Information Sciences Letters

Volume 11
Issue 2 *Mar. 2022*

Article 36

2022

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Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/isl/110205

Subgroup of the Jacobian of a Family of Superelliptic Curves

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Received: 10 Nov. 2021, Revised: 22 Dec. 2021, Accepted: 1 Jan. 2022
Published online: 1 Mar. 2022

Abstract: In this paper, we describe the structure of the subgroup generated by the images of the 2-sextactic points under the Abel-Jacobi map in the Jacobian of a 1-parameter family $(C_a)_{a\in\mathbb{C}\setminus\{0,1\}}$ of smooth projective plane curves of degree four. Each curve $C_a \subset \mathbb{P}^2(\mathbb{C})$ of $(C_a)_{a\in\mathbb{C}\setminus\{0,1\}}$ is given by

$$C_a : Y^4 = XZ(X-Z)(X-aZ),\ a\in\mathbb{C}\setminus\{0,1\}.$$

Keywords: Algebraic curves; Jacobian; Flex points; Sextactic points; Quartic curves; Elliptic curves.

1 Introduction

The finite set of q-Weierstrass points, $q \geq 1$, on a smooth algebraic curve has many interesting properties . An interesting problem that arises is about the structure of the subgroup of the Jacobian of the curve that one obtains from degree 0 divisors that are supported on such a finite set. It is well known that the 1-Weierstrass points on smooth quartic plane curves are nothing but flexes [1], and the 2-Weierstrass points on such curves are divided into flexes and sextactic points [2]. Most of the previous researches had studied the structure of the group W generated by the images, under the Abel-Jacobi map A_P, of 1-Weierstrass points in the Jacobian J_C of a smooth quartic plane curve C. In the case of a hyperelliptic curve, which is the simplest case, it is easy to see that $W = (\mathbb{Z}/2\mathbb{Z})^{2g} = J[2]$ (see, for instance, [3]). For non-hyperelliptic curves of genus 3; that is, plane quartics; some special cases have already been considered. Each of these curves has a large automorphism group. More precisely, for the Kuribayashi curve (given by $X^4+Y^4+Z^4+a(X^2Y^2+Y^2Z^2+Z^2X^2)=0$) when $a=3$, with 24 automorphisms, $W\cong(\mathbb{Z}/4\mathbb{Z})^5$ [4]. For the Klein curve (given by $X^3Y+Y^3Z+Z^3X=0$), with 168 automorphisms, $W\cong(\mathbb{Z}/2\mathbb{Z})\oplus(\mathbb{Z}/7\mathbb{Z})^3$ [5]. For the Fermat quartic (given by $X^4+Y^4=Z^4$), with 96 automorphisms, $W\cong(\mathbb{Z}/4\mathbb{Z})^5\oplus(\mathbb{Z}/2\mathbb{Z})$ [6]. For the Picard curve (given by $Y^3Z+Z^4=X^4$), with 48 automorphisms, $W\cong(\mathbb{Z}/4\mathbb{Z})^2\oplus(\mathbb{Z}/3\mathbb{Z})^5$ [7].

In [8, 9], the authors passed to study structure of the group G generated by the images, under A_P, of total sextactic points (which form a subset of 2-Weierstrass points [2]) in the Jacobian of Kuribayashi quartic curve when $a=14$, and a is a root of
$P(a)=a^3+68a^2-91a+98=0$,
respectively. More precisely, they found that $G\cong(\mathbb{Z}/2\mathbb{Z})\oplus(\mathbb{Z}/4\mathbb{Z})\oplus(\mathbb{Z}/8\mathbb{Z})^2$ if $a=14$, and $G\cong(\mathbb{Z}/8\mathbb{Z})^3$ if $P(a)=0$.

In this paper, we focus on a 1-parameter family $(C_a)_{a\in\mathbb{C}\setminus\{0,1\}}$ of superelliptic curves of genus 3 (smooth projective plane curves of degree 4) where each curve $C_a\subset\mathbb{P}^2(\mathbb{C})$ of $(C_a)_{a\in\mathbb{C}\setminus\{0,1\}}$ is given by

$$C_a : Y^4 = XZ(X-Z)(X-aZ),\ a\in\mathbb{C}\setminus\{0,1\}.$$

Note that C_a is not smooth if $a=0$ or 1. In [10], the authors studied the distribution of the 2-Weierstrass points which are nothing but flexes and sextactic points on such family. They have shown that if the parameter a is a root of the

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product $P(a)Q(a)$ where

$$P(a) = \left(a^2+a+1\right)\left(a^2-3a+3\right)\left(3a^2-3a+1\right),$$
$$Q(a) = \left(a^2+4a-4\right)\left(4a^2-4a-1\right)\left(a^2-6a+1\right),$$

then the curve C_a has s-sextactic points, where $s = 2$ or 3, and vice versa. Specifically, the six curves $(C_a)_{P(a)=0}$ are the only curves of the family $(C_a)_{a\in\mathbb{C}\setminus\{0,1\}}$ having sixteen 2-sextactic points while the six curves $(C_a)_{Q(a)=0}$ are the only curves possessing eight 3-sextactic points (see Theorem 1 of [10]).

In [11], the authors describe the group S generated by the images of the 3-sextactic (total sextactic) points under the Abel-Jacobi map in the Jacobian J_{C_a} of C_a where a is a root of $Q(a)$. They found that $S \cong (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})^2 \oplus (\mathbb{Z}/8\mathbb{Z})^2$. The study of the group structure of S was relatively easy because the generators of S are always of finite orders in J_{C_a}. Indeed, at a 3-sextactic point of C_a where a is a zero of $Q(a)$ there is a conic that intersects C_a only at that point. Thus, under the Abel-Jacobi map, the image of a 3-sextactic point in J_{C_a} is of order dividing 8 (see [11]).

In this manuscript, a 2-sextactic point $P_1 \in C_a$ where a is a root of $P(a)$ is taken as a base point of the Jacobian embedding A_{P_1}. Our goal is to describe the subgroup G of the Jacobian J_{C_a} of C_a which is generated by the images of the 2-sextactic points on C_a under A_{P_1}. Up to isomorphism the group G does not depend on the choice of the 2-sextactic point taking as a base point of A_{P_1}. This gives an interesting geometric invariant. When a is a root of $P(a)$ the curve C_a has 16 automorphisms [12]. We shall show the following

Theorem 1(Main Theorem). *Let a be a root of $P(a)$. Then, the group G generated by images of the 2-sextactic points in the Jacobian J_{C_a} of C_a satisfies*

$$G \cong (\mathbb{Z}/3\mathbb{Z})^2 \oplus (\mathbb{Z}/12\mathbb{Z}) \oplus (\mathbb{Z}/24\mathbb{Z}).$$

A point $P \in C_a$, where a is a root of $P(a)$, is a 2-sextactic point if and only if there exists an irreducible conic Δ_P satisfying $\text{div}(\Delta_P) = 7P + Q$ for some $Q \in C_a$ [2]. It is worth noting, which makes the distribution of 2-sextactic points on the curve C_a when a is a root of $P(a)$ a very rare case, that the point Q is also 2-sextactic point. This observation is essential to the proof of the main theorem of this paper and without it the proof becomes extremely complicated. So the paper gives a well developed example which enrich the subject.

This paper is organized as follows: In Section 2, we recall some basic definitions used throught the work. Then, we determine the 2-sextactic points on the curve C_a in Section 3. We also deduce some elementary geometric configurations involving the 2-sextactic points. Such geometric configurations enable us to restrict the number of generators. In Section 4, we study the structure of the Jacobian J_{C_a} of C_a and show the fact that the Jacobians of these curves are isogenous to the product of three elliptic curves. In Section 5, we study the image of the 2-sextactic points on each of the elliptic factors of J_{C_a}. We then deduce all possible relations among the 2-sextactic points. In Section 6, we prove that the relations that we have obtained are the only relations in the Jacobian. Finally, we conclude the paper by some open problems that can be solved using the same technique introduced in this paper. Then, we shall mention some applications. The computations in this paper are performed using the Maple software.

2 Preliminaries

Assume that C is a smooth algebraic plane curve of degree d at least three. Choosing a point $P_1 \in C$ as a base point, then the Abel-Jacobi map from C to its Jacobian J_C, is denoted by $A_{P_1} : C \to J_C$. It is defined by $P \mapsto [P - P_1]$, where $[D]$ denotes the class of the divisor D in $Pic^0(C)$, the group of degree zero divisor classes in C. This definition can be linearly extended to divisors in $\text{Div}(C)$, the group of all divisors on C. Furthermore, the Jacobian J_C is identified with $Pic^0(C)$. For more details we refer to Chapter VIII of [13].

A point $P \in C$ is a *flex point* if the tangent line T_P intersects C at P with intersection multiplicity at least three, i.e. $I_P(C,T_P) \geq 3$. Additionally, if $i = I_P(C,T_P) - 2$, then such point is called an *i-flex*. This positive integer i is called the *flex multiplicity* of C at P. In particular, by Bézout's Theorem, for quartic plane curves we have either $i = 1$ or $i = 2$. Moreover, the flex points on the curve C are the intersection points with its associated Hessian curve H_C. For more information about the flex points and their connection with the associated Hessian we refer to Chapter 9 of [14].

Let $P \in C$ be a non-flex point. Then there exists a unique smooth conic Δ_P with intersection multiplicity $I_P(C,\Delta_P) \geq 5$. Such conic Δ_P is called the *osculating conic* of C at P. This point P is called a *sextactic point* if the osculating conic Δ_P meets C at P with intersection multiplicity at least six, i.e., if $I_P(C,\Delta_P) \geq 6$ (in such case the osculating conic is called a *sextactic conic*). Furthermore, if $s = I_P(C,\Delta_P) - 5$, a sextactic point $P \in C$ is called *s-sextactic*. This positive integer s is called *the sextactic multiplicity* of C at P. Thorbergsson and Umehara showed, in Appendix C of [15], that if C possesses l flex points with multiplicities $m_1,\ldots,m_l$, then C has $3(5d-11)d - \sum_{i=1}^{l}(4m_i - 3)$ sextactic points, counted with multiplicities. By Bézout's Theorem, for quartic plane curves we have $s \in \{1,2,3\}$. In this note we are concerned with a 2-sextactic point P. The sextactic conic Δ_P, in this case, satisfies that $I_P(C,\Delta_P) = 7$ and Δ_P meets C transversely at an other point Q which is different from P. This means that the divisor of Δ_P on C is given by $\text{div}(\Delta_P) = 7P + Q$. For an explicit construction of the sextactic conic at a sextactic point on a smooth plane quartic curve consult Lemma 15 in [2] or Lemma 4 in [11].

