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Mathematical Modelling of the Co-dynamics of illicit Drug use and Terrorism

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Abstract: In recent years, illicit drug use and an act of terrorism has led to deaths of many people in different countries. This paper used mathematical modelling approach to studied the dynamics of illicit drug use and terrorism in a population. A model was developed and the basic properties of it were studied; the equilibrium points were obtained. Next generation method was used to calculate Illicit drug use and terrorism threshold ($\mathscr{R}_0$). Global stability of illicit drug use and terrorism-present equilibrium point was shown to be asymptotically stable analytically; this was achieved by construction of suitable Lyapunov function. The model exhibits backward bifurcation which was studied with by center manifold theory, this reveals to us the serious setback in eradicate illicit drug use and terrorism in the population, even when $\mathscr{R}_0 < 1$ except some condition are met. The normalized forward-sensitivity index of the variables was used to determine the contributory effects of each parameter on spread of the illicit drug use and terrorism menace in a population. Numerical simulations were done with the help of MATLAB, this was done to confirm these analytical results. The result indicates the parameters to targeted with corrective control and the one to be strengthen.

Keywords: Mathematical Model, Illicit drug, Terrorism, Basic properties, Illicit drug use and terrorism threshold and Bifurcation analysis, Sensitivity analysis.

1 Introduction

The impart of Illicit drug use and terrorism problem has significantly attract attention of many people, because of the rate at which it spreads all over the world in this recent time. More than 140 countries has report the problem of illicit drug use and its associated effects on their economic and the development [1], such effect includes different forms of illness and social challenges. From the report of World Health Organization (WHO), illicit drug use which is also known as drug abuse can be define as the harmful or hazardous use of psychoactive substances or a non-medical use of a drug that interferes with a healthy and productive life [2,3]. Similarly, illicit drug use which can also be refers to has the non-medical use of a variety of drugs that are prohibited by international law [4,6]. This illegal act has become a crucial issue in this recent years as it ruins the future of young generation which are hope of tomorrow of any developing nation. Examples of drugs that are commonly abuse by people are cannabis, cocaine, heroin, ecstasy alcohol, pregabalin, benzodiazepines, ketamine, nicotine, khat amphetamine-type stimulants, and other opioids [1,4].

Illicit drug use has continues to exert significant toll, with valuable human lives and productive years of many people being lost. An estimated between 162-324 million people with age ranging from 15 years to 64 years used an illicit drug globally in 2012 [4]. As reported by WHO, about 275 million people used illicit drug at least once in 2016 [1]. It was also stated that almost 31 million people suffer from drug use disorders leading to an estimated 3.3 million annual drug-related deaths [1].

The act of terrorism is one of the major and serious problem affecting the whole world this time, the population that this art has wasted in different nations of the worldwide is enormous. The word "terrorism" first heard during the French Revolution in the late 18th century, but becomes popular in the 1970s during the battle of Northern Ireland, the Basque Country, and Palestine. The increasing number of incidences of suicide

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bombing from the 1980s onwards is typified by the September attacks in New York City and Washington, D.C. in 2001 which made terrorism well-known across the globe. Terrorism is a physical process that requires little effort to agnise but very hard to delimit. Individuals definition of terrorism is based on the socio-economic and political state of the environment in which it occurs. According to Bruckberger [7] try to define terrorism as a form of sudden and reoccurring violent action, employed by individuals, clandestine groups or state actors, in which, in most cases, the victims are not the target but are used to send signals or warning to those that, are being targeted to influence a specific cause. Terrorism occurs based on many reasons which includes; political, religious, ethical, ideology, etc. The immediate unfortunate person of terrorism attacked are sometimes selected randomly but most often they are selected from the target population [8]. It is important to note that terrorism can not be discuss alone without mentioning the great effects of illicit drugs because such dangerous acts cannot be carried out without the use of hard-illicit drugs and a certain level of hypnotism such as brain-washing.

