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Geometrically Relative Convex Functions

*Muhammad Aslam Noor**, *Khalida Inayat Noor and Muhammad Uzair Awan*

Mathematics Department, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan

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Abstract: In this paper, some new concepts of geometrically relative convex sets and relative convex functions are defined. These new classes of geometrically relative convex functions unify several known and new classes of relative convex functions such as exponential convex functions. New Hermite-Hadamard type integral inequalities are derived for these new classes of geometrically relative convex functions and their variant forms. Some special cases, which can be obtained from our results, are discussed. Results proved in this paper represent significant improvements of the previously known results. We would like to emphasize that the results obtained and discussed in this paper may stimulate novel, innovative and potential applications of the geometrically relative convex functions in other fields.

Keywords: Convex functions, geometrically relative convex functions, Hermite-Hadamard inequality.
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1 Introduction

Recently convexity has seen a dramatic increase in its applications for solving a large number of problems which arise in various branches of pure and applied sciences. As a result of these activities, the concept of convexity has been extended and generalized in various directions using novel and innovative ideas see [1,6,7,9,10,11,12,13,14,15,16,18,19,22,23]. An important and significant generalization of the convex functions is the introduction of relative convex functions by Youness [6]. These relative convex functions plays an interesting role in optimization theory, since they provide a broader setting to study the optimization and programming problems. It is well known [5,6] that the relative convex sets and relative convex functions are nonconvex sets and nonconvex functions respectively. However it has been shown that the relative convex functions preserve some nice properties that the convex functions have. It has been shown by Noor [11] that the minimum of a differentiable relative convex functions on the relative convex set can be characterized by a class of variational inequalities, which are known as general variational inequalities. This shows that the concept of relative convexity plays the same role for general variational inequalities as classical convexity plays for variational inequalities. For the applications and other aspects of the relative convexity, see [10,11,12] and the references therein.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function with $a < b$ and $a,b \in I$. Then the following double inequality is known as Hermite-Hadamard inequality in the literature.

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}.$$

In recent years, much attention has been given to derive the Hermite-Hadamard type inequalities for various types of convex functions, see [1,2,4,17,19,20,21,22,23,24]. Motivated and inspired by the recent research going on in this field, we introduce and study a new class of relative convex functions, which is called the geometrically relative convex functions. We derive several Hermite-Hadamard type integral inequalities for these new geometrically relative convex functions. Several special cases are also discussed. The ideas and techniques of this paper may stimulate further research in this interesting field.

2 Preliminaries

In this section, we recall some known concepts and define the class of geometrically (GG) relative convex functions

* Corresponding author e-mail: noormaslam@hotmail.com

and *GA* relative convex functions. First of all let $\mathbb{R}^n$ be the finite dimensional space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $||.||$, respectively.

Definition 1. *Let $\mathscr{G} \subseteq (0,\infty)$. Then $\mathscr{G}$ is said to be geometrically relative convex set, if there exists an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$(g(x))^t (g(y))^{1-t} \in \mathscr{G}, \quad \forall g(x), g(y) \in \mathscr{G}, t \in [0,1].$$

Using $AM - GM$ inequality, we have

$$\begin{aligned} g(x), g(y) \in \mathscr{G}, t \in [0,1] \ &\Rightarrow (g(x))^t (g(y))^{1-t} \\ &\leq tg(x) + (1-t)g(y). \end{aligned}$$

Definition 2([6,19]). *A set $M_g \subseteq \mathbb{R}^n$ is said to be a relative convex (g-convex) set, if there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that,*

$$tg(x) + (1-t)g(y) \in M_g, \\ \forall x, y \in \mathbb{R}^n : g(x), g(y) \in M_g, t \in [0,1]. \quad (1)$$

Recently it has been shown in [5], that if M_g is a relative convex set then it is possible that it may not be a classical convex set.

Definition 3. *A function $f : \mathscr{G} \to \mathbb{R}$ (on subintervals of $(0,\infty)$) is said to be geometrically relative convex function (GG-relative convex function) if there exists an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that,*

$$f((g(x))^t (g(y))^{1-t}) \leq (f(g(x)))^t (f(g(y)))^{1-t}, \\ \forall g(x), g(y) \in \mathscr{G}, t \in [0,1]. \quad (2)$$

From (2), it follows that

$$\log f((g(x))^t (g(y))^{1-t}) \\ \leq t \log f(g(x)) + (1-t) \log f(g(y)), \\ \forall g(x), g(y) \in \mathscr{G}, t \in [0,1].$$

Using $AM - GM$ inequality, we have

$$\begin{aligned} f((g(x))^t (g(y))^{1-t}) \ &\leq (f(g(x)))^t (f(g(y)))^{1-t} \\ &\leq t f(g(x)) + (1-t) f(g(y)). \end{aligned}$$

This implies that every geometrically relative convex function (that is *GG*-relative convex function) is also *GA*-relative convex function, but the converse is not true.

Definition 4. *A function $f : \mathscr{G} \to \mathbb{R}$ (on subintervals of $(0,\infty)$) is said to be GA-relative convex function) if there exists an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that,*

$$f((g(x))^t (g(y))^{1-t}) \leq t f(g(x)) + (1-t) f(g(y)), \\ \forall g(x), g(y) \in \mathscr{G}, t \in [0,1]. \quad (3)$$

We note that for $f(x) = e^x$ Definition 4 reduces to one in [8]. That is

$$f(e^{tx+(1-t)y}) \leq t f(e^x) + (1-t) f(e^y), \\ \forall g(x), g(y) \in \mathscr{G}, t \in [0,1]. \quad (4)$$

From Definition 3 and Definition 4, it follows that $GG \Longrightarrow GA$, but the converse is not true.

