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A New Class of Life Distribution based on Laplace Transform and It's Applications

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Abstract: Based on the approach of Laplace transform, a new class of life distributions called used better than aged in increasing concave denoted by (UBAC(2)L) is introduced. The implication of our proposed class of life distribution with other classes is given. Some properties of UBAC(2)L class of life distribution are studied. By using the goodness of fit methodology, a new test statistic is proposed for testing exponentiality versus UBAC(2)L class of life distribution. Critical values of our test are calculated for complete and censored data. The power of the test and pitman's asymptotic efficiency (PAE) for some commonly used distributions in reliability are calculated. Finally, a set of real data is used as an example to elucidate the use of the proposed test statistic for practical reliability analysis.

Keywords: UBAC(2)L class of life distributions, Age-smooth, Survival functions, Increasing concave ordering, Convolution, Series system.

1 Introduction and Motivation

The concept of aging plays an important role in reliability analysis, which describes how can a component or system improves or depreciate with age. Many classes of life distributions are categorized and defined according to their aging is important in any reliability analysis. The definitions of these classes helped statisticians to introduce new test statistics which are defined based on the definition of these classes. The main aim of constructing new tests is to gain higher efficiencies and gives good power.

So, many authors introduced a lot of new classes and made a testing hypothesis for testing exponentiality versus these classes of life distribution. The most well-known classes of life distributions can be mentioned such as: such as (IFR, IFRA, DMRL, NBU, NBUE, UBA, UBAE, UBAC and UBACT) has got a good deal of attention in the literature [1,2,3,4,5,6,7]. Some authors took up testing exponentiality based on goodness of fit technique versus many classes of life distributions; see El-Bassiouny et al. [8] for MRL class of life distribution, Abu-Youssef [9] for DVRL(IVRL), Kayid et al. [10] for NBU(2), Ismail et al [11] for UBAC, Abu-Youssef et al. [12] for UBACT. Ali for UBAC(2) [13], Mahmoud et al. [14] for (NRBUL) and Abu-Youssef et al. [15] for (UBAL).

Let X be a nonnegative continuous random variable which represent equipment life with distribution function F; survival function $\overline{F} = 1 - F$ and finite mean $\mu = E[X] = \int_0^\infty \overline{F}(x)dx$. At age t, the random residual life is defined by X_t with survival function $\overline{F}_t = \frac{\overline{F}(t+x)}{\overline{F}(t)}$, x, $t \geq 0$. The mean residual life of $X_t = [X - t|X > t]$ is given by

$$\mu(t) = E[X_t] = \frac{\int_t^\infty \overline{F}(u)du}{\overline{F}(t)}, \qquad t \geq 0, \overline{F}(t) > 0.$$

Definition (1.1): The distribution function F is said to be UBA if

$$\frac{\bar{F}(x+t)}{\bar{F}(t)} \geq e^{-x}, \qquad x,t \geq 0, \tag{1}$$

See Alzaid [16].

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Definition (1.2): The distribution function F is said to be UBAC(2) if

$$\int_t^\infty \bar{F}(u)du - \int_{x+t}^\infty \bar{F}(u)du \geq (1-e^{-x})\bar{F}(t), \qquad x,t \geq 0, \tag{2}$$

which can be written as :

$$\nu(t) - \nu(x+t) \geq (1-e^{-x})\bar{F}(t). \tag{3}$$

See Ali [13].