Similarly, it is crucial to note that the problem of drug trafficking is the major factor responsible for acts of terrorism. The destructive impact, the intentions and sponsors of terrorist and drug traffickers are almost the same. [9]. Moreover, other crimes increase astronomically as drug trade increases. Although it can be denoted that illicit drug is an influencer for criminals, the case is different for terrorists [9]. First, leaders of terrorist groups often create an ideology and these ideologies are perceived to be true by their followers who are used to perpetuate the terrorist attacks. Since most individuals used to perpetrate terrorist attacks are drug addicts, it is often easy to introduce the much harder drugs to them to be able to control their minds and enable them to carry out the attacks [10].

Furthermore, terrorism is more of ideological than true, then it can be reduced to the barest minima if cooperate consideration is given to it worldwide. Serious measures such as: the national security policy against terrorism and illicit drug use should be strengthen and promotion against these acts should also be taken serious; this will help in curtailing the influence of terrorist and illicit drug user in the society. Also, the borders of each country should be strengthened to curtail the importation of these contrabands; illicit drugs and weapons. In addition, measures must be put in place to curtail money laundry by terrorist groups. A body should be organized in all countries of the World to promote, implement relevant universal measures against terrorism, drugs, organized crime. [11]. Such body will be in charge implementing and training people on the legal ways of dealing illicit drugs user, terrorists, and organized criminals.

A lot of works has been done in the area of using mathematical modeling approach to address physical problems. Some of them are : Akanni *et al.* [1] use a mathematical model approach to studied the effect of drug abuse in a population. The model was extensively studied, and the following observations were made, the model exhibits forward bifurcation property pointing that the problem can be control easily once effort are putting in place to lower the $\mathscr{R}_0$. In another development, Akanni [12], a studied the effect of substance abuse in a population using a mathematical model perspective, this was done to gain an insight to the dynamics spread of illicit drug use and banditry in a population. On the other way round, this study exhibits backward bifurcation, pointing that it is not only sufficient to lower the $\mathscr{R}_0$ below unity, but other things must be put into consideration. Also, sensitivity analysis was done to determine impart of parameters of the model on the spread of the drug abuse and banditry menace in the population.

Furthermore, Pang *et al.* [13] a mathematical model described by ordinary differential equations with a saturated incidence rate to explore the effect of controlling smoking by setting up designed smoking areas and raising the price of cigarettes was proposed and well studied. Similarly, Kalula and Nyabadza [14] formulated a six-state compartmental model which includes a core and non-core group, with fast and slow progression to addiction with the aim of qualitatively investigating the dynamics of substance abuse and predicting drug abuse trends. They suggested that the substance abuse epidemic can be reduced by intervention programmes targeted at light drug users and by increasing the uptake rate into treatment for those addicted.

In another development, Orwa and Nyabadza [15] confirmed that drug abuse poses a significant threat to the health and socio-economic fabric of individuals and nations. They suggested that social programs that raise awareness of the dangers posed by multiple substance abusers should promote, through educational campaigns in learning institutions, social media and health institutions, similarly, transmission control must focus on enhancing the quitting process while promoting support services to drug users during and after treatment to minimize cases of relapse. In Kanyaa *et al.* [16] formulated a model for drug abuse that explains the dynamics of the use and abuse of certain substances that are perceived as mood changing by commercial drivers. The drug model reveals that an increase in the recruitment rate of commercial drivers and the rate at which commercial drivers return to the use and abuse of drugs would cause an increase in the drug abuse number. The numerical results showed that an increase in the contact or imitation rate increased the population of drug users.

This work focuses on how illicit drug use affects terrorism and the relationship between illicit drug use and terrorism using the ideal of mathematical model, this focus are less or not cited in the existing works. A mathematical model governed by a system of nonlinear ordinary differential equations is studied to explore the influence of illicit drug use on terrorism in a population.

the remaining part of this paper is organized as follows: Section 2 is concerned with the Model formulation and analysis of the basic properties. In Section 3, Existence and stability analysis of equilibria are carried out. Furthermore, Sensitivity Analysis was extensively exploded in Section 4 and Numerical Simulation of the model was presented in Section 5. The Concluding remarks are given in Section 6.