In [16], page (326), Namba has shown the following Proposition.

Proposition 1. *The curve C_a is isomorphic to the curve C_b if and only if b is one of the following:*

$$a, \frac{1}{a}, 1-a, \frac{1}{1-a}, \frac{a-1}{a}, \frac{a}{a-1}.$$

We note that the zeros of $P(a) = \left(a^2+a+1\right)\left(a^2-3a+3\right)\left(3a^2-3a+1\right) = 0$ are

$$\begin{aligned} a &= \tfrac{-1+\sqrt{3}i}{2}, & \tfrac{1}{a} &= \tfrac{-1-\sqrt{3}i}{2}, \\ 1-a &= \tfrac{3-\sqrt{3}i}{2}, & \tfrac{1}{1-a} &= \tfrac{3+\sqrt{3}i}{6}, \\ \tfrac{a-1}{a} &= \tfrac{3+\sqrt{3}i}{2}, & \tfrac{a}{a-1} &= \tfrac{3-\sqrt{3}i}{6}, \end{aligned}$$

where $i = \sqrt{-1}$. Therefore by Proposition 1, the six curves $(C_a)_{P(a)=0}$ are all isomorphic to each other. Hence, it is enough to focus only on the curve C_ω where $\omega = \exp(\frac{2\pi i}{3}) = \frac{-1+\sqrt{3}i}{2}$.

3 2-sextactic points of C_ω

The curve C_ω possesses sixteen 2-sextactic points (Theorem 1 in [10]). Now, to determine locations of such points, we shall follow the steps 1 : 3 of the technique introduced in Section 4 of [17], as follows: The curve C_ω has a point at infinity, namely $\mathscr{R}_\infty = [1:0:0]$, which is 2-flex with tangent line $T_{\mathscr{R}_\infty} : Z = 0$. The 4-sheeted covering map $x : C_\omega \longrightarrow \mathbb{P}^1$ is ramified only at the points $\mathscr{R}_\infty, \mathscr{R}_1 = [0:0:1], \mathscr{R}_2 = [1:0:1]$ and $\mathscr{R}_3 = [\omega:0:1]$. Let $f(x,y) := y^4 - x(x-1)(x-\omega)$ be the affine equation of the curve C_ω. Computing the resultant of the affine polynomial $f(x,y)$ and its associated Hessian H_f with respect to y, one gets the locations and the multiplicities of flex points on C_ω

$$Res[f, H_f; y] = \text{Const}.x^2(x-1)^2(x-\omega)^2(F(x))^4,$$

where $F(x) = 3x^4 + 4\omega^2x^3 - 2\omega x^2 + 4x + 3\omega^2$. The discriminant of $F(x)$ is a non-zero constant. Hence C_ω has four 2-flexes at the points $\mathscr{R}_\infty, \mathscr{R}_1, \mathscr{R}_2, \mathscr{R}_3$ and C_ω has sixteen $(4\times 4 = 16)$ 1-flexes located over the zeros of the equation $F(x) = 0$.

Therefore, according to the formula due to Thorbergsson and Umehara which was stated in Section 2, C_ω has 72 sextactic points, counted with multiplicities. By computing the Wronskian $W_2(x,y)$ of $\{1, x, y, xy, x^2, y^2\}$ (see page 57 of [10]), one can determine the locations and the multiplicities of these sextactic points. We have

$$W_2(x,y) = Const.\frac{F(x)S_1(x)(S_2(x))^2}{y^{40}},$$

where

$$\begin{aligned} S_1(x) &= \left(x^2-\omega\right)\left(x^2-2x+\omega\right)\left(x^2-2\omega x+\omega\right) \\ &\quad\times\left(x^4+6\omega^2x^3+5\omega x^2+12x+\omega^2\right), \\ S_2(x) &= \left(x^2-2\left(1+\sqrt{3}\right)\omega^2x+\omega\right) \\ &\quad\times\left(x^2-2\left(1-\sqrt{3}\right)\omega^2x+\omega\right). \end{aligned}$$

Each of the polynomials $F(x), S_1(x)$ and $S_2(x)$ has no repeated roots, because its discriminant is a non-zero constant. Additionally, the resultant with respect to x of any two of $F(x), S_1(x)$ and $S_2(x)$ is a non-zero constant, therefore they have no common roots. Hence, there exist forty $(10\times 4 = 40)$ 1-sextactic points that are located over the zeros of the equation $S_1(x) = 0$. Furthermore, it is clear that the Wronskian $W_2(x,y)$ has a zero of multiplicity two if and only if x is a root of $S_2(x)$. Hence, the 2-sextactic points of C_ω are the points located over the four roots of $S_2(x)$, which are

$$\begin{aligned} P_1 &= [\alpha:\beta:1], & P_9 &= \left[\tfrac{\alpha-\omega}{\alpha-1} : \tfrac{\sqrt{\omega-1}\beta}{\alpha-1} : 1\right], \\ P_2 &= [\alpha:\beta i:1], & P_{10} &= \left[\tfrac{\alpha-\omega}{\alpha-1} : \tfrac{\sqrt{\omega-1}\beta}{\alpha-1}i : 1\right], \\ P_3 &= [\alpha:-\beta:1], & P_{11} &= \left[\tfrac{\alpha-\omega}{\alpha-1} : -\tfrac{\sqrt{\omega-1}\beta}{\alpha-1} : 1\right], \\ P_4 &= [\alpha:-\beta i:1], & P_{12} &= \left[\tfrac{\alpha-\omega}{\alpha-1} : -\tfrac{\sqrt{\omega-1}\beta}{\alpha-1}i : 1\right], \\ P_5 &= \left[\tfrac{\omega}{\alpha} : \tfrac{\sqrt{\omega}\beta}{\alpha} : 1\right], & P_{13} &= \left[\omega\left(\tfrac{\alpha-1}{\alpha-\omega}\right) : \tfrac{\sqrt{\omega^2-\omega}\beta}{\alpha-\omega} : 1\right], \\ P_6 &= \left[\tfrac{\omega}{\alpha} : \tfrac{\sqrt{\omega}\beta}{\alpha}i : 1\right], & P_{14} &= \left[\omega\left(\tfrac{\alpha-1}{\alpha-\omega}\right) : \tfrac{\sqrt{\omega^2-\omega}\beta}{\alpha-\omega}i : 1\right], \\ P_7 &= \left[\tfrac{\omega}{\alpha} : -\tfrac{\sqrt{\omega}\beta}{\alpha} : 1\right], & P_{15} &= \left[\omega\left(\tfrac{\alpha-1}{\alpha-\omega}\right) : -\tfrac{\sqrt{\omega^2-\omega}\beta}{\alpha-\omega} : 1\right], \\ P_8 &= \left[\tfrac{\omega}{\alpha} : -\tfrac{\sqrt{\omega}\beta}{\alpha}i : 1\right], & P_{16} &= \left[\omega\left(\tfrac{\alpha-1}{\alpha-\omega}\right) : -\tfrac{\sqrt{\omega^2-\omega}\beta}{\alpha-\omega}i : 1\right], \end{aligned}$$

where

$$\alpha = \left(1-\sqrt{3}+\sqrt{\left(3-2\sqrt{3}\right)}\right)\omega^2 \text{ and}$$

$$\beta = \sqrt{\left(1-\sqrt{3}\right)\sqrt{\left(3-2\sqrt{3}\right)}-2\sqrt{3}+3}.$$

The curve C_ω admits 16 automorphisms (see [12]) and its automorphisms group, $Aut(C_\omega)$, is generated by

$$\begin{aligned} \rho([X:Y:Z]) &= [X:iY:Z], \\ \sigma([X:Y:Z]) &= \left[\omega Z : \sqrt{\omega}Y : X\right], \\ \tau([X:Y:Z]) &= \left[\omega(X-Z) : \sqrt{\omega}\sqrt{\omega-1}\,Y : X-\omega Z\right]. \end{aligned}$$

Remark. (a) The sixteen 2-sextactic points of C_ω are in the same orbit. More precisely, we have

$$Orb(P_1) = \left\{\begin{array}{lll} P_1, & P_7 = \rho^2\sigma(P_1), & P_{13} = \tau(P_1) \\ P_2 = \rho(P_1), & P_8 = \rho^3\sigma(P_1), & P_{14} = \rho\tau(P_1) \\ P_3 = \rho^2(P_1), & P_9 = \sigma\tau(P_1), & P_{15} = \rho^2\tau(P_1) \\ P_4 = \rho^3(P_1), & P_{10} = \rho\sigma\tau(P_1), & P_{16} = \rho^3\tau(P_1) \\ P_5 = \sigma(P_1), & P_{11} = \rho^2\sigma\tau(P_1), & \\ P_6 = \rho\sigma(P_1), & P_{12} = \rho^3\sigma\tau(P_1), & \end{array}\right\}$$

(b) It is known that, for instance see Section $II.3$ in [18] , if $\mu \in Aut(C_\omega)$ and $D = \sum_{P\in C_\omega} n_P.P$ is a divisor on the curve C_ω, then $\mu(D) = \sum_{P\in C_\omega} n_P.\mu(P)$.