Definition (1.3): The distribution function F is said to be UBAC(2)L if

$$\int_0^\infty e^{-sx}[\nu(t)-\nu(x+t)]dx \geq \int_0^\infty e^{-sx}[(1-e^{-x})\bar{F}(t)]dx, s \geq 0. \tag{4}$$

After integration, we get:

$$\int_t^\infty e^{-sy}\bar{F}(y)dy \geq \frac{1}{s+1}e^{-st}\bar{F}(t), s \geq 0. \tag{5}$$

In many applications, it is important for the buyer of the purchasing used items with unknown age, to have an idea about the lifetime of it. Hence, it is important to compute the residual lifetime of the item with respect to its performance under the true age (UBA, UBAE, UBAC(2) and UBAC(2)L). Examples of criteria for comparing ages of electrical equipment like computers, radios, etc. can be found in Bhattacharjee [17] and Cline [18]. The implication of our proposed class of life distribution with other classes is

$$IFR \subset DMRL \subset UBA \subset UBAC(2) \subset UBAC(2)L$$

The proposed new class of life distribution UBAC(2)L which is the generalization of UBAC(2) class of life distribution is more efficient.

The paper is organized as follows: In section 2, definitions and relationships are given. In section 3, we study the UBAC(2)L property under the convolution, discrete mixture, and formation of a coherent system. A new test statistic technique is proposed in section 4 and critical values are tabulated. In section 5, PAE and The power estimates for the test are calculated. Testing of censored data and critical values are introduced in section 6. Applications for complete and censored data are studied in section 7. Finally, we give a conclusion for our work in section 8.

2 Definitions And Relationships

Definition (2.1): (Bhattacharjee [17]):
F is called age-smooth if for every support point x of F

$$\lim_{t\to\infty} \bar{F}_t(x) \in [0,1].$$

It is important to note That IFR and DFR classes are contained in the age-smooth class of life distribution.

Properties of age-smooth class :
(i) If F is age-smooth then

$\lim_{t\to\infty} \bar{F}_t(x) = e^{-\gamma x}$ for some $0 \leq \gamma \leq \infty$, and for every $x \geq 0$,

(see Bhattacharjee [17] and Alzaid [16]).
(ii) If F has a failure rate function $r(t)$, then $\lim_{t\to\infty} r(t) = \gamma$, (see Bhattacharjee [17])
(iii) If F has finite mean, then F is age-smooth iff

$$\lim_{t\to\infty} \mu(t) = \lim_{t\to\infty} \int_0^\infty \bar{F}_t(x)dx = \gamma^{-1},$$

(see Bhattacharjee [17]).
(iv) The finitely age-smooth distribution are closed under convolution. This implies that if

$$\lim_{t\to\infty} \frac{\bar{F}_i(x+y+t)}{\bar{F}_i(t)} = e^{-\gamma_i(x+y)} \qquad x,y \geq 0, i=1,2$$

for some $0 \leq \gamma_1 \leq \gamma_2 \leq \infty$ then

$$\lim_{t\to\infty} \frac{\overline{F_1 * F_2}(x+y+t)}{\overline{F_1 * F_2}(t)} = e^{-\gamma_i(x+y)} \qquad x,y \geq 0, i=1,2$$

where $F_1 * F_2$ is the convolution of F_1 with F_2, (see Embrechts et al [23]).

3 Closure Properties

In this section we discuss the closure properties of the UBAC(2)L under convolution, discrete mixing and formation of coherent system.

Theorem 1. The UBAC(2)L class of life distribution is closed under the convolution.

Proof:
Let F_1 and F_2 be UBAC(2)L, $F = F_1 * F_2$ is age-smooth class of life distribution. Then from the definition of UBAC(2)L class, we have:

$$\begin{aligned}
\int_t^\infty e^{-sy}\bar{F}(y)dy &= \int_t^\infty e^{-sy}\int_0^\infty \bar{F}_1(y-u)dF_2(u)dy,\\
&= \int_t^\infty \int_0^\infty e^{-sy}\bar{F}_1(y-u)dF_2(u)dy,\\
&= \int_0^\infty \int_t^\infty e^{-sy}\bar{F}_1(y-u)dF_2(u)dy,\\
&\geq \int_0^\infty \frac{e^{-st}}{s+1}\bar{F}_1(t-u)dF_2(u),\\
&\geq \frac{e^{-st}}{s+1}\int_0^\infty \bar{F}_1(t-u)dF_2(u),\\
&\geq \frac{e^{-st}}{s+1}\bar{F}(t).
\end{aligned} \tag{6}$$

The $\bar{F} = \overline{F_1 * F_2}$ is UBAC(2)L. This completes the proof.

