3.1053 (1.1502)	0.3838 (0.0274)	1.3741 (0.5323)	1.9430 (0.8237)	3.0995 (1.1412)	0.3870 (0.0281)	1.3915 (0.5373)	1.9432 (0.8259)
	C	3.2360 (1.3782)	0.3494 (0.0340)	1.2430 (0.5740)	2.0051 (1.0839)	3.2293 (1.3677)	0.3558 (0.0366)	1.2715 (0.5844)	2.0040 (1.0787)
(40,30)	A	3.1272 (1.1938)	0.3740 (0.0231)	1.3189 (0.5738)	1.8822 (0.7535)	3.1206 (1.1824)	0.3767 (0.0238)	1.3370 (0.5790)	1.8820 (0.7542)
	B	3.1718 (1.2414)	0.3700 (0.0238)	1.3212 (0.5452)	1.8729 (0.7033)	3.1638 (1.2304)	0.3734 (0.0243)	1.3436 (0.5492)	1.8735 (0.7052)
	C	3.1892 (1.2000)	0.3884 (0.0287)	1.3370 (0.5330)	1.7718 (0.7514)	3.1806 (1.1878)	0.3932 (0.0303)	1.3639 (0.5416)	1.7724 (0.7522)
(60,40)	A	3.0440 (1.0265)	0.3785 (0.0185)	1.3406 (0.5225)	1.9167 (0.7151)	3.0354 (1.0134)	0.3819 (0.0189)	1.3672 (0.5297)	1.9171 (0.7172)
	B	3.0409 (1.0288)	0.3736 (0.0192)	1.3655 (0.5129)	1.9553 (0.6031)	3.0304 (1.0119)	0.3787 (0.0197)	1.4014 (0.5233)	1.9551 (0.6024)
	C	3.1580 (1.1387)	0.3764 (0.0225)	1.3266 (0.4919)	1.8555 (0.7247)	3.1484 (1.1221)	0.3835 (0.0242)	1.3697 (0.5041)	1.8547 (0.7230)

Table 7. MSE of ML and MCMC estimates for the parameters with $(\alpha,\theta,\beta,\lambda) = (2.5,0.4,1.5,3)$ at $\tau = 0.5$.

		MLE				*MCMC*			
(n,m)	*CS*	α	θ	β	λ	α	θ	β	λ
(30,20)	*A*	3.2105 (1.3417)	0.3612 (0.0271)	1.0811 (0.8135)	2.4376 (1.3822)	3.2092 (1.3382)	0.3615 (0.0273)	1.0753 (0.8055)	2.4363 (1.3793)
	B	3.2255 (1.3471)	0.3711 (0.0313)	1.1532 (0.7633)	2.4981 (1.2372)	3.2240 (1.3487)	0.3721 (0.0318)	1.1499 (0.7568)	2.4966 (1.2384)
	C	3.2592 (1.3853)	0.3779 (0.0356)	1.1354 (0.7406)	2.3946 (1.4982)	3.2574 (1.3799)	0.3796 (0.0366)	1.1323 (0.7310)	2.3911 (1.4927)
(40,20)	*A*	3.2983 (1.4231)	0.3478 (0.0283)	1.0101 (0.7928)	2.4290 (1.2835)	3.2975 (1.4238)	0.3483 (0.0286)	1.0089 (0.7938)	2.4247 (1.2764)
	B	3.1729 (1.3412)	0.3733 (0.0288)	1.1054 (0.7521)	2.5103 (1.3222)	3.1711 (1.3368)	0.3757 (0.0296)	1.1138 (0.7488)	2.5097 (1.3285)
	C	3.2492 (1.4129)	0.3820 (0.0391)	1.0849 (0.7644)	2.4365 (1.6098)	3.2483 (1.4154)	0.3863 (0.0415)	1.0955 (0.7688)	2.4347 (1.6153)
(40,30)	*A*	3.2112 (1.2604)	0.3508 (0.0204)	1.1278 (0.7443)	2.5707 (1.2662)	3.2081 (1.2577)	0.3521 (0.0207)	1.1269 (0.7426)	2.5674 (1.2652)
	B	3.1446 (1.2847)	0.3673 (0.0252)	1.1885 (0.7430)	2.6103 (1.2326)	3.1404 (1.2771)	0.3701 (0.0260)	1.1978 (0.7377)	2.6091 (1.2332)
	C	3.2066 (1.2761)	0.3798 (0.0290)	1.1698 (0.7204)	2.4863 (1.4306)	3.2013 (1.2697)	0.3832 (0.0303)	1.1788 (0.7136)	2.4871 (1.4347)
(60,40)	*A*	3.1793 (1.2245)	0.3596 (0.0200)	1.1518 (0.7293)	2.5429 (1.1937)	3.1746 (1.2185)	0.3624 (0.0205)	1.1678 (0.7262)	2.5417 (1.1914)
	B	3.1213 (1.0857)	0.3652 (0.0181)	1.1784 (0.6858)	2.5900 (1.1607)	3.1139 (1.0752)	0.3700 (0.0187)	1.2071 (0.6877)	2.5921 (1.1601)
	C	3.1769 (1.2038)	0.3842 (0.0266)	1.1476 (0.6796)	2.4384 (1.2943)	3.1675 (1.1870)	0.3925 (0.0294)	1.1858 (0.6870)	2.4371 (1.2953)