We note that the lines

$$T_1 = \left(\left(3\sqrt{3}-4\right)\omega^2\alpha+\omega\right)X + \left(2-\sqrt{3}+\left(6\sqrt{3}-11\right)\omega\alpha+\frac{1}{4}\beta\right)Z + \left(\frac{1}{2}\left(5-\sqrt{3}\right)\omega^2\alpha^2+6(i-\omega)\alpha + \frac{1-\sqrt{3}}{2}-2\sqrt{3}\alpha\right)\beta Y,$$

$$\begin{array}{ll} & T_9=\sigma\tau(T_1),\\ T_2=\rho(T_1), & T_{10}=\rho\sigma\tau(T_1)),\\ T_3=\rho^2(T_1), & T_{11}=\rho^2\sigma\tau(T_1),\\ T_4=\rho^3(T_1), & T_{12}=\rho^3\sigma\tau(T_1),\\ T_5=\sigma(T_1), & T_{13}=\tau(T_1),\\ T_6=\rho\sigma(T_1), & T_{14}=\rho\tau(T_1),\\ T_7=\rho^2\sigma(T_1), & T_{15}=\rho^2\tau(T_1),\\ T_8=\rho^3\sigma(T_1), & T_{16}=\rho^3\tau(T_1),\end{array}$$

are the tangent lines at the points P_j, $j=1,2,\dots,16$, respectively. Using Maple software, one can find that the intersection divisor of the tangent line T_1 on C_ω which is $\operatorname{div}(T_1)=2P_1+P_{11}+P_{16}$. Now, considering Remark 3, it is not difficult to see that the intersection divisors of the other tangent lines on the curve C_ω are given by

$$\begin{array}{l} \operatorname{div}(T_2)=2P_2+P_{12}+P_{13},\\ \operatorname{div}(T_3)=2P_3+P_9+P_{14},\\ \operatorname{div}(T_4)=2P_4+P_{10}+P_{15},\\ \operatorname{div}(T_5)=2P_5+P_{12}+P_{15},\\ \operatorname{div}(T_6)=2P_6+P_9+P_{16},\\ \operatorname{div}(T_7)=2P_7+P_{10}+P_{13},\\ \operatorname{div}(T_8)=2P_8+P_{11}+P_{14},\\ \operatorname{div}(T_9)=2P_9+P_1+P_8,\\ \operatorname{div}(T_{10})=2P_{10}+P_2+P_5,\\ \operatorname{div}(T_{11})=2P_{11}+P_3+P_6,\\ \operatorname{div}(T_{12})=2P_{12}+P_4+P_7,\\ \operatorname{div}(T_{13})=2P_{13}+P_4+P_5,\\ \operatorname{div}(T_{14})=2P_{14}+P_1+P_6,\\ \operatorname{div}(T_{15})=2P_{15}+P_2+P_7,\\ \operatorname{div}(T_{16})=2P_{16}+P_3+P_8.\end{array}$$

We can determine the equation of each sextactic conic Δ_j associated to P_j (see Lemma 15 in [2] or Lemma 4 in [11]). We find that the equations of these conics are given by

$$\Delta_1 = X^2+\left(1-\sqrt{3}\right)\omega^2\alpha Y^2+\left(2\left(\sqrt{3}-1\right)\alpha\omega^2+\omega\right)Z^2 +4\left(\left(1+\sqrt{3}\right)\alpha+2\omega^2\right)\beta XY +\left(\left(1-\sqrt{3}\right)\omega^2+2\left(-2+\sqrt{3}\right)\alpha\right)XZ +\left(\left(1+\sqrt{3}\right)\omega+\omega^2\right)\beta YZ,$$

$$\begin{array}{ll} & \Delta_9=\sigma\tau(\Delta_1),\\ \Delta_2=\rho(\Delta_1), & \Delta_{10}=\rho\sigma\tau(\Delta_1)),\\ \Delta_3=\rho^2(\Delta_1), & \Delta_{11}=\rho^2\sigma\tau(\Delta_1),\\ \Delta_4=\rho^3(\Delta_1), & \Delta_{12}=\rho^3\sigma\tau(\Delta_1),\\ \Delta_5=\sigma(\Delta_1), & \Delta_{13}=\tau(\Delta_1),\\ \Delta_6=\rho\sigma(\Delta_1), & \Delta_{14}=\rho\tau(\Delta_1),\\ \Delta_7=\rho^2\sigma(\Delta_1), & \Delta_{15}=\rho^2\tau(\Delta_1),\\ \Delta_8=\rho^3\sigma(\Delta_1), & \Delta_{16}=\rho^3\tau(\Delta_1).\end{array}$$

Using Maple software, one can find that the intersection divisor of the sextactic conics Δ_1 on C_ω is $\operatorname{div}(\Delta_1)=7P_1+P_3$. By considering Remark 3 once again, we see that the intersection divisors of the other sextactic conics Δ_j on C_ω are given by

$$\begin{array}{ll} \operatorname{div}(\Delta_2)=7P_2+P_4, & \operatorname{div}(\Delta_9)=2P_9+P_{11},\\ \operatorname{div}(\Delta_3)=7P_3+P_1, & \operatorname{div}(\Delta_{10})=7P_{10}+P_{12},\\ \operatorname{div}(\Delta_4)=7P_4+P_2, & \operatorname{div}(\Delta_{11})=7P_{11}+P_9,\\ \operatorname{div}(\Delta_5)=7P_5+P_7, & \operatorname{div}(\Delta_{12})=7P_{12}+P_{10},\\ \operatorname{div}(\Delta_6)=7P_6+P_8, & \operatorname{div}(\Delta_{13})=7P_{13}+P_{15},\\ \operatorname{div}(\Delta_7)=7P_7+P_5, & \operatorname{div}(\Delta_{14})=7P_{14}+P_{16},\\ \operatorname{div}(\Delta_8)=7P_8+P_6, & \operatorname{div}(\Delta_{15})=7P_{15}+P_{13},\\ & \operatorname{div}(\Delta_{16})=7P_{16}+P_{14}.\end{array}$$

Let $\ell_1: X-\alpha Z=0, \ell_2: X-\frac{\omega}{\alpha}Z=0, \ell_3: X-\left(\frac{\alpha-\omega}{\alpha-1}\right)Z=0$ and $\ell_4: X-\omega\left(\frac{\alpha-1}{\alpha-\omega}\right)Z=0$. Then the intersection divisors of these four lines on the curve C_ω are given by

$$\begin{array}{l} \operatorname{div}(\ell_1)=P_1+P_2+P_3+P_4,\\ \operatorname{div}(\ell_2)=P_5+P_6+P_7+P_8,\\ \operatorname{div}(\ell_3)=P_9+P_{10}+P_{11}+P_{12},\\ \operatorname{div}(\ell_4)=P_{13}+P_{14}+P_{15}+P_{16}.\end{array}$$

Using these divisors, one gets some relations among the 2-sextactic points images in the Jacobian J_{C_ω}. Let's select the point P_1 as a base point for the Abel-Jacobi map. The embedding A_{P_1} in the Jacobian is $P\mapsto[P-P_1]$. By abuse of notation, a point and its image under A_{P_1} are denoted by the same way. In particular, $\sum_{P\in C} n_P P=0$ coincides with the fact that a divisor $\sum_{P\in C} n_P P-(\sum_{P\in C} n_P)P_1$ is in the kernel of A_{P_1}, i.e. that $\sum_{P\in C} n_P P-(\sum_{P\in C} n_P)P_1$ is principal.

Under this convention we have: $P_1=0$. Now we shall try to reduce the number of generating elements of the group G. Taking into consideration the principal divisors $\operatorname{div}(\frac{\ell_i}{T_1})$, for $i=1,2,3,4$, respectively, we obtain that

$$\begin{array}{l} P_2=P_{11}+P_{16}-P_3-P_4,\\ P_5=P_{11}+P_{16}-P_6-P_7-P_8,\\ P_9=P_{16}-P_{10}-P_{12},\\ P_{13}=P_{11}-P_{14}-P_{15}.\end{array}$$

Using the principal divisors $\operatorname{div}(\frac{T_8}{T_1})$ and $\operatorname{div}(\frac{T_{16}}{T_1})$ we have

$$\begin{array}{l} P_{14}=P_{16}-2P_8,\\ P_{11}=P_3+P_8+P_{16}.\end{array} \tag{1}$$

The last two relations lead to

$$\begin{array}{l} P_2=2P_{16}+P_8-P_4,\\ P_5=2P_{16}+P_3-P_6-P_7,\\ P_9=P_{16}-P_{10}-P_{12},\\ P_{13}=P_3+3P_8-P_{15}.\end{array} \tag{2}$$

Considering the principal divisors $\operatorname{div}(\frac{T_2}{T_1})$ and $\operatorname{div}(\frac{T_{11}}{T_1})$, and substituting about P_2, P_{11} and P_{13} from (1) and (2) we find that

$$\begin{array}{l} P_{15}=4P_8+P_{12}+2P_{16}-2P_4,\\ P_6=-2P_3-P_8.\end{array}$$

The last two relations imply that

$$\begin{aligned} P_2 &= 2P_{16}+P_8-P_4, \\ P_5 &= 2P_{16}+3P_3+P_8-P_7, \\ P_9 &= P_{16}-P_{10}-P_{12}, \\ P_{13} &= P_3+2P_4-P_8-P_{12}-2P_{16}. \end{aligned} \quad (3)$$

Using the principal divisor $\mathrm{div}(\frac{T_{12}}{T_1})$ and substituting about P_{11} from (1) imply that

$$P_7 = P_3+P_8+2P_{16}-P_4-2P_{12}.$$

So, we have

$$\begin{aligned} P_2 &= 2P_{16}+P_8-P_4, \\ P_5 &= 2P_3+P_4+2P_{12}, \\ P_9 &= P_{16}-P_{10}-P_{12}, \\ P_{13} &= P_3+2P_4-P_8-P_{12}-2P_{16}. \end{aligned} \quad (4)$$

Taking the principal divisor $\mathrm{div}(\frac{T_3}{T_1})$ into our account and substituting about P_9, P_{11} and P_{14} from (1) and (4) we get

$$P_{10} = P_3-P_{12}-3P_8 \quad (5)$$

The last relation implies that

$$\begin{aligned} P_2 &= 2P_{16}+P_8-P_4, \\ P_5 &= 2P_3+P_4+2P_{12}, \\ P_9 &= P_{16}+3P_8-P_3, \\ P_{13} &= P_3+2P_4-P_8-P_{12}-2P_{16}. \end{aligned}$$