Theorem 2. Let F_i, $i = 1,2,...,n$ be UBAC(2)L life distribution with $x \geq 0$ and $0 \leq \alpha_i \leq 1$ such that

$\Sigma_{i=1}^{n}\alpha_i = 1$, then $\bar{F}(x) = \sum_{i=1}^{n}\alpha_i\bar{F}_i(x)$ is UBAC(2)L.

Proof:
Since $\alpha_1 \leq \alpha_2 \leq ... \leq \alpha_n$. Let $F_1, F_2, ...F_n$ are UBAC(2)L, then we have

$$\int_t^\infty e^{-sy}[\alpha_i\bar{F}_i(y)]dy \geq \frac{e^{-st}}{s+1}[\alpha_i\bar{F}_i(t)]$$

For i=1 :

$$\int_t^\infty e^{-sy}[\alpha_1\bar{F}_1(y)]dy \geq \frac{e^{-st}}{s+1}[\alpha_1\bar{F}_1(t)]$$

For i=2 :

$$\int_t^\infty e^{-sy}[\alpha_2\bar{F}_2(y)]dy \geq \frac{e^{-st}}{s+1}[\alpha_2\bar{F}_2(t)]$$

For i=n :

$$\int_t^\infty e^{-sy}[\alpha_n\bar{F}_n(y)]dy \geq \frac{e^{-st}}{s+1}[\alpha_n\bar{F}_n(t)]$$

For all i :

$$\int_t^\infty e^{-sy}[\Sigma_{i=1}^{n}\alpha_i\bar{F}_i(y)]dy \geq \frac{e^{-st}}{s+1}[\Sigma_{i=1}^{n}\alpha_i\bar{F}_i(t)]$$

And from $F(x) = \Sigma_{i=1}^{n}\alpha_i F_i(x)$, we have

$$\int_t^\infty e^{-sy}\bar{F}(y)dy \geq \frac{e^{-st}}{s+1}\bar{F}(t),$$

This result holds for every finite n. This completes the proof of the theorem.

Theorem 3. A series system of n independent UBAC(2)L components is UBAC(2)L.

Proof:
Let $X_1, X_2, ..., X_n$ be independent UBAC(2)L components. Then

$$\int_t^\infty e^{-sy}\frac{P(\min(X_1,...,X_n \geq y))}{P(\min(X_1,...,X_n \geq t))}dy = \int_t^\infty e^{-sy}\prod_{i=1}^{n}\frac{P(X_i \geq y)}{P(X_i \geq t)}dy$$

$$= \int_t^\infty e^{-sy}\prod_{i=1}^{n}\frac{\bar{F}_i(y)}{\bar{F}_i(t))}dy \quad \text{since } \bar{F}_i \text{ is UBA}$$

$$\geq \int_t^\infty e^{-sy}\prod_{i=1}^{n}e^{-(y-t)}dy.$$

Since

$$\int_t^\infty e^{-sy}\prod_{i=1}^{n}e^{-(y-t)}dy = e^t\int_t^\infty e^{-(s+1)y}dy,$$

$$= \frac{1}{s+1}e^{-st},$$

then

$$\int_t^\infty e^{-sy}\frac{P(\min(X_1,...,X_n \geq y))}{P(\min(X_1,...,X_n \geq t))}dy \geq \frac{1}{s+1}e^{-st}$$

This implies that a series system $X_1, X_2, ..., X_n$ is UBAC(2)L.

4 Testing of hypotheses

In this section, a new test statistic based on goodness of fit for testing $H_0 : \overline{F}$ is standard exponential against $H_1 : \overline{F}$ is UBAC(2)L and isn't exponential is studied for a random sample $X_1, X_2, ..., X_n$ from a population with distribution function F is proposed.