Again using the $AM - GM$ inequality from Definition 3, we have the following known concept of relative convex functions.

Definition 5([6,19]). *A function f is said to be a relative convex (g-convex) function (that is AA relative convex function) on a relative convex (g-convex) set M_g, if and only if, there exists a function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that,*

$$f((1-t)g(x) + tg(y)) \leq (1-t) f(g(x)) + t f(g(y)), \\ \forall x, y \in \mathbb{R}^n : g(x), g(y) \in M_g, t \in [0,1]. \quad (5)$$

It is known [6] that every convex function f on a convex set is a relative convex function, but the converse is not true. There are functions which are relative convex function but may not be a convex function in the classical sense.
Noor [11] proved the optimality condition for the differentiable relative convex functions on relative convex set can be characterized by a class of variational inequality which is called as general variational inequality, for the applications and other aspects of general variational inequalities, see [9,10,11,12].
Now we define the concept of relative log convex functions.

Definition 6. *A function $f : M_g \to \mathbb{R}$ (on subintervals of $(0,\infty)$) is said to be relative* log *convex function (that is AG-relative convex function) if there exists an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that,*

$$f(tg(x) + (1-t)g(y)) \leq (f(g(x)))^t (f(g(y)))^{1-t}, \\ \forall g(x), g(y) \in M_g, t \in [0,1]. \quad (6)$$

From (6) it follows that

$$\log f(tg(x) + (1-t)g(y)) \\ \leq t \log f(g(x)) + (1-t) \log f(g(y)), \\ \forall g(x), g(y) \in M_g, t \in [0,1].$$

Definition 7. *A function $f : M_g \to \mathbb{R}$ (on subintervals of $(0,\infty)$) is said to be relative geometrically quasi-convex function if there exists an arbitrary function $g : \mathbb{R}^n \to \mathbb{R}^n$ such that,*

$$f((g(x))^t (g(y))^{1-t}) \leq \max\{f(g(x)), f(g(y))\}, \\ \forall g(x), g(y) \in M_g, t \in [0,1]. \quad (7)$$

Next we define the concept of geometrically relative convex functions on an interval.

Definition 8. *Let I be a subinterval of $(0,\infty)$. Then f is geometrically relative convex function, if and only if,*

$$\begin{vmatrix} 1 & 1 & 1 \\ \log g(a) & \log g(x) & \log g(b) \\ \log f(g(a)) & \log f(g(x)) & \log f(g(b)) \end{vmatrix} \geq 0,$$

where $g(a) \leq g(x) \leq g(a)$.

One can easily show that the following are equivalent:

1. f is geometrically relative semi-convex function on relative convex set.
2. $f(g(a))^{\log(g(b))} f(g(x))^{\log(g(a))} f(g(b))^{\log(g(x))} \geq f(g(a))^{\log(g(x))} f(g(x))^{\log(g(b))} f(g(b))^{\log(g(a))}$.

where $g(x) = g(a)^t g(b)^{1-t}$ and $t \in [0,1]$.

For $g(x) = x$ the Definition 8 reduces to the definition for geometrically convex functions see [3].

3 Main Results

In this section we discuss our main results. For this purpose we need following lemmas which play a key part in proving our main results.
Essentially using the technique of [21] one can prove the following result.

Lemma 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I) and $g : \mathbb{R} \to \mathbb{R}$ be arbitrary function. If $f' \in \mathscr{L}[g(a),g(b)]$ for $g(a),g(b) \in I$ with $g(a) < g(b)$. Then*

$$\frac{f(g(a))+f(g(b))}{2} - \frac{1}{g(b)-g(a)} \int\limits_{g(a)}^{g(b)} f(g(x))dg(x)$$

$$= \frac{g(b)-g(a)}{2} \int\limits_0^1 (1-2t) f'(tg(a)+(1-t)g(b))dt.$$

Lemma 2([17,24]). *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I) and $g : \mathbb{R} \to \mathbb{R}$ be arbitrary function. If $f'' \in \mathscr{L}[g(a),g(b)]$ for $g(a),g(b) \in I$ with $g(a) < g(b)$. Then*

$$\frac{1}{g(b)-g(a)} \int\limits_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)$$

$$= (g(b)-g(a)) \left[\int\limits_0^1 \mu(t) f'(tg(a)+(1-t)g(b))dt \right],$$

where

$$\mu(t) = \begin{cases} t, & [0,\frac{1}{2}), \\ t-1, & [\frac{1}{2},1]. \end{cases}$$

Using the technique of [20], one cane prove the following lemma.

Lemma 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° (the interior of I) and $g : \mathbb{R} \to \mathbb{R}$ be arbitrary function. If $f' \in \mathscr{L}[g(a),g(b)]$ for $g(a),g(b) \in I$ with $g(a) < g(b)$. Then*

$$\frac{f(g(a))+f(g(b))}{2} - \frac{1}{g(b)-g(a)} \int\limits_{g(a)}^{g(b)} f(g(x))dg(x)$$

$$= \frac{[g(b)-g(a)]^2}{2} \int\limits_0^1 t(1-t) f''(tg(a)+(1-t)g(b))dt.$$

Using the technique of [23] one can prove the following lemma.