Considering the principal divisor $\mathrm{div}(\frac{\Delta_{16}}{\Delta_{12}})$ and substituting about P_{14} and P_{10} from (1) and (5) we get

$$P_3 = 8P_{16}+P_8-6P_{12}. \quad (6)$$

Summarizing above we find that

$$\begin{aligned} P_1 &= 0, \\ P_2 &= 2P_{16}+P_8-P_4, \\ P_3 &= 8P_{16}+P_8-6P_{12}, \\ P_5 &= 16P_{16}+2P_8+P_4-10P_{12}, \\ P_6 &= 12P_{12}-3P_8-16P_{16}, \\ P_7 &= 10P_{16}+2P_8-8P_{12}-P_4, \\ P_9 &= 2P_8+6P_{12}-7P_{16}, \\ P_{10} &= 8P_{16}-2P_8-7P_{12}, \\ P_{11} &= 9P_{16}+2P_8-6P_{12}, \\ P_{13} &= 6P_{16}+2P_4-7P_{12}, \\ P_{14} &= P_{16}-2P_8, \\ P_{15} &= 4P_8+P_{12}+2P_{16}-2P_4. \end{aligned} \quad (*)$$

Hence, the subgroup G generated by the 2-sextactic points images in the Jacobian J_{C_ω} is only generated by P_4,P_8,P_{12} and P_{16}. Moreover, these generators are of finite order. Indeed, by considering the principal divisor $\mathrm{div}(\frac{T_9}{T_{14}})$ and taking into account relations in $(*)$ we obtain $12P_8 = 0$ (which means $12[P_8-P_1] = 0$). This shows that the order of the generator $[P_8-P_1]$ divides 12. Using this fact together with the following relations (which one can finds them by using principal divisors $\mathrm{div}(\frac{T_{10}}{T_7}),\mathrm{div}(\frac{T_7}{T_1})$ and $\mathrm{div}(\frac{T_{13}}{T_{10}})$, respectively, and using appropriate substitutions from $(*)$)

$$\begin{aligned} 6P_{12}-3P_8 &= 0, \\ 24P_{16}-24P_{12} &= 0, \\ 6P_4+3P_8-6P_{16} &= 0, \end{aligned}$$

we find that each of the generators $[P_4-P_1]$, $[P_{12}-P_1]$ and $[P_{16}-P_1]$ has order dividing 24. Hence, we get that the subgroup G is a quotient of $(\mathbb{Z}/12\mathbb{Z})\oplus(\mathbb{Z}/24\mathbb{Z})^3$.

Lemma 1. *The following relations hold in G:*

$$\begin{aligned} 6P_4+6P_{12}-6P_{16} &= 0, \\ 3P_8-6P_{12} &= 0. \end{aligned}$$

Proof. Taking into consideration that the divisors $\mathrm{div}(\frac{\Delta_4}{\Delta_1})$ and $\mathrm{div}(\frac{\Delta_1}{\Delta_{16}})$ are principal, then, substituting about P_2,P_3 and P_{14} from $(*)$, yields the lemma.

Note that if $P_8-2P_{12}=0$ which equivalents to say $P_1+P_8-2P_{12}=0$, then there is a non-constant rational function g on C_ω such that $div(g) = P_1+P_8-2P_{12}$. This implies the existence of a degree 2 morphism $C_\omega \to \mathbb{P}^1$, contradicting the fact that C_ω is not a hyperelliptic curve. Consequently, the order of $[P_8-2P_{12}]$ in J_{C_ω} is exactly 3. As a result of Lemma 1 one obtains

Corollary 1. *G is a quotient of*

$$(\mathbb{Z}/3\mathbb{Z})\oplus(\mathbb{Z}/6\mathbb{Z})\oplus(\mathbb{Z}/24\mathbb{Z})^2.$$

Proof. We can take P_{12} and P_{16} of order 24, together with P_8-2P_{12} of order 3 and $P_4+P_{12}-P_{16}$ of order 6. Clearly, that G is generated by these elements. Furthermore, their orders are simply produced by the relations given in Lemma 1.

The more difficult part is to find more relations and to prove that there are no other relations.

4 Structure of the Jacobian of C_ω

For a generalization of all results given in this section, we refer to Section 4 of [11]. As we mentioned in Section 3 the curve C_ω admits 16 automorphisms and the group of automorphisms $Aut(C_\omega)$ is generated by

$$\begin{cases} \rho: (x,y) \longmapsto & (x,iy) \\ \sigma: (x,y) \longmapsto & \left(\frac{\omega}{x},\frac{\sqrt{\omega}\,y}{x}\right) \\ \tau: (x,y) \longmapsto & \left(\frac{\omega(x-1)}{x-\omega},\frac{\sqrt{\omega}\sqrt{\omega-1}\,y}{x-\omega}\right) \end{cases}$$

By identifying points that belong to the same orbit for the action of these automorphisms, we get the quotient of C_ω by the group generated by one of the automorphism ρ^2, σ and τ which are elliptic curves. We will denote them by $E_1 = C_\omega/\langle\rho^2\rangle$, $E_2 = C_\omega/\langle\sigma\rangle$ and $E_3 = C_\omega/\langle\tau\rangle$. These elliptic curves are defined by the equations

$$\begin{aligned} &E_1: y^2 = x(x-1)(x-\omega), \\ &E_2: y^2 = x^3+16\left(1+\sqrt{\omega}\right)^2 x, \\ &E_3: y^2 = x^3-16\left(2\omega-1+2\sqrt{\omega^2-\omega}\right)x. \end{aligned}$$

Let D_1 be the elliptic curve defined by the equation $D_1 : v^2 = u(u-1)(u-\omega)$. This curve is isomorphic to the elliptic curve E_1. Furthermore, there exists a degree 2 morphism $\varphi_1 : C_\omega \to D_1$ where
$\varphi_1 : (x,y) \mapsto (u := x, v := y^2)$. We would like to show that there is an isogeny between the Jacobian of C_ω and the product $E_1 \times E_2 \times E_3$. For this purpose the following two lemmas are required (for proofs see the Appendix in [11]).

Lemma 2. *There is a morphism φ_2 from C_ω to an elliptic curve D_2 of equation*

$$v^2 = -4\left(1+\sqrt{\omega}\right)^2 u^4 + 1$$

given by $(x,y) \longmapsto (u,v)$ *where*

$$u = \frac{y}{x+\sqrt{\omega}} \text{ and } v = \frac{\left(x-\sqrt{\omega}\right)^2 - 2\left(1+\omega\right)x}{\left(x+\sqrt{\omega}\right)^2}.$$

This elliptic curve D_2 is birational to the curve E_2. Moreover, there is a degree 2 morphism ψ_2 from C_ω to the elliptic curve E_2 can be described by

$$(x,y) \longmapsto \left(\frac{4y^2}{x}, \frac{8\left(x+\sqrt{\omega}\right)y}{x}\right).$$

Lemma 3. *There is a morphism φ_3 from C_ω to an elliptic curve D_3 of affine equation*

$$v^2 = 4\left(-1+2\omega+2\sqrt{\omega}\sqrt{\omega-1}\right)u^4 + 1$$

given by $(x,y) \longmapsto (u,v)$ *such that*

$$u = \frac{y}{\omega+\sqrt{\omega^2-\omega}-x} \text{ and}$$
$$v = \frac{\left(x+\omega+\sqrt{\omega^2-\omega}\right)^2 - 2x - 2\left(\omega+\sqrt{\omega^2-\omega}\right)^2}{\left(\omega+\sqrt{\omega^2-\omega}-x\right)^2}.$$

This elliptic curve D_3 is birational to the curve E_3. Moreover, there is a degree 2 morphism ψ_3 from C_ω to the elliptic curve E_3 can be described by

$$(x,y) \longmapsto \left(\frac{4y^2}{x-\omega}, \frac{8\left(x-\omega-\sqrt{\omega^2-\omega}\right)y}{x-\omega}\right).$$

Proposition 2. *The Jacobian J_{C_ω} of C_ω is isogenous to* $E_1 \times E_2 \times E_3$.

Proof. We refer the reader to Section 4 of [11].

5 Images on the elliptic curves

To check whether there are more relations among the sixteen 2-sextactic points on C_ω, we shall use the fact that the Jacobian J_{C_ω} of C_ω is isogenous to $E_1 \times E_2 \times E_3$. More precisely, we shall apply the following technique on each elliptic curve E_i, $i = 1,2,3$.

1. Compute the image of 2-sextactic points under the degree 2 morphism ψ_i from C_ω to the elliptic curve E_i.
2. Determine from which 2-sextactic points each of these points, that we have in step 1, arises.
3. For the group law on E_i, take the point at infinity, denoted by ∞_i, as an identity element. Then, use the elliptic curve group law to deduce the relations among these points (that we got in step 1) on the elliptic curve E_i.
4. Use the fact that the Jacobian of the elliptic curve E_i is isomorphic to the elliptic curve itself (see [13], chap. VIII, sec. 5) to obtain the principal divisor classes D_j on E_i.
5. The pullback $(\psi_i)^*(D_j)$ are also principal on C_ω. Look at the image of the $(\psi_i)^*(D_j)$, under A_{P_1}, in the Jacobian J_{C_ω} of C_ω, and get the relations in G among $P_2, P_3, \ldots, P_{16}$.
6. Finally, use relations $(*)$ (given in Section 3) to find the relations in G among the generators P_4, P_8, P_{12} and P_{16}.

Note that when we apply this technique on E_1 and E_2 we shall only keep principal divisor classes on them (that we obtained in step 4) that affect on the structure of G, and we leave to reader to verify that the other classes do not change the structure of G. On E_3, we will completely apply the technique. Therefore, the reader can apply the technique in the same way on both E_1 and E_2.