According to Eq. (5) we proposed the following as a measure of departure from H_0

$$\begin{aligned}\delta_{U_{2L}}(s) &= \int_0^\infty [\int_t^\infty e^{-sy}\bar{F}(y)dy - \frac{1}{s+1}e^{-st}\bar{F}(t)]dF(t) \\ &= \int_0^\infty [\int_t^\infty e^{-sy}\bar{F}(y)dy - \frac{1}{s+1}e^{-st}\bar{F}(t)]e^{-t}dt \\ &= \int_0^\infty \int_t^\infty e^{-sy}e^{-t}\bar{F}(y)dy - \frac{1}{s+1}\int_0^\infty e^{-(s+1)t}\bar{F}(t)dt \\ &= \frac{1}{s}[1-\phi(s)] - \frac{s+2}{(s+1)^2}[1-\phi(s+1)] \qquad s>0, \quad (7)\end{aligned}$$

where $\phi(s) = Ee^{-sT}$.

The empirical estimator $\hat{\delta}_{U_{2L}}(s)$ of our test statistic can be given as follows:

$$\hat{\delta}_{U_{2L}}(s) = \frac{1}{n}\sum_i[\frac{1}{s}[1-e^{-sX_i}] - \frac{s+2}{(s+1)^2}[1-e^{-(s+1)X_i}]], \qquad s>0. \quad (8)$$

To make the test statistic scale invariant set

$$\hat{\Delta}_{U_{2L}}(s) = \frac{\hat{\delta}_{U_{2L}}(s)}{\overline{X}}. \quad (9)$$

Eq. (8) can be rewritten as follows;

$$\hat{\delta}_{U_{2L}}(s) = \frac{1}{n}\sum_i \wp(X_i), \quad (10)$$

where $\wp(X_i) = \frac{1}{s}[1-e^{-sX_i}] - \frac{s+2}{(s+1)^2}[1-e^{-(s+1)X_i}]$.

Set i = 1, then

$$\wp(X_1) = \frac{1}{s}[1-e^{-sX_1}] - \frac{s+2}{(s+1)^2}[1-e^{-(s+1)X_1}]. \quad (11)$$

Then $\hat{\delta}_{U_{2L}}(s)$ is a classical U-statistics, see Lee [24].

Theorem 4. As $n \to \infty$, $\sqrt{n}\langle\hat{\delta}_{U_{2L}}(s) - \delta_{U_{2L}}(s)\rangle/\sigma(s)$ is asymptotically normal with mean 0 and variance $\sigma^2(s) = var[\wp(X_1)]$, where $\wp(X_1)$ is given in Eq. (11).
Under H_0

$$\sigma^2(s) = \frac{1}{(s+1)^3(4s^4+8s+3)}. \quad (12)$$

Proof:
Using the theory of standard U-statistics and by direct calculations,we can find the mean equal 0 and the variance is given by

$$\sigma^2(s) = var[\wp(X_1)].$$

Then

$$\sigma^2(s) = E[\frac{1}{s}[1-e^{-sX_1}] - \frac{s+2}{(s+1)^2}[1-e^{-(s+1)X_1}]]^2. \quad (13)$$

Under H_0: $E(\wp(X_1)) = 0$ and

$$\sigma^2(s) = E[\wp(X_1)]^2,$$
$$= \int_0^\infty [\frac{1}{s}[1-e^{-sX_1}] - \frac{s+2}{(s+1)^2}[1-e^{-(s+1)X_1}]]^2 e^{-x} dx. \quad (14)$$

Then (12) is given and the theorem is proved.

when $s = 1 \rightarrow \sigma^2(1) = \frac{1}{120}$.

$$\hat{\Delta}_{U_{2L}}(1) = \frac{1}{n\bar{X}}[\frac{1}{2}[1-e^{-X_1}] - \frac{3}{8}[1-e^{-2X_1}]]. \quad (15)$$