Lemma 4. *Let f be a differentiable function on $(g(a),g(b))$ with $g(a) < g(b)$ where $g : \mathbb{R}^n \to \mathbb{R}^n$ is any arbitrary function. If $f \in \mathscr{L}[g(a),g(b)]$. Then the following identity holds:*

$$[g(b)f(g(b)) - g(a)f(g(a))] - \int\limits_{g(a)}^{g(b)} f(g(x))dg(x)$$

$$= (\ln g(b) - \ln g(a)) \int\limits_0^1 (g(a))^{2t}(g(b))^{2(1-t)} \times f'((g(a))^t(g(b))^{1-t})dt.$$

Now we are in a position to derive our main results. First of all, we prove the results for the class of geometrically relative convex functions (*GG*).

Theorem 1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I) and $g : \mathbb{R} \to \mathbb{R}$ be arbitrary function. Also $f' \in \mathscr{L}[g(a),g(b)]$ for $g(a),g(b) \in I$ with $g(a) < g(b)$. If $|f'|$ is decreasing and geometrically relative convex function. Then*

$$\left| \frac{f(g(a))+f(g(b))}{2} - \frac{1}{g(b)-g(a)} \int\limits_{g(a)}^{g(b)} f(g(x))dg(x) \right|$$

$$\leq \frac{g(b)-g(a)}{2} |f'(g(b))| \Psi(w),$$

where $w = \frac{|f'(g(a))|}{|f'(g(b))|}$ and $\Psi(w) = \frac{w\ln(w)+4\sqrt{w}-2w-\ln(w)-2}{\ln(w)^2}$.

Proof. Using Lemma 1 and the fact that $|f'|$ is geometrically relative convex function, we have

$$\left| \frac{f(g(a))+f(g(b))}{2} - \frac{1}{g(b)-g(a)} \int\limits_{g(a)}^{g(b)} f(g(x))dg(x) \right|$$

$$\leq \frac{g(b)-g(a)}{2} \int\limits_0^1 |1-2t| |f'(tg(a)+(1-t)g(b))| dt$$

$$\leq \frac{g(b)-g(a)}{2} \int\limits_0^1 |1-2t| \left| f'\left(g(a)^t g(b)^{1-t}\right) \right| dt$$

$$\leq \frac{g(b)-g(a)}{2}|f'(g(b))|\int_0^1 |1-2t|\left(\frac{|f'(g(a))|}{|f'(g(b))|}\right)^t dt$$

$$= \frac{g(b)-g(a)}{2}|f'(g(b))|\int_0^1 |1-2t|w^t dt$$

$$= \frac{g(b)-g(a)}{2}|f'(g(b))|\frac{w\ln(w)+4\sqrt{w}-2w-\ln(w)-2}{\ln(w)^2}.$$

This completes the proof. □

Theorem 2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I) and $g : \mathbb{R} \to \mathbb{R}$ be arbitrary function. Also $f' \in \mathscr{L}[g(a),g(b)]$ for $g(a),g(b) \in I$ with $g(a) < g(b)$. If $|f'|^q$ is decreasing and geometrically relative convex function for $p,q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$. Then*

$$\left|\frac{f(g(a))+f(g(b))}{2} - \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|$$

$$\leq \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{w^q-1}{q\ln(w)}\right)^{\frac{1}{q}},$$

where $w = \frac{|f'(g(a))|}{|f'(g(b))|}$.

Proof. Using Lemma 1, well known Holder's inequality and the fact that $|f'|^q$ is geometrically relative convex function, we have

$$\left|\frac{f(g(a))+f(g(b))}{2} - \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|$$

$$\leq \frac{g(b)-g(a)}{2}\int_0^1 |1-2t||f'(tg(a)+(1-t)g(b))|dt$$

$$\leq \frac{g(b)-g(a)}{2}\left(\int_0^1 |1-2t|^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left|f'\left(g(a)^t g(b)^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left(\frac{|f'(g(a))|}{|f'(g(b))|}\right)^{qt} dt\right)^{\frac{1}{q}}$$

$$= \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_0^1 w^{qt}dt\right)^{\frac{1}{q}}$$

$$= \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{w^q-1}{q\ln(w)}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 3. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I) and $g : \mathbb{R} \to \mathbb{R}$ be arbitrary function. Also $f' \in \mathscr{L}[g(a),g(b)]$ for $g(a),g(b) \in I$ with $g(a) < g(b)$. If $|f'|^q$ is decreasing and geometrically relative convex function for $q > 1$. Then*

$$\left|\frac{f(g(a))+f(g(b))}{2} - \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|$$

$$\leq \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{1}{2}\right)^{\frac{1}{p}}(\Psi(w))^{\frac{1}{q}},$$

where $w = \frac{|f'(g(a))|}{|f'(g(b))|}$ *and*

$\Psi(w) = \frac{w^q\ln(w)q-2w^q+4w^{\frac{1}{2}q}-\ln(w)q-2}{\ln(w)^2q^2}$.