5.1 On the first elliptic curve E_1

Applying the first step of the previous technique, we find that under the degree two morphism $\psi_1 : (x,y) \mapsto (u := x, v := y^2)$ from C_ω to the elliptic curve E_1, the image of a 2-sextactic point is among the following eight points on the elliptic curve E_1 :

$$\begin{aligned}
Q_{1,1} &= \left(\alpha, \beta^2\right),\\
Q_{1,2} &= \left(\alpha, -\beta^2\right),\\
Q_{1,3} &= \left(\tfrac{\omega}{\alpha}, \left(\tfrac{\sqrt{\omega}\beta}{\alpha}\right)^2\right),\\
Q_{1,4} &= \left(\tfrac{\omega}{\alpha}, -\left(\tfrac{\sqrt{\omega}\beta}{\alpha}\right)^2\right),\\
Q_{1,5} &= \left(\tfrac{\alpha-\omega}{\alpha-1}, \left(\tfrac{\sqrt{\omega-1}\beta}{\alpha-\omega}\right)^2\right),\\
Q_{1,6} &= \left(\tfrac{\alpha-\omega}{\alpha-1}, -\left(\tfrac{\sqrt{\omega-1}\beta}{\alpha-\omega}\right)^2\right),\\
Q_{1,7} &= \left(\omega\left(\tfrac{\alpha-1}{\alpha-\omega}\right), \left(\tfrac{\sqrt{\omega^2-\omega}\beta}{\alpha-\omega}\right)^2\right),\\
Q_{1,8} &= \left(\omega\left(\tfrac{\alpha-1}{\alpha-\omega}\right), -\left(\tfrac{\sqrt{\omega^2-\omega}\beta}{\alpha-\omega}\right)^2\right),
\end{aligned}$$

where α and β are as in Section 3. It is important to know from which 2-sextactic points each of these particular points arises. Since ψ_1 is a degree two map and $\psi_1(P_1) = \psi_1(P_3) = Q_{1,1} \in E_1$, we get $(\psi_1)^*(Q_{1,1}) = P_1 + P_3$. Analogous computations provide the following table: For the group law on E_1, take the point at infinity, denoted by ∞_1, as an identity element. On the elliptic curve E_1, we get the following relations:

Table 1: Pullback of the 2-sextactic points images on E_1 .

Q	$(\psi_1)^*(Q)$	Q	$(\psi_1)^*(Q)$
$Q_{1,1}$	P_1+P_3	$Q_{1,5}$	P_9+P_{11}
$Q_{1,2}$	P_2+P_4	$Q_{1,6}$	$P_{10}+P_{12}$
$Q_{1,3}$	P_5+P_7	$Q_{1,7}$	$P_{13}+P_{15}$
$Q_{1,4}$	P_6+P_8	$Q_{1,8}$	$P_{14}+P_{16}$

$$\begin{cases} Q_{1,1}+Q_{1,2}=Q_{1,3}+Q_{1,4}=Q_{1,5}+Q_{1,6}=Q_{1,7}+Q_{1,8}=\infty_1, \\ Q_{1,1}+Q_{1,3}=Q_{1,2}+Q_{1,4}=Q_{1,5}+Q_{1,7}=Q_{1,6}+Q_{1,8}=(0,0), \\ Q_{1,1}+Q_{1,7}=Q_{1,2}+Q_{1,8}=Q_{1,3}+Q_{1,5}=Q_{1,4}+Q_{1,6}=(\omega,0), \\ Q_{1,1}+Q_{1,6}=Q_{1,2}+Q_{1,5}=Q_{1,3}+Q_{1,8}=Q_{1,4}+Q_{1,7}=(1,0), \\ 2Q_{1,1}+2Q_{1,5}=2Q_{1,1}+2Q_{1,8}=2Q_{1,2}+2Q_{1,3} \\ =2Q_{1,2}+2Q_{1,6}=2Q_{1,2}+2Q_{1,7}=2Q_{1,3}+2Q_{1,6} \\ =2Q_{1,3}+2Q_{1,7}=2Q_{1,4}+2Q_{1,5}=2Q_{1,4}+2Q_{1,8} \\ =2Q_{1,5}+2Q_{1,8}=2Q_{1,6}+2Q_{1,7}=4Q_{1,1} \\ =4Q_{1,2}=4Q_{1,3}=4Q_{1,4}=4Q_{1,5}=4Q_{1,6} \\ =4Q_{1,7}=4Q_{1,8}=(0,0), \\ 2(0,0)=2(1,0)=2(\omega,0)=\infty_1. \end{cases}$$

The Abel-Jacobi mapping A_{∞_1} on E_1 sending a formal sum to the actual sum. Thus, we on E_1 get about 99 principal divisor classes, for instance,
$D_1=[Q_{1,3}+Q_{1,4}-Q_{1,1}-Q_{1,2}]$,
$D_2=[Q_{1,5}+Q_{1,6}-Q_{1,1}-Q_{1,2}]$ and so on. Now, by applying step 5 and step 6 of our technique on these classes, we get relations in G among the generators P_4,P_8,P_{12} and P_{16}. By examining these relations, we find that all of these relations do not lead to anything new except one relation that produced from the principal divisor class $D=[Q_{1,3}+Q_{1,5}-Q_{1,1}-Q_{1,7}]$.

Lemma 4. *In G we have* $12P_{16}-12P_1=0$.

Proof. On the elliptic curve E_1, the Abel–Jacobi map A_{∞_1} sends a formal sum to the actual sum. Therefore, we have on E_1 the equality

$$\begin{aligned} A_{\infty_1}(Q_{1,3}+Q_{1,5}-(Q_{1,1}+Q_{1,7})) &= \\ Q_{1,3}+Q_{1,5}-(Q_{1,1}+Q_{1,7}) &= \\ (\omega,0)-(\omega,0) &= \infty_1. \end{aligned}$$

Then the divisor $Q_{1,3}+Q_{1,5}-Q_{1,1}-Q_{1,7}$ on E_1 is principal. Therefore the divisor

$$\begin{aligned} &(\psi_1)^*(Q_{1,3}+Q_{1,5}-Q_{1,1}-Q_{1,7}) \\ &=P_5+P_7+P_9+P_{11}-P_1-P_3-P_{13}-P_{15} \end{aligned}$$

on C_ω is principal as well. Looking at the image of this divisor by A_{P_1} in the Jacobian of C_ω, the lemma yields from relations $(*)$ from Section 3 and recalling that $3P_8-6P_{12}=0$ (remember Lemma 1).

Lemma 4 shows that the generator P_{16} is of order dividing 12 instead of 24. Therefore, as a consequence of Corollary 1 we obtain

Corollary 2. *The subgroup G is a quotient of*

$$(\mathbb{Z}/3\mathbb{Z})\oplus(\mathbb{Z}/6\mathbb{Z})\oplus(\mathbb{Z}/12\mathbb{Z})\oplus(\mathbb{Z}/24\mathbb{Z}).$$

5.2 On the second elliptic curve E_2

We will follow our technique. We find that under the degree two morphism ψ_2 from C_ω to the elliptic curve E_2, the image of a 2-sextactic point is among the following eight points on the elliptic curve E_2 :

$$\begin{aligned} Q_{2,1}&=\left(\frac{4\beta^2}{\alpha},\frac{8(\alpha+\sqrt{\omega})\beta}{\alpha}\right), \\ Q_{2,2}&=\left(-\frac{4\beta^2}{\alpha},\frac{8i(\alpha+\sqrt{\omega})\beta}{\alpha}\right), \\ Q_{2,3}&=\left(\frac{4\beta^2}{\alpha},-\frac{8(\alpha+\sqrt{\omega})\beta}{\alpha}\right), \\ Q_{2,4}&=\left(-\frac{4\beta^2}{\alpha},-\frac{8i(\alpha+\sqrt{\omega})\beta}{\alpha}\right), \\ Q_{2,5}&=\left(-\frac{4(\omega-1)\beta^2}{(\alpha-1)(\omega-\alpha)},-\frac{8\sqrt{\omega-1}(\sqrt{\omega}+1)(\alpha-\sqrt{\omega})\beta}{(\alpha-1)(\omega-\alpha)}\right), \\ Q_{2,6}&=\left(\frac{4(\omega-1)\beta^2}{(\alpha-1)(\omega-\alpha)},-\frac{8i\sqrt{\omega-1}(\sqrt{\omega}+1)(\alpha-\sqrt{\omega})\beta}{(\alpha-1)(\omega-\alpha)}\right), \\ Q_{2,7}&=\left(-\frac{4(\omega-1)\beta^2}{(\alpha-1)(\omega-\alpha)},\frac{8\sqrt{\omega-1}(\sqrt{\omega}+1)(\alpha-\sqrt{\omega})\beta}{(\alpha-1)(\omega-\alpha)}\right), \\ Q_{2,8}&=\left(\frac{4(\omega-1)\beta^2}{(\alpha-1)(\omega-\alpha)},\frac{8i\sqrt{\omega-1}(\sqrt{\omega}+1)(\alpha-\sqrt{\omega})\beta}{(\alpha-1)(\omega-\alpha)}\right), \end{aligned}$$

where α and β are as in Section 3. Since ψ_2 is a degree two map and $\psi_2(P_1)=\psi_2(P_5)=Q_{2,1}\in E_2$,
we get $(\psi_2)^*(Q_{2,1})=P_1+P_5$. Analogous computations produce the following table: For the group law on E_2, take the point at infinity, denoted by ∞_2, as an identity element. On the elliptic curve E_2, we get the following relations:

Table 2: Pullback of the 2-sextactic points images on E_2 .

Q	$(\psi_2)^*(Q)$	Q	$(\psi_2)^*(Q)$
$Q_{2,1}$	P_1+P_5	$Q_{2,5}$	P_9+P_{13}
$Q_{2,2}$	P_2+P_6	$Q_{2,6}$	$P_{10}+P_{14}$
$Q_{2,3}$	P_3+P_7	$Q_{2,7}$	$P_{11}+P_{15}$
$Q_{2,4}$	P_4+P_8	$Q_{2,8}$	$P_{12}+P_{16}$

$$\begin{cases} Q_{2,1}+Q_{2,3}=Q_{2,2}+Q_{2,4}=Q_{2,5}+Q_{2,7}=Q_{2,6}+Q_{2,8}=\infty_2, \\ 3Q_{2,1}=3Q_{2,2}=3Q_{2,3}=3Q_{2,4}=3Q_{2,5} \\ =3Q_{2,6}=3Q_{2,7}=3Q_{2,8}=\infty_2. \end{cases}$$

Now, using the fact that on E_2 the Abel-Jacobi mapping A_{∞_2} sending a formal sum to the actual sum. We thus on E_2 obtained about 33 principal divisor classes, for instance,
$D_1=[3Q_{2,2}-3Q_{2,1}]$,
$D_2=[Q_{2,5}+Q_{2,7}-Q_{2,1}-Q_{2,3}]$ and so on. Applying step 5 and step 6 of our technique on these classes, we get relations in G among the generators P_4,P_8,P_{12} and P_{16}. By examining these relations, we find that all of these relations do not have effect on the structure of G except one relation that produced from the principal divisor class $D=[3Q_{2,5}-3Q_{2,1}]$. We illustrate this in the following lemma.