To use the above test, calculate $\sqrt{n}\hat{\Delta}_{U_{2L}}(s)/\sigma(s)$ and reject H_0 if this exceeds the normal variate value $Z_{1-\alpha}$. To illustrate the test, we calculate, by using Monte Carlo Method, the empirical critical values of $\hat{\Delta}_{U_{2L}}(1)$ in (15) for sample sizes 5(5)100. Table 1 gives the percentile points for 1%, 5%, 10%, 90%, 95%, 99%. The calculations are based on 10000 simulated samples of sizes n = 5(5)100. These values will be the criteria for dividing the samples space into acceptance or rejection region for H_0.

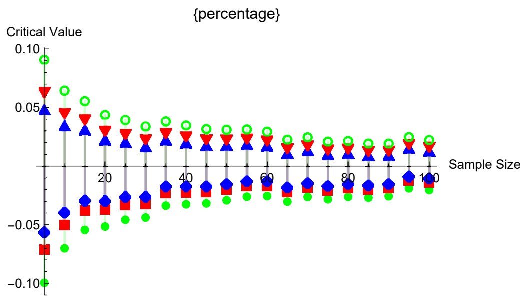

Fig. 1: The relation between sample space and critical values

It is clear from Table 1 and Fig. 1 that the values of the percentiles changes slowly as n increases.

5 Pitman's asymptotic efficiency

In this section, we calculate PAE for UBAC(2)L class of life distributions and compare our proposed test with tests of other well-known classes of life distribution on basis of PAE.

Here we choose K^* presented by Hollander and Prochan [25] for (DMRL), $\hat{\delta}_2$ presented by Ahmad [21] for (UBAE) class, $\hat{\Delta}_{U_T}$ presented by Abu-Youssef et al [26] for (UBACT) class of life distribution based on U-Statistics, Λ_n introduced by Mahmoud et al [27] for (ODL) and $\delta(\theta)$ introduced by Ali [13] for UBAC(2).

PAE of $\delta_{U_{2L}}(s)$ is given by :

$$PAE(\delta_{U_{2L}}(s,\theta)) = \frac{1}{\sigma(s)}\left|\frac{d}{d\theta}\delta_{U_{2L}}(s,\theta)|_{\theta\rightarrow\theta_0}\right|. \quad (16)$$

Two of the most commonly used alternatives (cf. Hollander and Proschan [28]) are:

(i) Linear Failure Rate (LFR) : $\bar{F}_\theta = e^{-x-\frac{\theta x^2}{2}}$, $\quad x > 0, \theta > 0$
(ii) Makeham : $\bar{F}_\theta = e^{-x-\theta(x+e^{-x}-1)}$, $\quad x > 0, \theta > 0$

The null hypothesis is at $\theta = 0$ for LFR and Makham families. The PAE's of these alternatives of our procedure are, respectively:

$$PAE(\delta_{U_{2L}}(s), LFR) = \frac{1}{\sigma_0}\left|\frac{-3-2s}{(s+1)^3(s+2)^2}\right|, \quad s \geq 0. \quad (17)$$

$$PAE(\delta_{U_{2L}}(s), Makeham) = \frac{1}{\sigma_0}\left|\frac{-2}{(s+1)^2(s+2)(s+3)}\right|, \quad s \geq 0. \quad (18)$$

From Table 2, our test statistic $\delta_{U_{2L}}(s)$ is more efficient than K^*, $\hat{\delta}_2$, $\hat{\Delta}_{U_T}$, Λ_n and $\delta(\theta)$ for LFR and Makeham families.

Finally, the power of the test statistic $\delta_{U_{2L}}(s)$ is considered for 95% percentiles in Table 3 for three of the most commonly used alternatives [see Hollander and Proschan [28]], they are

(i)	Linear failure rate	: $\bar{F}_\theta = e^{-x-\frac{\theta x^2}{2}}$,	$x > 0, \theta > 0$
(ii)	Makeham	: $\bar{F}_\theta = e^{-x-\theta(x+e^{-x}-1)}$,	$x \geq 0, \theta > 0$
(iii)	Weibull	: $\bar{F}_\theta = e^{-x^\theta}$,	$x \geq 0, \theta > 0$

These distributions are reduced to exponential distribution for appropriate values of θ.