Proof. Using Lemma 1, well known Power mean inequality and the fact that $|f'|^q$ is geometrically relative convex function, we have

$$\left|\frac{f(g(a))+f(g(b))}{2} - \frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|$$

$$\leq \frac{g(b)-g(a)}{2}\int_0^1 |1-2t||f'(tg(a)+(1-t)g(b))|dt$$

$$\leq \frac{g(b)-g(a)}{2}\left(\int_0^1 |1-2t|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 |1-2t|\left|f'\left(g(a)^t g(b)^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{1}{2}\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 |1-2t|\left(\frac{|f'(g(a))|}{|f'(g(b))|}\right)^{qt} dt\right)^{\frac{1}{q}}$$

$$= \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{1}{2}\right)^{\frac{1}{p}}\left(\int_0^1 |1-2t|w^{qt}dt\right)^{\frac{1}{q}}$$

$$= \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{1}{2}\right)^{\frac{1}{p}}$$

$$\times\left(\frac{w^q\ln(w)q-2w^q+4w^{\frac{1}{2}q}-\ln(w)q-2}{\ln(w)^2q^2}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Remark. For $q = 1$ Theorem 3 reduces to Theorem 1.

Theorem 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I° (the interior of I) and $g : \mathbb{R} \to \mathbb{R}$ be arbitrary function. Also $f' \in \mathscr{L}[g(a),g(b)]$ for $g(a),g(b) \in I$ with $g(a) < g(b)$. If $|f'|$ is decreasing and geometrically relative convex function. Then*

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right|$$

$$\leq (g(b)-g(a))|f'(b)|[\Psi_1(w)+\Psi_2(w)],$$

where $w = \frac{|f'(g(a))|}{|f'(g(b))|}$, $\Psi_1(w) = \frac{-w^{\frac{1}{2}}+\frac{1}{2}w^{\frac{1}{2}}\ln(w)+1}{\ln(w)^2}$ *and* $\Psi_3(w) = \frac{w-\frac{1}{2}w^{\frac{1}{2}}\ln(w)-w^{\frac{1}{2}}}{\ln(w)^2}$.

Proof. Using Lemma 2 and the fact that $|f'|$ is geometrically relative convex function, we have

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$= \left|(g(b)-g(a))\left[\int_0^1 \mu(t) f'(tg(a)+(1-t)g(b))dt\right]\right|$$
$$\leq (g(b)-g(a))\left[\int_0^{\frac{1}{2}} |t||f'((g(a))^t(g(b))^{1-t})|dt + \int_{\frac{1}{2}}^1 |(t-1)||f'((g(a))^t(g(b))^{1-t})|dt\right]$$
$$\leq (g(b)-g(a))|f'(b)| \times \left[\int_0^{\frac{1}{2}} \left\{t\left(\frac{|f'(a)|}{|f'(b)|}\right)^t\right\}dt + \int_{\frac{1}{2}}^1 \left\{(1-t)\left(\frac{|f'(a)|}{|f'(b)|}\right)^t\right\}dt\right]$$
$$\leq (g(b)-g(a))|f'(b)|\left[\int_0^{\frac{1}{2}} tw^t dt + \int_{\frac{1}{2}}^1 (1-t)w^t dt\right]$$
$$= (g(b)-g(a))|f'(b)| \times \left[\frac{-w^{\frac{1}{2}}+\frac{1}{2}w^{\frac{1}{2}}\ln(w)+1}{\ln(w)^2} + \frac{w-\frac{1}{2}w^{\frac{1}{2}}\ln(w)-w^{\frac{1}{2}}}{\ln(w)^2}\right].$$

This completes the proof. □

Theorem 5. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *(the interior of I) and* $g : \mathbb{R} \to \mathbb{R}$ *be arbitrary function. Also* $f' \in \mathscr{L}[g(a),g(b)]$ *for* $g(a),g(b) \in I$ *with* $g(a) < g(b)$. *If* $|f'|^q$ *is decreasing and geometrically relative convex function for* $p,q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$. *Then*

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$\leq (g(b)-g(a))|f'(b)|\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}} \times[(\Psi_1(w))^{\frac{1}{q}}+(\Psi_2(w))^{\frac{1}{q}}],$$

where $w = \frac{|f'(g(a))|}{|f'(g(b))|}$, $\Psi_1 = \frac{w^{\frac{1}{2}q}-1}{q\ln(w)}$ *and* $\Psi_2(w) = \frac{w^q-w^{\frac{1}{2}q}}{q\ln(w)}$.

Proof. Using Lemma 2, well known Holder's inequality and the fact that $|f'|^q$ is geometrically relative convex function, we have

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$= \left|(g(b)-g(a))\left[\int_0^1 \mu(t) f'(tg(a)+(1-t)g(b))dt\right]\right|$$
$$\leq (g(b)-g(a))\left[\int_0^{\frac{1}{2}} |t||f'((g(a))^t(g(b))^{1-t})|dt + \int_{\frac{1}{2}}^1 |(t-1)||f'((g(a))^t(g(b))^{1-t})|dt\right]$$
$$\leq (g(b)-g(a))\left[\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}} \times\left(\int_0^{\frac{1}{2}} |f'((g(a))^t(g(b))^{1-t})|^q dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 |t-1|^p dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 |f'((g(a))^t(g(b))^{1-t})|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leq (g(b)-g(a))|f'(b)|\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}} \times\left[\left(\int_0^{\frac{1}{2}} w^{qt}dt\right)^{\frac{1}{q}} + \left(\int_{\frac{1}{2}}^1 w^{qt}dt\right)^{\frac{1}{q}}\right]$$
$$= (g(b)-g(a))|f'(b)|\left(\frac{1}{(p+1)2^{p+1}}\right)^{\frac{1}{p}} \times\left[\left(\frac{w^{\frac{1}{2}q}-1}{q\ln(w)}\right)^{\frac{1}{q}} + \left(\frac{w^q-w^{\frac{1}{2}q}}{q\ln(w)}\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Theorem 6. *Let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable function on* I° *(the interior of I) and* $g : \mathbb{R} \to \mathbb{R}$ *be arbitrary function. Also* $f' \in \mathscr{L}[g(a),g(b)]$ *for* $g(a),g(b) \in I$ *with* $g(a) < g(b)$. *If* $|f'|^q$ *is decreasing and geometrically relative convex function for* $q > 1$. *Then*