Lemma 5. *In G we have* $3P_4+3P_{12}-3P_{16}-3P_1=0$.

Proof. Applying step 4 of the previous technique, we thus on E_2 obtain that

$$A_{\infty_2}(3Q_{2,5}-3Q_{2,1})=3Q_{2,5}-3Q_{2,1}=\infty_2-\infty_2=\infty_2,$$

then the divisor $3Q_{2,5} - 3Q_{2,1}$ on E_2 is principal. Therefore the divisor

$$(\psi_2)^*(3Q_{2,5} - 3Q_{2,1}) = 3P_9 + 3P_{13} - 3P_1 - 3P_5$$

on C_ω is also principal. Looking at the image by A_{P_1} of this divisor in the Jacobian of C_ω and using relations $(*)$ given in Section 3 we get

$$3P_4 + 27P_{12} - 51P_{16} = 0.$$

Since the orders of P_{12} and P_{16} divide 24 and 12, respectively, the relation $3P_4 + 27P_{12} - 51P_{16} = 0$ becomes $3P_4 + 3P_{12} - 3P_{16} = 0$.

According to Lemma 5, the order of the element $P_4 + P_{12} - P_{16}$ in J_{C_ω} is exactly 3. Indeed, the relation $3P_4 + 3P_{12} - 3P_{16} = 0$ implies that the order of the element $P_4 + P_{12} - P_{16}$ in J_{C_ω} divides 3 (instead of 6, remember Lemma 1). Note that if $P_4 + P_{12} - P_{16} = 0$ which equivalents to say $P_4 + P_{12} - P_{16} - P_1 = 0$. This implies the existence of a degree two morphism $C_\omega \to \mathbb{P}^1$, contradicting the fact that C_ω is not a hyperelliptic curve. As a result of Corollary 1 and Corollary 2 we obtain

Corollary 3. *The subgroup G is a quotient of* $(\mathbb{Z}/3\mathbb{Z})^2 \oplus (\mathbb{Z}/12\mathbb{Z}) \oplus (\mathbb{Z}/24\mathbb{Z})$.

5.3 On the third elliptic curve E_3

The degree two map ψ_3 sending a 2-sextactic point on C_ω to one of the following eight points on the elliptic curve E_3:

$$Q_{3,1} = \left(-\frac{4\beta^2}{\omega-\alpha}, \frac{8\left(\sqrt{\omega}\sqrt{\omega-1}+\omega-\alpha\right)\beta}{\omega-\alpha}\right),$$
$$Q_{3,2} = \left(\frac{4\beta^2}{\omega-\alpha}, \frac{8i\left(\sqrt{\omega}\sqrt{\omega-1}+\omega-\alpha\right)\beta}{\omega-\alpha}\right),$$
$$Q_{3,3} = \left(-\frac{4\beta^2}{\omega-\alpha}, -\frac{8\left(\sqrt{\omega}\sqrt{\omega-1}+\omega-\alpha\right)\beta}{\omega-\alpha}\right),$$
$$Q_{3,4} = \left(\frac{4\beta^2}{\omega-\alpha}, -\frac{8i\left(\sqrt{\omega}\sqrt{\omega-1}+\omega-\alpha\right)\beta}{\omega-\alpha}\right),$$
$$Q_{3,5} = \left(-\frac{4\beta^2}{\alpha(\alpha-1)}, \frac{8\left(\sqrt{\omega}\sqrt{\omega-1}\alpha+\omega\alpha-\omega\right)\beta}{\alpha\sqrt{\omega}(\alpha-1)}\right),$$
$$Q_{3,6} = \left(\frac{4\beta^2}{\alpha(\alpha-1)}, \frac{8i\left(\sqrt{\omega}\sqrt{\omega-1}\alpha+\omega\alpha-\omega\right)\beta}{\alpha\sqrt{\omega}(\alpha-1)}\right),$$
$$Q_{3,7} = \left(-\frac{4\beta^2}{\alpha(\alpha-1)}, -\frac{8\left(\sqrt{\omega}\sqrt{\omega-1}\alpha+\omega\alpha-\omega\right)\beta}{\alpha\sqrt{\omega}(\alpha-1)}\right),$$
$$Q_{3,8} = \left(\frac{4\beta^2}{\alpha(\alpha-1)}, -\frac{8i\left(\sqrt{\omega}\sqrt{\omega-1}\alpha+\omega\alpha-\omega\right)\beta}{\alpha\sqrt{\omega}(\alpha-1)}\right),$$

where α and β are as in Section 3. Here, we have For the group law on E_3, take the point at infinity, denoted by ∞_3, as an identity element. On the elliptic curve E_3, we get the following relations:

Table 3: Pullback of the 2-sextactic points images on E_3 .

Q	$(\psi_3)^*(Q)$	Q	$(\psi_3)^*(Q)$
$Q_{3,1}$	P_1+P_{15}	$Q_{3,5}$	P_5+P_9
$Q_{3,2}$	P_2+P_{16}	$Q_{3,6}$	P_6+P_{10}
$Q_{3,3}$	P_3+P_{13}	$Q_{3,7}$	P_7+P_{11}
$Q_{3,4}$	P_4+P_{14}	$Q_{3,8}$	P_8+P_{12}

$$\begin{cases} Q_{3,1}+Q_{3,3}=Q_{3,2}+Q_{3,4}=Q_{3,5}+Q_{3,7}=Q_{3,6}+Q_{3,8}=\infty_3, \\ 3Q_{3,1}= 3Q_{3,3}= 3Q_{3,6}= 3Q_{3,8}= (4(i-\alpha)(i+\omega),0), \\ 3Q_{3,2}= 3Q_{3,4}= 3Q_{3,5}= 3Q_{3,7}= (-4(i-\alpha)(i+\omega),0), \\ 2(4(i-\alpha)(i+\omega),0) = 2(-4(i-\alpha)(i+\omega),0), = \infty_3. \end{cases}$$

where α is as in Section 3. On E_3, the Abel-Jacobi mapping A_{∞_3} sends a formal sum to the actual sum. Hence, we on E_3 get the following principal divisor classes:

$$\begin{array}{ll} D_1= [3Q_{3,3}-3Q_{3,1}], & D_{10}= [3Q_{3,5}-3Q_{3,4}], \\ D_2= [3Q_{3,6}-3Q_{3,1}], & D_{11}= [3Q_{3,7}-3Q_{3,4}], \\ D_3= [3Q_{3,8}-3Q_{3,1}], & D_{12}= [3Q_{3,7}-3Q_{3,5}], \\ D_4= [3Q_{3,6}-3Q_{3,3}], & D_{13}= [Q_{3,2}+Q_{3,4}-Q_{3,1}-Q_{3,3}], \\ D_5= [3Q_{3,8}-3Q_{3,3}], & D_{14}= [Q_{3,5}+Q_{3,7}-Q_{3,1}-Q_{3,3}], \\ D_6= [3Q_{3,8}-3Q_{3,6}], & D_{15}= [Q_{3,6}+Q_{3,8}-Q_{3,1}-Q_{3,3}] \\ D_7= [3Q_{3,4}-3Q_{3,2}], & D_{16}= [Q_{3,5}+Q_{3,7}-Q_{3,2}-Q_{3,4}], \\ D_8= [3Q_{3,5}-3Q_{3,2}], & D_{17}= [Q_{3,6}+Q_{3,8}-Q_{3,2}-Q_{3,4}], \\ D_9= [3Q_{3,7}-3Q_{3,2}], & D_{18}= [Q_{3,6}+Q_{3,8}-Q_{3,5}-Q_{3,7}]. \end{array}$$

As the pullback $(\psi_3)^*(D_j)$, for $1 \le j \le 18$, on C_ω is also principal on the curve C_ω, we have the following relations in the group G among the generators $P_2, P_3, \ldots, P_{16}$

(1)$3P_3+3P_{13}-3P_{15}= 0$,
(2)$3P_6+3P_{10}-3P_{15}= 0$,
(3)$3P_8+3P_{12}-3P_{15}= 0$,
(4)$3P_6+3P_{10}-3P_3-3P_{13}= 0$,
(5)$3P_8+3P_{12}-3P_3-3P_{13}= 0$,
(6)$3P_8+3P_{12}-3P_6-3P_{10}= 0$,
(7)$3P_4+3P_{14}-3P_2-3P_{16}= 0$,
(8)$3P_5+3P_9-3P_2-3P_{16}= 0$,
(9)$3P_7+3P_{11}-3P_2-3P_{16}= 0$,
(10)$3P_5+3P_9-3P_4-3P_{14}= 0$,
(11)$3P_7+3P_{11}-3P_4-3P_{14}= 0$,
(12)$3P_7+3P_{11}-3P_5-3P_9= 0$,
(13)$P_2+P_{16}+P_4+P_{14}-P_{15}-P_3-P_{13}= 0$,
(14)$P_5+P_9+P_7+P_{11}-P_{15}-P_3-P_{13}= 0$,
(15)$P_6+P_{10}+P_8+P_{12}-P_{15}-P_3-P_{13}= 0$,
(16)$P_5+P_9+P_7+P_{11}-P_2-P_{16}-P_4-P_{14}= 0$,
(17)$P_6+P_{10}+P_8+P_{12}-P_2-P_{16}-P_4-P_{14}= 0$,
(18)$P_6+P_{10}+P_8+P_{12}-P_5-P_9-P_7-P_{11}= 0$.

Using relations given in $(*)$ we find that

$$\begin{aligned}
&(1)12P_4-9P_8-42P_{12}+36P_{16}=0,\\
&(2)6P_4-27P_8+12P_{12}-30P_{16}=0,\\
&(3)6P_4-9P_8-6P_{16}=0,\\
&(4)54P_{12}-18P_8-6P_4-66P_{16}=0,\\
&(5)42P_{12}-6P_4-42P_{16}=0,\\
&(6)18P_8-12P_{12}+24P_{16}=0,\\
&(7)6P_4-9P_8-6P_{16}=0,\\
&(8)6P_4+9P_8-12P_{12}+18P_{16}=0,\\
&(9)9P_8-42P_{12}+48P_{16}=0,\\
&(10)18P_8-12P_{12}+24P_{16}=0,\\
&(11)18P_8-6P_4-42P_{12}+54P_{16}=0,\\
&(12)30P_{16}-30P_{12}-6P_4=0,\\
&(13)12P_{12}-6P_8-12P_{16}=0,\\
&(14)3P_8-6P_{12}+12P_{16}=0,\\
&(15)18P_{12}-9P_8-24P_{16}=0,\\
&(16)9P_8-18P_{12}+24P_{16}=0,\\
&(17)6P_{12}-3P_8-12P_{16}=0,\\
&(18)24P_{12}-12P_8-36P_{16}=0.
\end{aligned}$$

Note that these relations do not affect on the structure of the group G.