It is clear from Table 3 that the power increases as n and θ increase.

6 Testing for Censored Data

In this section, a test statistic is proposed to test $H_0 : \overline{F}$ is standard exponential distribution versus, $H_1 : \overline{F}$ is $UBAC(2)L$ and not exponential distribution, with randomly right-censored data (RR-C)in many experiments. Censored data is usually the only information available in a life-testing model or in a clinical study where patients may be lost (censored) before the completion of a study. This experimental situation can be modeled as following:

Suppose n items are put on test, and $X_1, X_2, ..., X_n$ are independent and identically distributed (i.i.d) random variables according to a continuous life distribution F which denote their true life time. Let $Y_1, Y_2, ..., Y_n$ be (i.i.d) according to a continuous life distribution G and assume that $X's$ and $Y's$ are independent. In the randomly right-censored model, we observe the pairs (Z_i, δ_i), $i = 1, ..., n$, where $Z_i = \min(X_i, Y_i)$ and

$$\delta_i = \begin{cases} 1, & if\ Z_i = X_i \quad (i\text{-}th\ observation\ is\ uncensored) \\ 0, & if\ Z_i = Y_i \quad (i\text{-}th\ observation\ is\ censored) \end{cases} \quad (19)$$

Let $Z_{(0)} < Z_{(1)} < ... < Z_{(n)}$ denoted the ordered of Z's and δ_i is the δ corresponding to $Z_{(i)}$, respectively. Using the Kaplan and Meier estimator [29] in the case of censored data (Z_i, δ_i), $i = 1, 2, ..., n$, then the proposed test statistic is given by (7) can be written using right censored data as

$$\hat{\delta}^{c}_{U_{2L}}(s) = \frac{1}{s}(1-\zeta) - \frac{s+2}{(s+1)^2}(1-\beta) \qquad (20)$$

where

$$\zeta = \sum_{j=1}^{n} e^{-sZ_{(j)}}[\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}]$$

$$\beta = \sum_{j=1}^{n} e^{-(s+1)Z_{(j)}}[\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}]$$

and

$$dF_n(Z_j) = \bar{F}_n(Z_{j-1}) - \bar{F}_n(Z_{j-2})$$

$$c_k = \frac{n-k}{n-k+1}$$

To make the test invariant, let

$$\hat{\Delta}^{c}_{U_{2L}}(s) = \frac{\hat{\delta}^{c}_{U_{2L}}}{\bar{Z}}, \bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i. \qquad (21)$$

Table 4 shows the critical values percentiles of $\hat{\Delta}^{c}_{U_{2L}}(s)$ for sample size n=2(2)20(10)100.

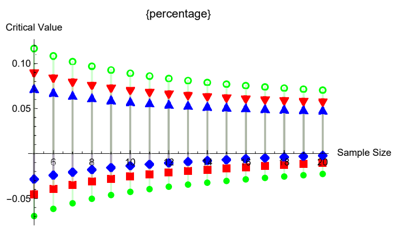

Fig. 2: The relation between sample space and critical values

According to Table 4 and Fig. 2, the critical values decreases when the sample size increases. These values will be the criteria for dividing the samples space into acceptance or rejection region for H_0.

7 Applying the test

7.1 Applications for Complete Data

Example 1: The following data represent 39 liver cancers patients taken from El Minia Cancer Center Ministry of Health Egypt A. F. Attia [30] The ordered life times (in days) are:

107 , 18 , 74 , 20 , 23 , 20 , 23 , 24 , 52 , 105 , 60 , 31 , 75 , 107 , 71 , 107 , 14 , 49 , 10 , 15 , 30 , 26 , 14 , 87 , 51 , 17 , 116 , 67 , 20 , 14 , 40 , 14 , 30 , 96 , 20 , 20 , 61 , 150 , 14.