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x) - f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$\leq (g(b)-g(a))|f'(b)|\left(\frac{1}{8}\right)^{1-\frac{1}{q}} \times\left[\left(\frac{-w^{\frac{1}{2}q}+\frac{1}{2}w^{\frac{1}{2}q}q\ln(w)+1}{q^2\ln(w)^2}\right)^{\frac{1}{q}} + \left(\frac{w^q-\frac{1}{2}w^{\frac{1}{2}q}q\ln(w)-w^{\frac{1}{2}q}}{q^2\ln(w)^2}\right)^{\frac{1}{q}}\right],$$

where $w = \frac{|f'(g(a))|}{|f'(g(b))|}$.

Proof. Using Lemma 2, well known Power mean inequality and the fact that $|f'|^q$ is geometrically relative convex function, we have

$$\left|\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)-f\left(\frac{g(a)+g(b)}{2}\right)\right|$$
$$=\left|(g(b)-g(a))\left[\int_0^1 \mu(t)f'(tg(a)+(1-t)g(b))dt\right]\right|$$
$$\leq (g(b)-g(a))\left[\int_0^{\frac{1}{2}}|t||f'((g(a))^t(g(b))^{1-t})|dt\right.$$
$$\left.+\int_{\frac{1}{2}}^1 |(t-1)||f'((g(a))^t(g(b))^{1-t})|dt\right]$$
$$\leq (g(b)-g(a))$$
$$\times\left[\left(\int_0^{\frac{1}{2}} tdt\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}|t||f'((g(a))^t(g(b))^{1-t})|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1(1-t)dt\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^1|t-1||f'((g(a))^t(g(b))^{1-t})|^q dt\right)^{\frac{1}{q}}\right]$$
$$\leq (g(b)-g(a))|f'(b)|\left(\frac{1}{8}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_0^{\frac{1}{2}} tw^{qt}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^1(1-t)w^{qt}dt\right)^{\frac{1}{q}}\right]$$
$$=(g(b)-g(a))|f'(b)|\left(\frac{1}{8}\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\frac{-w^{\frac{1}{2}q}+\frac{1}{2}w^{\frac{1}{2}q}q\ln(w)+1}{q^2\ln(w)^2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{w^q-\frac{1}{2}w^{\frac{1}{2}q}q\ln(w)-w^{\frac{1}{2}q}}{q^2\ln(w)^2}\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Remark. For $q=1$ Theorem 6 reduces to Theorem 4.

Theorem 7. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be twice differentiable function on I° (the interior of I) and $g:\mathbb{R}\to\mathbb{R}$ be arbitrary function. Also $f''\in\mathscr{L}[g(a),g(b)]$ for $g(a),g(b)\in I$ with $g(a)<g(b)$. If $|f'|$ is decreasing and geometrically relative convex function. Then*

$$\left|\frac{f(g(a))+f(g(b))}{2}-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|$$
$$\leq\frac{g(b)-g(a)}{2}|f''(g(b))|\Psi(k),$$

where $k=\frac{|f''(g(a))|}{|f''(g(b))|}$ and $\Psi(k)=\frac{k\ln(k)-2k+\ln(k)+2}{\ln(k)^3}$.

Proof. Using Lemma 3 and the fact that $|f''|$ is geometrically relative convex function, we have

$$\left|\frac{f(g(a))+f(g(b))}{2}-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|$$
$$\leq\frac{g(b)-g(a)}{2}\int_0^1 t(1-t)|f''(tg(a)+(1-t)g(b))|dt$$
$$\leq\frac{g(b)-g(a)}{2}\int_0^1 t(1-t)\left|f''\left(g(a)^t g(b)^{1-t}\right)\right|dt$$
$$\leq\frac{g(b)-g(a)}{2}|f'(g(b))|\int_0^1 t(1-t)\left(\frac{|f''(g(a))|}{|f''(g(b))|}\right)^t dt$$
$$=\frac{g(b)-g(a)}{2}|f'(g(b))|\int_0^1 t(1-t)k^t dt$$
$$=\frac{g(b)-g(a)}{2}|f''(g(b))|\frac{k\ln(k)-2k+\ln(k)+2}{\ln(k)^3}.$$

This completes the proof. □

Theorem 8. *Let $f: I\subseteq\mathbb{R}\to\mathbb{R}$ be twice differentiable function on I° (the interior of I) and $g:\mathbb{R}\to\mathbb{R}$ be arbitrary function. Also $f''\in\mathscr{L}[g(a),g(b)]$ for $g(a),g(b)\in I$ with $g(a)<g(b)$. If $|f''|^q$ is decreasing and geometrically relative convex function for $p,q>1$, $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\left|\frac{f(g(a))+f(g(b))}{2}-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|$$
$$\leq\frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{2^{(-1-2p)}\sqrt{\pi}\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}}$$
$$\times\left(\frac{k^q-1}{q\ln(k)}\right)^{\frac{1}{q}},$$

where $k=\frac{|f''(g(a))|}{|f''(g(b))|}$.