6 Proof of The Main Theorem

Initially, we briefly recall what we need about Weierstrass points on quartic curves which will be useful to prove (iii) of Lemma 6 below. Let C be a non-singular projective quartic plane curve. For any divisor D on C, the Riemann-Roch space $L(D)$ is defined as $\{f\in\mathbb{C}(C)\,|\,\text{div}(f)+D\geq 0\}$. A Weierstrass point on C is a point Q for which there exists a non-constant rational function on C with a pole of order at most three at Q and no poles everywhere else, or equivalently, $L(3Q)$ has at least dimension 2. It is well known that Weierstrass points on C are nothing but flexes (Vermeulen [1]). Lemma 6 and Lemma 7 below verify that the order of the generators $[P_{12}-P_1]$ and $[P_{16}-P_1]$ of G are exactly 24 and 12, respectively.

Lemma 6. *For any two different 2-sextactic points P and Q on C_ω, we get*

$$(i)\ [P-Q]\neq 0,\qquad (ii)\ [2P-2Q]\neq 0,\qquad (iii)\ [3P-3Q]\neq 0.$$

Proof. (i) Suppose, to the contrary, that $[P-Q]=0$. Then there is a non-constant rational function on C_ω with a simple pole at Q. This implies that the curve is isomorphic to the projective line, which is a contradiction.

(ii) Assume that $[2P-2Q]=0$. Hence, there is a non-constant rational function f on C_ω satisfying that $\text{div}(f)=2P-2Q$. This implies the existence of a degree two morphism $C_\omega\to\mathbb{P}^1$, contradicting the fact that C_ω is not a hyperelliptic curve.

(iii) Let $[3P-3Q]=0$. Then there is a non-constant rational function on C_ω with a pole of order three at Q and no poles everywhere else, which implies that the point Q is a Weierstrass point, or equivalently, Q is a flex point. This is impossible since $Q\in C_\omega$ is a 2-sextactic point.

Lemma 7. *In G we have*

$$\begin{aligned}
&(i)\ [4P_{16}-4P_1]\neq 0,\\
&(ii)\ [6P_{16}-6P_1]\neq 0,\\
&(iii)\ [12P_{12}-12P_1]\neq 0,\\
&(iv)\ [6P_{12}-6P_1]\neq 0,\\
&(v)\ [4P_{12}-4P_1]\neq 0,\\
&(vi)\ [12P_{12}+6P_{16}-18P_1]\neq 0.
\end{aligned}$$

Proof. (i) We show that $[8P_{16}-8P_1]\neq 0$. Suppose that the divisor $8P_{16}-8P_1$ is principal, then so is $8P_{16}-8P_1+\text{div}(\frac{\Delta_1}{\Delta_{16}})=P_3+P_{16}-P_1-P_{14}$. This implies the existence of a rational function of degree two on C_ω contradicting the fact that C_ω is not a hyperelliptic curve. Now since $[8P_{16}-8P_1]$ is a twice of $[4P_{16}-4P_1]$, so if the former does not vanish, neither can the latter.

(ii) In a similar way as in (i), if the divisor $6P_{16}-6P_1$ is principal, then so is $6P_{16}-6P_1+\text{div}(\frac{\Delta_1}{\Delta_{16}})=P_1+P_3-P_{14}-P_{16}$. This implies the existence of a rational function degree 2 on C_ω, so we get the same contradiction as (i).

(iii) It is a well-known fact that the canonical linear system on a smooth plane quartic curve is cut out by lines in $\mathbb{P}^2$. If the divisor $12P_{12}-12P_1$ is principal, then so is $12P_{12}-12P_1+\text{div}(\frac{\Delta_1^2}{\Delta_{12}^2})=2P_1+2\ P_3-2P_{10}-2P_{12}$. This implies the existence of a rational function $\mathfrak{f}$ on C_ω with $\text{div}(\mathfrak{f})=2P_1+2P_3-2P_{10}-2P_{12}$, it follows that $\mathfrak{f}\in L(2P_{10}+2P_{12})$. Let K be a canonical divisor on C_ω. If $E=2P_{10}+2P_{12}$, then the divisor E is not linearly equivalent to K (since otherwise, there is a bitangent line to C_ω at P_{10} and P_{12} and this is not true) and $deg(K-E)=0$. Therefore, the vector space $L(K-E)$ has dimension zero (see Lemma 1.2 page 295 in [19]). Riemann-Roch Theorem implies that the vector space $L(E)$ is of dimension two. So, we may consider that $L(E)$ is generated by the rational functions 1 and $\mathfrak{g}=\frac{T_9T_{11}}{\ell_3^2}$, where $\text{div}(\mathfrak{g})=P_1+P_3+P_6+P_8-2P_{10}-2P_{12}$. Particularly, $\mathfrak{f}$ can be written as

$$\mathfrak{f}=c.1+\mathfrak{g}=\frac{c\ell_3^2+T_9T_{11}}{\ell_3^2},$$

for some constant $c\in\mathbb{C}$. P_3 is a zero of $\mathfrak{f}$ if and only if $c=-\mathfrak{g}(P_3)=0$. Therefore $\mathfrak{f}=\mathfrak{g}$, a contradiction.

(iv) Note that the expression in (iii) is twice that of (iv), so if the former does not vanish, neither can the latter.

(v) Also, the expression in (iii) is three times that of (v), so if the former does not vanish, neither can the latter.

(vi) Suppose, to the contrary, that the divisor $12P_{12}+6P_{16}-18P_1$ is principal, then so is

$$\begin{aligned}
&12P_{12}+6P_{16}-18P_1+\text{div}(\frac{\Delta_1^3T_{10}T_{12}\ell_3}{\Delta_{16}\Delta_{12}^2T_1T_3\ell_1})\\
&=P_5+P_7+P_{10}+P_{12}-2P_{14}-2P_{16}.
\end{aligned}$$

This implies the existence of a rational function h on C_ω such that $div(h)=P_5+P_7+P_{10}+P_{12}-2P_{14}-2P_{16}$, it follows that $h\in L(2P_{14}+2P_{16})$. In a similar method as in (iii), the vector space $L(2P_{14}+2P_{16})$ is two dimensional. It is generated by the rational functions 1 and $k=\frac{\ell_1\ell_2}{T_{14}T_{16}}$, where $\text{div}(k)=P_2+P_4+P_5+P_7-2P_{14}-2P_{16}$. Particularly, h can be written in the form

$$h=b.1+k=\frac{bT_{14}T_{16}+\ell_1\ell_2}{T_{14}T_{16}},$$

for some complex number b. The point P_5 is a zero of h if and only if $b = -k(P_5) = 0$. Therefore $h = k$ and this is a contradiction.

Now we can finish the proof of the main result.

6.1 Proof of Theorem 1

We will show that there is no more relations among the generators $[P_4 - P_1]$, $[P_8 - P_1]$, $[P_{12} - P_1]$ and $[P_{16} - P_1]$ of G. We will find which elements of the subgroup G are in the kernel of the isogeny from the Jacobian J_{C_ω} to the product $E_1 \times E_2 \times E_3$. Assume that for some integers c_4, c_8, c_{12} and c_{16} we have

$$M := \left[\begin{array}{c} c_4P_4 + c_8P_8 + c_{12}P_{12} + c_{16}P_{16} \\ -(c_4 + c_8 + c_{12} + c_{16})P_1 \end{array} \right] = 0,$$

i.e., this divisor is in the kernel of the isogeny from the Jacobian J_{C_ω} to the product $E_1 \times E_2 \times E_3$. We compute the image of this divisor on each of the elliptic curves E_1, E_2 and E_3.

Looking at $\psi_1(M)$ and using the results in Subsection 5.1 we get

$$\begin{aligned} & c_4 (2Q_{1,2}) + c_8 (Q_{1,2} + Q_{1,4}) + c_{12} (Q_{1,2} + Q_{1,6}) \\ & + c_{16} (Q_{1,2} + Q_{1,8}) = \infty_1. \end{aligned}$$

Since $2Q_{1,2}$ and $Q_{1,2} + Q_{1,6}$ are of order 4, $Q_{1,2} + Q_{1,4}$ and $Q_{1,2} + Q_{1,8}$ are of order 2, moreover $2Q_{1,2} \neq Q_{1,2} + Q_{1,4} \neq Q_{1,2} + Q_{1,6} \neq Q_{1,2} + Q_{1,8}$. This implies that $c_4 \equiv 0 \bmod 4$, $c_8 \equiv 0 \bmod 2$, $c_{12} \equiv 0 \bmod 4$ and that $c_{16} \equiv 0 \bmod 2$.

Looking at $\psi_2(M)$ and using the results in Subsection 5.2 we have

$$(c_4 + c_8)(Q_{2,3} + Q_{2,4}) + (c_{12} + c_{16})(Q_{2,3} + Q_{2,8}) = \infty_2.$$

As $Q_{2,3}$, $Q_{2,4}$ and $Q_{2,8}$ are of order 3, as well as $Q_{2,3} + Q_{2,4} \neq Q_{2,3} + Q_{2,8}$. This implies that $(c_4 + c_8) \equiv 0 \bmod 3$ and that $(c_4 + c_{16}) \equiv 0 \bmod 3$.