Table 8. Comparisons of ACL and CP of 95% CIs for the parameters with $(\alpha,\theta,\beta,\lambda) = (2.5,0.4,1.5,2)$ at $\tau = 0.5$.

		MLE				*MCMC*			
(n,m)	*CS*	α	θ	β	λ	α	θ	β	λ
(30,20)	*A*	10.4792 (0.9489)	1.2548 (0.9538)	12.2017 (0.9393)	8.7697 (0.9342)	0.0695 (0.9407)	0.3157 (0.9325)	0.0848 (0.9329)	0.0599 (0.9583)
	B	9.9974 (0.9598)	1.2323 (0.9667)	11.4378 (0.9532)	7.6266 (0.9582)	0.0668 (0.9344)	0.3170 (0.9493)	0.0825 (0.9531)	0.0510 (0.9380)
	C	11.4543 (0.9428)	1.7692 (0.9400)	13.5299 (0.9374)	8.0375 (0.9570)	0.0748 (0.9304)	0.3307 (0.9524)	0.0950 (0.9677)	0.0545 (0.9608)
(40,20)	*A*	9.5655 (0.9636)	1.1506 (0.9545)	11.3585 (0.9459)	8.1057 (0.9585)	0.0667 (0.9637)	0.3139 (0.9445)	0.0774 (0.9641)	0.0556 (0.9527)
	B	9.0243 (0.9668)	1.2187 (0.9477)	10.4495 (0.9519)	7.2425 (0.9334)	0.0607 (0.9360)	0.3216 (0.9519)	0.0729 (0.9419)	0.0493 (0.9387)
	C	10.5390 (0.9535)	2.1387 (0.9452)	12.4174 (0.9345)	7.3565 (0.9688)	0.0711 (0.9466)	0.3254 (0.9622)	0.0887 (0.9362)	0.0507 (0.9583)
(40,30)	*A*	9.3978 (0.9447)	1.0789 (0.9658)	11.3493 (0.9526)	7.9712 (0.9469)	0.0644 (0.9525)	0.2557 (0.9421)	0.0805 (0.9524)	0.0528 (0.9337)
	B	8.6929 (0.9362)	1.0571 (0.9535)	10.5407 (0.9479)	7.1796 (0.9663)	0.0605 (0.9621)	0.2619 (0.9466)	0.0776 (0.9316)	0.0479 (0.9482)
	C	9.5133 (0.9462)	1.3462 (0.9465)	11.9067 (0.9627)	7.4482 (0.9663)	0.0648 (0.9596)	0.2736 (0.9494)	0.0871 (0.9585)	0.0501 (0.9500)
(60,40)	*A*	8.2498 (0.9577)	0.9356 (0.9345)	10.2342 (0.9616)	7.0203 (0.9497)	0.0557 (0.9425)	0.2214 (0.9563)	0.0748 (0.9626)	0.0471 (0.9374)
	B	7.6968 (0.9474)	0.9283 (0.9451)	9.1651 (0.9516)	6.2891 (0.9369)	0.0525 (0.9464)	0.2254 (0.9403)	0.0754 (0.9567)	0.0416 (0.9611)
	C	8.2351 (0.9559)	1.2439 (0.9351)	10.1585 (0.9664)	6.2907 (0.9614)	0.0568 (0.9329)	0.2393 (0.9623)	0.0866 (0.9402)	0.0431 (0.9671)