Proof. Using Lemma 3, well known Holder's inequality and the fact that $|f''|^q$ is geometrically relative convex function, we have

$$\left|\frac{f(g(a))+f(g(b))}{2}-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|$$
$$\leq\frac{g(b)-g(a)}{2}\int_0^1 t(1-t)|f''(tg(a)+(1-t)g(b))|dt$$
$$\leq\frac{g(b)-g(a)}{2}\left(\int_0^1(t(1-t))^p dt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left|f''\left(g(a)^t g(b)^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{2^{(-1-2p)}\sqrt{\pi}\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \left(\frac{|f''(g(a))|}{|f''(g(b))|}\right)^{qt} dt\right)^{\frac{1}{q}}$$

$$= \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{2^{(-1-2p)}\sqrt{\pi}\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 k^{qt} dt\right)^{\frac{1}{q}}$$

$$= \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{2^{(-1-2p)}\sqrt{\pi}\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}\right)^{\frac{1}{p}}$$

$$\times\left(\frac{k^q-1}{q\ln(k)}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 9. *Let $f:\mathscr{G}\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable function on I° (the interior of I) and $g:\mathbb{R}\to\mathbb{R}$ be arbitrary function. Also $f''\in\mathscr{L}[g(a),g(b)]$ for $g(a),g(b)\in I$ with $g(a)<g(b)$. If $|f''|^q$ is decreasing and geometrically relative convex function for $q>1$. Then*

$$\left|\frac{f(g(a))+f(g(b))}{2}-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|$$

$$\leq \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{1}{6}\right)^{\frac{1}{p}}\Psi(k),$$

where $k=\frac{|f''(g(a))|}{|f''(g(b))|}$ and

$$\Psi(k)=\left(\frac{k^q q\ln(k)-2k^q+q\ln(k)+2}{q^3\ln(k)^3}\right)^{\frac{1}{q}}.$$

Proof. Using Lemma 3, well known Power mean inequality and the fact that $|f''|^q$ is geometrically relative convex function, we have

$$\left|\frac{f(g(a))+f(g(b))}{2}-\frac{1}{g(b)-g(a)}\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|$$

$$\leq \frac{g(b)-g(a)}{2}\int_0^1 t(1-t)|f'(tg(a)+(1-t)g(b))|dt$$

$$\leq \frac{g(b)-g(a)}{2}\left(\int_0^1 t(1-t)dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 t(1-t)\left|f'\left(g(a)^t g(b)^{1-t}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$\leq \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{1}{6}\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 t(1-t)\left(\frac{|f'(g(a))|}{|f'(g(b))|}\right)^{qt} dt\right)^{\frac{1}{q}}$$

$$= \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{1}{6}\right)^{\frac{1}{p}}\left(\int_0^1 t(1-t)k^{qt}dt\right)^{\frac{1}{q}}$$

$$= \frac{g(b)-g(a)}{2}|f'(g(b))|\left(\frac{1}{6}\right)^{\frac{1}{p}}$$

$$\times\left(\frac{k^q q\ln(k)-2k^q+q\ln(k)+2}{q^3\ln(k)^3}\right)^{\frac{1}{q}}.$$

This completes the proof. □

Next we prove the results for the class of *GA*-relative convex functions.

Theorem 10. *Let $f:\mathscr{G}\to\mathbb{R}$ be GA-relative convex function such that $g(a),g(b)\in\mathscr{G}$ with $g(a)<g(b)$. Then the following inequality holds:*

$$f\sqrt{g(a)g(b)} \leq \frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)}\frac{f(g(x))}{g(x)}dg(x)$$
$$\leq \frac{f(g(a))+f(g(b))}{2}. \tag{8}$$

Proof. Since f is *GA*-relative convex function. Thus

$$f(\sqrt{g(x)g(y)})\leq\frac{f(g(x))+f(g(y))}{2}.$$

Let $g(x)=(g(a))^{1-t}(g(b))^t$ and $g(y)=(g(a))^t(g(b))^{1-t}$. Then this implies

$$f(\sqrt{g(a)g(b)})$$
$$=\int_o^t f(\sqrt{g(a)g(b)})dt$$
$$\leq\int_o^t \frac{f((g(a))^{1-t}(g(b))^t)+f((g(a))^t(g(b))^{1-t})}{2}dt$$
$$=\frac{1}{\ln g(b)-\ln g(a)}\int_{g(a)}^{g(b)}\frac{f(g(x))}{g(x)}dg(x)$$
$$=\int_0^1 f((g(a))^t(g(b))^{1-t})dt$$
$$\leq\frac{f(g(a))+f(g(b))}{2}.$$