Looking at $\psi_3(M)$ and using the results in Subsection 5.3 we obtain

$$\begin{aligned} & c_4 (Q_{3,3} + Q_{3,4}) + (c_8 + c_{12})(Q_{3,3} + Q_{3,8}) \\ & + c_{16} (Q_{3,2} + Q_{3,3}) = \infty_3. \end{aligned}$$

Since $Q_{3,3} + Q_{3,4}$ and $Q_{3,2} + Q_{3,3}$ are of order 6, $Q_{3,3} + Q_{3,8}$ is of order 3, furthermore $Q_{3,2} + Q_{3,3}$, $Q_{3,3} + Q_{3,4}$ and $Q_{3,3} + Q_{3,8}$ are mutually distinct. This implies that $c_4 \equiv 0 \bmod 6$, $c_8 + c_{12} \equiv 0 \bmod 3$ and that $c_{16} \equiv 0 \bmod 6$.

Summarizing above, we have the following system

$$\begin{aligned} & c_4 \equiv c_{12} \equiv 0 \bmod 4, c_8 \equiv c_{16} \equiv 0 \bmod 2, \\ & (c_4 + c_8) \equiv (c_{12} + c_{16}) \equiv 0 \bmod 3, \\ & c_4 \equiv c_{16} \equiv 0 \bmod 6, c_8 + c_{12} \equiv 0 \bmod 3. \end{aligned} \qquad (\circledast)$$

Recall that we want to prove that no non-trivial element of the group from Corollary 3 are trivial in G. This group is of order $3^4 \cdot 2^5$, but we found some restrictions for triviality in the subgroup G (the congruences modulo 2, 3, 4 and 6 on these sums which satisfies $(\circledast)$), therefore we only need to verify that non-zero elements of the group from Corollary 3 that satisfy these restrictions still non-trivial in G. For this objective we consider

$$[P_4 + P_{12} - P_{16} - P_1], [P_1 + P_8 - 2P_{12}], [P_{12} - P_1], [P_{16} - P_1],$$

as a basis for the subgroup G, so that the element M could be expressed by this basis using the respective coefficients b_4, b_8, b_{12} and b_{16}. It is known that both b_4 and b_8 are residues modulo 3, b_{12} is well-defined modulo 24, and b_{16} is well-defined modulo 12 (this is Corollary 3 explicitly). we get
$c_4 = b_4, c_8 = b_8$,
$c_{12} = b_4 - 2b_8 + b_{12}$ and $c_{16} = b_{16} - b_4$.

The congruence modulo 4 in $(\circledast)$, $c_4 \equiv 0 \bmod 4$, implies that b_4 is divisible by 4. Therefore, the congruences modulo 2 in $(\circledast)$ imply that b_8 and b_{16} must be even. The congruence modulo 4 in $(\circledast)$, $c_{12} \equiv 0 \bmod 4$, thus implies that b_{12} must be divided by 4 while the congruence modulo 3, $(c_8 + c_{12}) \equiv 0 \bmod 3$, implies that b_{12} is also divisible by 3. The congruence modulo 3, $(c_{12} + c_{16}) \equiv 0 \bmod 3$, implies that 3 must divide b_{16}. Note that the congruence modulo 3 in $(\circledast)$, $(c_4 + c_8) \equiv 0 \bmod 3$, and the congruences modulo 6, $c_4 \equiv c_{16} \equiv 0 \bmod 6$, do yield nothing new. This specifies elements of the group from Corollary 3 that become trivial in the subgroup G under each ψ_j. Actually, we can write $b_4 = 12a_4$, $b_8 = 6a_8$, $b_{12} = 12a_{12}$ and $b_{16} = 6a_{16}$ with each of these a_is being well-defined modulo 2. It follows that an element of the group from Corollary 3 that satisfies $(\circledast)$ is generated by

$$\begin{aligned} & [12P_4 + 12P_{12} - 12P_{16} - 12P_1], [6P_1 + 6P_8 - 12P_{12}], \\ & [12P_{12} - 12P_1], [6P_{16} - 6P_1], \end{aligned}$$

with respective coefficients a_4, a_8, a_{12} and a_{16}, all well-defined only modulo 2. Lemma 1 and Lemma 5 show that $[6P_1 + 6P_8 - 12P_{12}]$ and $[12P_4 + 12P_{12} - 12P_{16} - 12P_1]$ are trivial, respectively. Therefore, the only non-trivial elements in the kernel are $[12P_{12} - 12P_1]$, $[6P_{16} - 6P_1]$ and their sum $[12P_{12} + 6P_{16} - 18P_1]$. Lemma 7 above explains that these classes are non-trivialities in G. This finishes the proof of Theorem 1. ■

Open Problems. Now, it is interesting to mention to some open problems that can be handled using the same technique proposed in this paper. One of these problems is investigation the distribution and locations of sextactic points on the n-th Fermat curve, defined by

$$F_n \colon X^n + Y^n = Z^n, \qquad n \geq 4,$$

to study the structure of the subgroup generated by images of these points under the Abel- Jacobi map in the Jacobian of such curve F_n. A second problem is studing the subgroup generated by the 2-Weiertrass points in the Jacobian of the family of smooth quartic curves given by the equation:

$$C_a \colon Y^4 = XZ(X - Z)(X - aZ), \qquad a \in \mathbb{C} \setminus \{0, 1\}.$$

A third problem is determining the structure of the group generated by the 2-Weiertrass points in the Jacobian of the family of Kuribayashi quartic curves given by the equation:

$$\begin{aligned} K_t \colon \quad & X^4 + Y^4 + Z^4 + t(X^2Y^2 + X^2Z^2 + Y^2Z^2) = 0, \\ & \text{where } t \in \mathbb{C} \setminus \{-1, \pm 2\}. \end{aligned}$$

Application. Finally, we mention to the fact that studying smooth proper algebraic curves which carry numerous finite sets of points with special properties has a large number of applications in different fields and leads to better understanding of the general behaviour of these systems, such as thermal stability and crystallization kinetics of the semiconducting [20], uniform algebraic hyperbolic [21], differential-algebraic systems with power series coefficients and reducing algorithm for differential-algebraic systems [22]. Also, it can be used for multicategory support vector machine as well as sequential testing procedure for the parameter of left truncated exponential distribution [23].

Acknowledgement

The first author wishes to extend his sincere gratitude to the Deanship of Scientific Research at the Islamic University of Madinah for the support provided to the Post-Publishing Program 1.
The second and the third authors were funded by the Academy Scientific Research and Technology, Egypt, under ScienceUP Grant No. 6492.

Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

References

[1] A. Vermeulen. *Weierstrass points of weight two on curves of genus three*. Ph.D. Thesis, University of Amsterdam, (1983).
[2] K. Alwaleed and F. Sakai, Geometry and computation of 2-Weierstrass points of Kuribayashi quartic curves, *Saitama Math. J.,* **26**, 67-82 (2009).
[3] D. Mumford. *Tata lectures on theta. II*, Birkhauser Boston Inc., Boston, MA, Jacobian theta functions and differential equations. MR 86b:14017 (1984).
[4] M. Girard and P. Tzermias. Group generated by the Weierstrass points of a plane quartic. *Proceedings of the American Mathematical Society*, **130**(3), 667-672 (2002).
[5] D. T. Prapavessi. On the Jacobian of the Klein curve, *Proc. Amer. Math. Soc.,* **122**(4), 971-978 (1994).
[6] D. E. Rohrlich. Points at infinity on the Fermat curves, *Invent. Math.,* **39**(2), 95-127 (1977).
[7] M. J. Klassen and E. F. Schaefer. Arithmetic and geometry of the curve $y^3+1=x^4$, *Acta Arith.,* **74**(3), 241-257 (1996).
[8] K. Alwaleed. Group generated by total sextactic points of Kuribayashi quartic curve, J*ournal of Algebra and Its Applications,* **20**(10), 2150184 (2021): DOI: 10.1142/S021949882150184X.
[9] K. Alwaleed and W. K. Elshareef. On the Jacobian of Kuribayashi curves, *Communications in Algebra,* **48**(1), 291-298 (2020).
[10] K. Alwaleed and M. Kawasaki. 2-Weierstrass points of certain plane curves of genus three, *Saitama Math. J.,* **26**, 49-65 (2009).
[11] Alwaleed, K., Saleem M. A. and Elshareef, W. K.: Group generated by total sextactic points of a family of quartic curves, Communications in Algebra, (2021), DOI: 10.1080/00927872.2021.1980796.
[12] A. Kuribayashi and K. Yoshida. On the non-hyperelliptic compact Riemann surfaces defined by $Y^4=X(X-1)(X-t)$, *Bull. Fac. Sci. Eng., Chuo Uni.,* Ser. I, **21**, 13-27 (1978).
[13] R. Miranda. Algebraic Curves and Riemann Surfaces, American Mathematical Society, USA, (1995).
[14] E. Kunz and R. G. Belshoff. *Introduction to plane algebraic curves*, Birkhäuser, (2005).
[15] G. Thorbergsson and M. Umehara. Sextactic points on a simple closed curve, *Nagoya Math J.,* **167**, 55-94 (2002).
[16] M. Namba. Equivalence problem and automorphism groups of certain compact riemann surfaces, *Tsukuba Journal of Mathematics*, **5**(2), 319-338 (1981).
[17] K. Alwaleed and W. K. Elshareef. Weierstrass points of order three on smooth quartic curves, *Journal of Algebra and Its Applications*, **18**(01), 1950020 (2019).
[18] J. H. Silverman. *The arithmetic of elliptic curves*, Vol(106), New York: Springer, (2009).
[19] R. Hartshorne. *Algebraic Geometry*, *Graduate texts in Mathematics*, (Vol. 52), Springer, USA, 2010.
[20] M. Shapaan. DC Conductivity, Thermal Stability and Crystallization Kinetics of the Semiconducting 30P2O5 (50-x)V2O5 xB2O3 20Fe2O3 Oxide Glasses, I*nt. J. of Thin Film Science and Technology,* **5**, 143-153 (2016).
[21] Y. Quintana, W. Ramírez and A. Urieles. Euler Matrices and their Algebraic Properties Revisited, *Appl. Math. Inf. Sci.* **14**, 583-596(2020): DOI:10.18576/amis/140407.
[22] S. Thota and S. D. Kumar. A New Reduction Algorithm for Differential-Algebraic Systems with Power Series Coefficients, *Inf Sci. Lett.* ,**10,** 59-66 (2021): doi:10.18576/isl/100108.
[23] Y. Liu. Extended Bayesian Framework for Multicategory Support Vector Machine, *J. Stat. Appl. Prob.,* **9**, 1-11(2020): doi:10.18576/jsap/090101