Table 9. Comparisons of ACL and CP of 95% CIs for the parameters with $(\alpha,\theta,\beta,\lambda)=(2.5,0.4,1.5,2)$ at $\tau=0.7$.

		MLE				MCMC			
(n,m)	CS	α	θ	β	λ	α	θ	β	λ
$(30,20)$	A	8.5831 (0.9592)	1.1294 (0.9586)	9.9531 (0.9479)	7.6929 (0.9603)	0.0571 (0.9665)	0.3325 (0.9335)	0.0685 (0.9694)	0.0511 (0.9348)
	B	8.2770 (0.9404)	1.1986 (0.9672)	9.0829 (0.9606)	6.7076 (0.9336)	0.0550 (0.9543)	0.3403 (0.9583)	0.0641 (0.9523)	0.0450 (0.9604)
	C	8.7695 (0.9305)	1.6389 (0.9690)	10.0805 (0.9680)	7.1370 (0.9552)	0.0587 (0.9346)	0.3361 (0.9698)	0.0717 (0.9687)	0.0469 (0.9466)
$(40,20)$	A	7.8163 (0.9689)	1.0358 (0.9463)	8.8197 (0.9526)	6.9696 (0.9498)	0.0532 (0.9618)	0.3288 (0.9503)	0.0619 (0.9507)	0.0482 (0.9499)
	B	6.9551 (0.9456)	1.1193 (0.9504)	8.0884 (0.9382)	6.1133 (0.9664)	0.0473 (0.9365)	0.3380 (0.9563)	0.0586 (0.9477)	0.0409 (0.9540)
	C	7.8857 (0.9693)	1.9280 (0.9450)	10.0374 (0.9429)	7.1131 (0.9576)	0.0545 (0.9613)	0.3112 (0.9669)	0.0781 (0.9524)	0.0470 (0.9564)
$(40,30)$	A	7.2025 (0.9355)	0.9396 (0.9598)	8.0338 (0.9641)	6.1997 (0.9678)	0.0492 (0.9528)	0.2688 (0.9339)	0.0577 (0.9352)	0.0414 (0.9420)
	B	7.2256 (0.9399)	0.9419 (0.9488)	7.6113 (0.9469)	5.7719 (0.9554)	0.0492 (0.9645)	0.2667 (0.9545)	0.0586 (0.9359)	0.0380 (0.9413)
	C	7.3175 (0.9624)	1.2022 (0.9694)	8.1875 (0.9453)	5.6696 (0.9412)	0.0501 (0.9474)	0.2809 (0.9570)	0.0660 (0.9695)	0.0384 (0.9404)
$(60,40)$	A	5.6812 (0.9503)	0.7768 (0.9642)	6.5501 (0.9464)	5.2450 (0.9586)	0.0394 (0.9608)	0.2364 (0.9469)	0.0548 (0.9416)	0.0359 (0.9689)
	B	5.4851 (0.9657)	0.7961 (0.9302)	6.2372 (0.9303)	4.8869 (0.9492)	0.0392 (0.9370)	0.2345 (0.9524)	0.0608 (0.9448)	0.0331 (0.9631)
	C	5.8448 (0.9615)	1.0365 (0.9472)	6.6294 (0.9484)	4.7979 (0.9494)	0.0414 (0.9577)	0.2377 (0.9503)	0.0701 (0.9636)	0.0321 (0.9366)

Table 10. Comparisons of ACL and CP of 95% CIs for the parameters with $(\alpha,\theta,\beta,\lambda)=(2.5,0.4,1.5,3)$ at $\tau=0.5$.