This completes the proof. □

Theorem 11. *Let $f:\mathscr{G}\to\mathbb{R}$ be differentiable function on $(g(a),g(b))$ with $g(a)<g(b)$ and $f'\in\mathscr{L}[g(a),g(b)]$. If $|f'|^q$ is GA-relative convex function for $q\geq 1$, then following inequality holds:*

$$\left|[g(b)f(g(b))-g(a)f(g(a))]-\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|$$

$$\le (g(b))^2(\ln g(b)-\ln g(a))\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\left[\frac{h^2-1}{\ln(h)}\right]^{1-\frac{1}{q}}$$
$$\times\left[\frac{2h^2\ln(h)-h^2+1}{\ln(h)^2}|f'(a)|^q+\frac{h^2-2\ln(h)-1}{\ln(h)^2}|f'(b)|^q\right]^{\frac{1}{q}}.$$

Proof. Since $|f'|^q$ is *GA*-relative convex function. Then from lemma 4 and well known power mean inequality, we have

$$\left|[g(b)f(g(b))-g(a)f(g(a))]-\int_{g(a)}^{g(b)}f(g(x))dg(x)\right|$$
$$\le (g(b))^2(\ln g(b)-\ln g(a))\int_0^1\left(\frac{g(a)}{g(b)}\right)^{2t}\times|f'((g(a))^t(g(b))^{1-t})|dt$$
$$\le (g(b))^2(\ln g(b)-\ln g(a))\left[\int_0^1\left(\frac{g(a)}{g(b)}\right)^{2t}dt\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1\left(\frac{g(a)}{g(b)}\right)^{2t}\{t|f'(a)|^q+(1-t)|f'(b)|^q\}dt\right]^{\frac{1}{q}}$$
$$=(g(b))^2(\ln g(b)-\ln g(a))\left[\int_0^1 h^{2t}dt\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1 h^{2t}\{t|f'(a)|^q+(1-t)|f'(b)|^q\}dt\right]^{\frac{1}{q}}$$
$$=(g(b))^2(\ln g(b)-\ln g(a))\left[\frac{1}{2}\frac{h^2-1}{\ln(h)}\right]^{1-\frac{1}{q}}$$
$$\times\left[\frac{1}{4}\frac{2h^2\ln(h)-h^2+1}{\ln(h)^2}|f'(a)|^q+\frac{1}{4}\frac{h^2-2\ln(h)-1}{\ln(h)^2}|f'(b)|^q\right]^{\frac{1}{q}}$$
$$=(g(b))^2(\ln g(b)-\ln g(a))\left(\frac{1}{2}\right)^{1+\frac{1}{q}}\left[\frac{h^2-1}{\ln(h)}\right]^{1-\frac{1}{q}}$$
$$\times\left[\frac{2h^2\ln(h)-h^2+1}{\ln(h)^2}|f'(a)|^q+\frac{h^2-2\ln(h)-1}{\ln(h)^2}|f'(b)|^q\right]^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 1. *Under the assumptions of Theorem 11, if* $q=1$, *we have*

$$\left|[g(b)f(g(b))-g(a)f(g(a))]-\int_{g(a)}^{g(b)}f(g(x))dg(x)\right|$$
$$\le (g(b))^2(\ln g(b)-\ln g(a))$$
$$\times\frac{1}{4}\left[\frac{2h^2\ln(h)-h^2+1}{\ln(h)^2}|f'(a)|+\frac{h^2-2\ln(h)-1}{\ln(h)^2}|f'(b)|\right].$$

Theorem 12. *Let* $f:\mathscr{G}\to\mathbb{R}$ *be differentiable function on* $(g(a),g(b))$ *with* $g(a)<g(b)$ *and* $f'\in\mathscr{L}[g(a),g(b)]$. *If* $|f'|^q$ *is GA-relative convex function for* $q>1$, *then following inequality holds:*

$$\left|[g(b)f(g(b))-g(a)f(g(a))]-\int_{g(a)}^{g(b)}f(g(x))dg(x)\right|$$
$$\le(\ln g(b)-\ln g(a))\left[L((g(a))^{\frac{2q}{q-1}},(g(b))^{\frac{2q}{q-1}})\right]^{1-\frac{1}{q}}$$
$$\times\left[A(|f'(a)|^q,|f'(b)|^q)\right]^{\frac{1}{q}}.$$

Proof. Since $|f'|^q$ is *GA*-relative convex function. Then from lemma 4 and well known Holder's inequality, we have

$$\left|[g(b)f(g(b))-g(a)f(g(a))]-\int_{g(a)}^{g(b)}f(g(x))dg(x)\right|$$
$$\le (g(b))^2(\ln g(b)-\ln g(a))\int_0^1\left(\frac{g(a)}{g(b)}\right)^{2t}\times|f'((g(a))^t(g(b))^{1-t})|dt$$
$$\le (g(b))^2(\ln g(b)-\ln g(a))\left[\int_0^1\left(\frac{g(a)}{g(b)}\right)^{\frac{2tq}{q-1}}dt\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1\{t|f'(a)|^q+(1-t)|f'(b)|^q\}dt\right]^{\frac{1}{q}}$$
$$=(\ln g(b)-\ln g(a))\left[\frac{(q-1)\left((g(b))^{\frac{2q}{q-1}}-(g(a))^{\frac{2q}{q-1}}\right)}{2q(\ln g(b)-\ln g(a))}\right]^{1-\frac{1}{q}}$$
$$\times\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}$$
$$=(\ln g(b)-\ln g(a))\left[L((g(a))^{\frac{2q}{q-1}},(g(b))^{\frac{2q}{q-1}})\right]^{1-\frac{1}{q}}$$
$$\times\left[A(|f'(a)|^q,|f'(b)|^q)\right]^{\frac{1}{q}}.$$