Using (15), the value of test statistic based on the above data is $\hat{\Delta}_{U_{2L}}(1) = 0.0419031$. the critical value at $\alpha = 0.05$ is 0.024426, then we reject H_0 at $\alpha = 0.05$.Therefore the data have UBAC(2)L property.

Example 2: Consider the data in Abouammoh et al [31]. These data represent set of 40 patients suffering from blood cancer (Leukemia) from one ministry of health hospital in Saudi Arabia. The ordered life times (in day) are:
0.315, 0.496, 0.699, 1.145, 1.208, 1.263, 1.414, 2.025, 2.036, 2.162, 2.211, 2.370, 2.532, 2.693, 2.805, 2.910, 2.912, 3.192, 3.263, 3.348, 3.348, 3.427, 3.499, 3.534, 3.718, 3.751, 3.858, 3.986, 4.049, 4.244, 4.323, 4.323, 4.381, 4.392, 4.397, 4.647, 4.753, 4.929, 4.973, 5.074.

The value of test statistic based on the above data is $\hat{\Delta}_{U_{2L}}(1) = 0.0779788$. the critical value at $\alpha = 0.05$ is 0.0242422. This value leads to the rejecting of H_0 at the significance level $\alpha = 0.05$. Therefore the data have UBAC(2)L property.

Example 3: In an experiment at Florida state university to study the effect of methyl mercury poisoning on the life lengths of fish goldfish were subjected to various dosages of methyl mercury (Kochar [32]). At one dosage level the ordered times to death in week are:
6, 6.143, 7.286, 8.714, 9.429, 9.857, 10.143, 11.571, 11.714, 11.714

The value of test statistics, based on the above data is $\hat{\Delta}_{U_{2L}}(1) = 0.0688283$. the critical value at $\alpha = 0.05$ is 0.0461727. Then H_0 at the significance level $\alpha = 0.05$ is rejected. Therefore the data have UBAC(2)L property.

7.2 Applications for Censored Data

Example 1: Consider the following data in Mahmoud and Abdul Alim [33] that represent 51 liver cancers patients taken from the El Minia Cancer Center Ministry of Health in Egypt. Of them 39 represent whole life times (non-censored data) and the others represent censored data. The ordered lifetimes (in days) are:

(i) Non-censored data

10, 14, 14, 14, 14, 14, 15, 17, 18, 20, 20, 20, 20, 20, 23, 23, 24, 26, 30, 30, 31, 40, 49, 51, 52, 60, 61, 67, 71, 74, 75, 87, 96, 105, 107, 107, 107, 116, 150.

(ii) Censored data

30, 30, 30, 30, 30, 60, 150, 150, 150, 150, 150, 185.

It is found that the test statistic for the set of data $\hat{\Delta}^{c}_{U_{2L}}(1) = 0.0251201$. The critical value is 0.0395898, so we accept H_0 which states that the set of data haven't UBAC(2)L property under significant level $\alpha = 0.5$.