		MLE				MCMC			
(n,m)	CS	α	θ	β	λ	α	θ	β	λ
$(30,20)$	A	9.9533 (0.9634)	1.1508 (0.9302)	10.9302 (0.9600)	11.2387 (0.9486)	0.0655 (0.9686)	0.3158 (0.9378)	0.0753 (0.9650)	0.0768 (0.9557)
	B	10.2560 (0.9650)	1.3081 (0.9627)	12.0157 (0.9389)	11.2822 (0.9341)	0.0705 (0.9316)	0.3255 (0.9556)	0.0839 (0.9504)	0.0751 (0.9410)
	C	11.5007 (0.9672)	1.8467 (0.9332)	14.2324 (0.9642)	12.1692 (0.9692)	0.0768 (0.9495)	0.3325 (0.9367)	0.1044 (0.9632)	0.0844 (0.9631)
$(40,20)$	A	9.9061 (0.9311)	1.0987 (0.9486)	10.5940 (0.9603)	11.559 (0.9337)	0.0674 (0.9323)	0.3043 (0.9457)	0.0697 (0.9327)	0.0766 (0.9622)
	B	9.9373 (0.9658)	1.3614 (0.9593)	11.3537 (0.9330)	11.1615 (0.9362)	0.0660 (0.9619)	0.3284 (0.9301)	0.0794 (0.9306)	0.0751 (0.9441)
	C	10.6516 (0.9561)	2.3701 (0.9481)	13.4752 (0.9611)	11.0654 (0.9381)	0.0707 (0.9563)	0.3391 (0.9430)	0.0955 (0.9348)	0.0761 (0.9560)
$(40,30)$	A	9.8193 (0.9363)	1.0968 (0.9548)	11.7321 (0.9635)	12.1359 (0.9356)	0.0667 (0.9429)	0.2518 (0.9414)	0.0826 (0.9352)	0.0816 (0.9540)
	B	9.0671 (0.9467)	1.1348 (0.9457)	11.2377 (0.9390)	11.0045 (0.9522)	0.0617 (0.9406)	0.2651 (0.9487)	0.0802 (0.9338)	0.0724 (0.9651)
	C	10.0315 (0.9492)	1.3861 (0.9426)	11.9100 (0.9500)	11.0529 (0.9475)	0.0674 (0.9568)	0.2745 (0.9645)	0.0876 (0.9606)	0.0744 (0.9451)
$(60,40)$	A	8.1445 (0.9574)	0.9380 (0.9340)	10.2425 (0.9606)	10.2517 (0.9469)	0.0550 (0.9438)	0.2249 (0.9550)	0.0743 (0.9322)	0.0691 (0.9670)
	B	7.7193 (0.9325)	0.9613 (0.9690)	9.6541 (0.9609)	9.4565 (0.9317)	0.0532 (0.9308)	0.2295 (0.9424)	0.0785 (0.9322)	0.0637 (0.9315)
	C	7.9766 (0.9500)	1.2382 (0.9454)	9.5908 (0.9540)	8.7440 (0.9409)	0.0547 (0.9329)	0.2439 (0.9552)	0.0827 (0.9395)	0.0593 (0.9338)

8 Conclusion

Using PROG-II-C samples, the analysis of the SSPALT of WG failure model is performed based on Bayes and non-Bayes methods. Four types of bootstrap confidence intervals are used to obtain 95% confidence intervals for the unknown parameters. The importance of MCMC technique was noticeable in Bayesian estimation using Metropolis-Hastings method. A simulated data set is presented to show how the MCMC and parametric bootstrap methods work. A simulation study is computerized to inspect and compare the rendition of the proposed methods for different sample sizes, different CSs, different acceleration factors and different change stress τ. From the results, we observe the following:

(1)The increase in the values of n and m would be helpful and will effect on MSEs and average interval lengths.
(2)For the parameters θ and λ the increase of τ leads to the increase of their MSEs.
(3)The increase of τ leads to decreasing the average width of the CIs.
(4)The MSEs of the estimators and the width of the CIs increase as λ increases.
(5)The width of MCMC CRIs is shorter than approximate CIs for different sample sizes, schemes, τ and different λ.

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