This completes the proof. □

Theorem 13. *Let* $f:\mathscr{G}\to\mathbb{R}$ *be differentiable function on* $(g(a),g(b))$ *with* $g(a)<g(b)$ *and* $f'\in\mathscr{L}[g(a),g(b)]$. *If* $|f'|^q$ *is GA-relative convex function for* $q\ge 1$, *then following inequality holds:*

$$\left|[g(b)f(g(b))-g(a)f(g(a))]-\int_{g(a)}^{g(b)}f(g(x))dg(x)\right|$$
$$\le\frac{(\ln g(b)-\ln g(a))^{1-\frac{1}{q}}}{(2q)^{\frac{1}{q}}}$$
$$\times\left\{\left[(g(b))^{2q}-L((g(a))^{2q},(g(b))^{2q})|f'(g(a))|^q\right]\right.$$
$$\left.+\left[L((g(a))^{2q},(g(b))^{2q})-(g(a))^{2q}\right]|f'(b)|^q\right\}^{\frac{1}{q}}.$$

Proof. Since $|f'|^q$ is *GA*-relative convex function. Then from lemma 4 and well known Holder's inequality, we have

$$\left|[g(b)f(g(b))-g(a)f(g(a))]-\int_{g(a)}^{g(b)} f(g(x))dg(x)\right|$$
$$\leq (g(b))^2(\ln g(b)-\ln g(a))\int_0^1\left(\frac{g(a)}{g(b)}\right)^{2t}$$
$$\times|f'((g(a))^t(g(b))^{1-t})|dt$$
$$\leq (g(b))^2(\ln g(b)-\ln g(a))\left[\int_0^1 1dt\right]^{1-\frac{1}{q}}$$
$$\times\left[\int_0^1\left(\frac{g(a)}{g(b)}\right)^{2qt}\{t|f'(a)|^q+(1-t)|f'(b)|^q\}dt\right]^{\frac{1}{q}}$$
$$=(\ln g(b)-\ln g(a))$$
$$\times\left[\frac{(g(b))^{2q}(\ln(g(b))^{2q}-\ln(g(a))^{2q})-(g(b))^{2q}+(g(a))^{2q}}{(\ln(g(b))^{2q}-\ln(g(a))^{2q})^2}\right.$$
$$\times|f'(g(a))|^q$$
$$+\frac{(g(b))^{2q}-(g(a))^{2q}(\ln(g(b))^{2q}-\ln(g(a))^{2q})-(g(a))^{2q}}{(\ln(g(b))^{2q}-\ln(g(a))^{2q})^2}$$
$$\left.\times|f'(g(b))|^q\right]^{\frac{1}{q}}$$
$$\leq\frac{(\ln g(b)-\ln g(a))^{1-\frac{1}{q}}}{(2q)^{\frac{1}{q}}}$$
$$\times\left\{\left[(g(b))^{2q}-L((g(a))^{2q},(g(b))^{2q})|f'(g(a))|^q\right]\right.$$
$$\left.+\left[L((g(a))^{2q},(g(b))^{2q})-(g(a))^{2q}\right]|f'(b)|^q\right\}^{\frac{1}{q}}.$$

This completes the proof. □

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Muhammad Aslam Noor earned his PhD degree from Brunel University, London, UK (1975) in the field of Applied Mathematics(Numerical Analysis and Optimization). He has vast experience of teaching and research at university levels in various countries including Pakistan, Iran, Canada, Saudi Arabia and United Arab Emirates. His field of interest and specialization is versatile in nature. It covers many areas of Mathematical and Engineering sciences such as Variational Inequalities, Operations Research and Numerical Analysis. He has been awarded by the President of Pakistan: President's Award for pride of performance on August 14, 2008, in recognition of his contributions in the field of Mathematical Sciences. He was awarded HEC Best Research paper award in 2009. He has supervised successfully several Ph.D and MS/M.Phil students. He is currently member of the Editorial Board of several reputed international journals of Mathematics and Engineering sciences. He has more than 750 research papers to his credit which were published in leading world class journals.

Khalida Inayat Noor is a leading world-known figure in mathematics and is presently employed as HEC Foreign Professor at CIIT, Islamabad. She obtained her PhD from Wales University (UK). She has a vast experience of teaching and research at university levels in various countries including Iran, Pakistan, Saudi Arabia, Canada and United Arab Emirates. She was awarded HEC best research paper award in 2009 and CIIT Medal for innovation in 2009. She has been awarded by the President of Pakistan: Presidents Award for pride of performance on August 14, 2010 for her outstanding contributions in mathematical sciences and other fields. Her field of interest and specialization is Complex analysis, Geometric function theory, Functional and Convex analysis. She introduced a new technique, now called as Noor Integral Operator which proved to be an innovation in the field of geometric function theory and has brought new dimensions in the realm of research in this area. She has been personally instrumental in establishing PhD/MS programs at CIIT. Dr. Khalida Inayat Noor has supervised successfully several Ph.D and MS/M.Phil students. She has been an invited speaker of number of conferences and has published more than 400 (Four hundred) research articles in reputed international journals of mathematical and engineering sciences. She is member of editorial boards of several international journals of mathematical and engineering sciences.

Muhammad Uzair Awan has earned his MS degree from COMSATS Institute of Information Technology, Islamabad, Pakistan. His field of interest is Convex Analysis and Numerical Optimization.