Table 1: Critical values of $\hat{\Delta}_{U_{2L}}(s)$

n	1%	5%	10%	90%	95%	99%
5	-0.0952711	-0.0673061	-0.052405	0.0521065	0.0670076	0.0949726
10	-0.0685757	-0.0488014	-0.0382648	0.0356361	0.0461727	0.065947
15	-0.0568782	-0.0407326	-0.0321295	0.0282103	0.0368135	0.0529591
20	-0.0457499	-0.0317674	-0.0243168	0.0279389	0.0353895	0.049372
25	-0.0427624	-0.0302561	-0.0235921	0.0231469	0.0298109	0.0423172
30	-0.040619	-0.0292023	-0.023119	0.0195477	0.025631	0.0370477
35	-0.0353131	-0.0247433	-0.0191113	0.0203904	0.0260225	0.0365922
40	-0.033132	-0.0232449	-0.0179765	0.0189739	0.0242422	0.0341293
45	-0.0319505	-0.0226289	-0.0176618	0.0171753	0.0221424	0.031464
50	-0.0297344	-0.0208911	-0.016179	0.0168705	0.0215826	0.0304259
55	-0.0281054	-0.0196737	-0.0151808	0.0163306	0.0208234	0.0292552
60	-0.0274241	-0.0193513	-0.0150497	0.0151201	0.0194217	0.0274945
65	-0.0274398	-0.0196837	-0.0155509	0.0134354	0.0175682	0.0253243
70	-0.0267357	-0.0192617	-0.0152792	0.0126527	0.0166351	0.0241091
75	-0.0249314	-0.0177108	-0.0138634	0.0131214	0.0169688	0.0241894
80	-0.0265205	-0.0195293	-0.015804	0.0103239	0.0140491	0.0210404
85	-0.0244289	-0.0176464	-0.0140323	0.0113155	0.0149295	0.021712
90	-0.0234292	-0.0168378	-0.0133256	0.011308	0.0148202	0.0214116
95	-0.0222067	-0.0157911	-0.0123725	0.011604	0.0150226	0.0214382
100	-0.0206546	-0.0144014	-0.0110695	0.0123	0.015632	0.0218852

Table 2: PAE of $\delta_{U_{2L}}(s)$

Dist.	K^*	$\hat{\delta}_2$	$\hat{\Delta}_{U_T}$	Λ_n	$\delta(\theta)$	$\delta_{U_{2L}}(s)$
F_1 LFR	0.81	0.63	0.748	0.982	0.918	1.3
F_2 Makeham	0.29	0.385	0.248	0.218	0.510	0.58

Table 3: Power Estimate of $\delta_{U_{2L}}(s)$

		Sample Size		
Distribution	θ	n=10	n=20	n=30
F_1 Linear failure rate	2	1	1	1
	3	1	1	1
	4	1	1	1
F_2 Makham	2	1	1	1
	3	1	1	1
	4	1	1	1
F_3 Weibull	2	0.9924	0.9971	1
	3	0.9952	0.9986	1
	4	0.9955	1	1

Table 4: Critical Values of $\hat{\Delta}^c_{U_{2L}}(s)$

n	90%	95%	99%
2	0.104148	0.127709	0.171925
4	0.0799974	0.0966573	0.127923
6	0.069334	0.0829367	0.108465
8	0.0630111	0.0747914	0.0968997
10	0.0587317	0.0692684	0.0890426
12	0.0556135	0.0652321	0.0832835
14	0.0532404	0.0621455	0.0788578
16	0.0513953	0.0597253	0.0753582
18	0.049969	0.0578225	0.0725614
20	0.0489429	0.0563934	0.0703759
30	0.0421667	0.04825	0.0596667
40	0.0393085	0.0445769	0.054464
50	0.0346142	0.0393263	0.0481697
60	0.034869	0.0391706	0.0472434
70	0.0341083	0.0380908	0.0455647
80	0.0333767	0.0371019	0.0440932
90	0.0327311	0.0362433	0.0428347
100	0.0321669	0.0354988	0.041752

8 Conclusion

In this paper, a new class of life distribution denoted by UBAC(2)L is introduced. We studied the UBAC(2)L property under convolution, discrete mixture and formation of a coherent system. A new test statistic technique for testing exponentiality versus UBAC(2)L class of life distribution based on the goodness of fit test is proposed. Pitman's asymptotic efficiencies of our proposed test are calculated for LFR and makeham families. It is proved that our test is more efficient than other tests. Critical values of this test are tabulated for complete and censored data and the powers of this test are estimated for some famously alternative distributions in reliability such as LFR and makeham. Finally, examples in different areas are used as practical applications of our proposed test.

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Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

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