2 Model Formulation

The illicit drug use and terrorism dynamic in the population was studied with the total population denoted by $N(t)$, at time t, which was sub-divided into seven well-defined classes of individuals that are susceptible $S(t)$ (someone who does not indulge in the use of illicit drug but move with drug users), Prisoner $C(t)$ (someone who is in correctional center due to the use of illicit drug and a suspected terrorist), individuals under detention $D(t)$ (someone who is in the police custody due to the use of illicit drug use and a suspected terrorist), suspected terrorist $S_E(t)$ (a terrorist), Illicit drug user $I(t)$ (someone who utilize illicit drug often or those that who absolutely depends on the use of illicit drugs) and rehabilitate population $R(t)$ (someone who is under treatment or rehabilitation due to drug use or art of being involved in terrorism) and quitters $Q(t)$ (individuals who quit or stop taking illicit drugs and terrorism). The population is not a constant population size because the recruitment (births) is not balanced by removal (natural deaths) rates due to induced death rate δ_i, where $i = 1,2,3$. Figure 1 represents the schematic diagram for the illicit drug use and terrorism population dynamics, which the nonlinear ordinary differential equations (1) is the governed system.

This work is an extension of [5], the assumptions that guide the flows of the model were given in details in [5], the extension was to study how class R progresses to Q, S and I classes at the rate σ_5, ρ and α. Table 1 gives the description of variables and parameters of the illicit drug use and terrorism model.

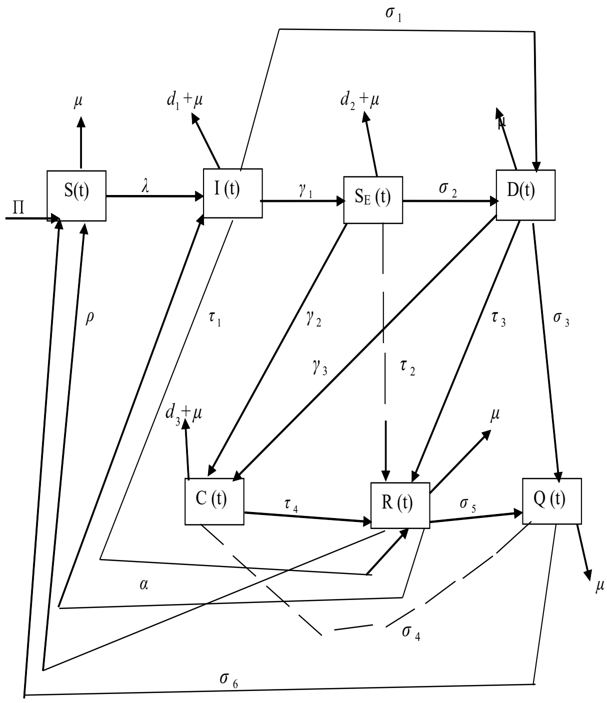


Fig. 1: Schematic diagram for illicit drug use and terrorism dynamics

$$
\begin{aligned}
\frac{dS}{dt} &= \Pi - \frac{\beta(I(t)+\eta S_E(t))}{N(t)}S(t) + \sigma_6 Q(t) + \rho R(t) \\
&\quad -\mu S(t) \\
\frac{dI}{dt} &= \frac{\beta(I(t)+\eta S_E(t))}{N(t)}S(t) + \alpha R(t) - (\sigma_1 + \tau_1 + \gamma_1 \\
&\quad +\delta_1 + \mu)I(t) \\
\frac{dS_E}{dt} &= \gamma_1 I(t) - (\sigma_2 + \tau_2 + \gamma_2 + \delta_2 + \mu)S_E(t) \\
\frac{dD}{dt} &= \sigma_1 I(t) + \sigma_2 S_E(t) - (\sigma_3 + \gamma_3 + \tau_3 + \mu)D(t) \\
\frac{dC}{dt} &= \gamma_2 S_E(t) + \gamma_3 D(t) - (\sigma_4 + \tau_4 + \delta_4 + \mu)C(t) \\
\frac{dR}{dt} &= \tau_1 I(t) + \tau_2 S_E(t) + \tau_3 D(t) + \tau_4 C(t) - (\sigma_5 + \alpha \\
&\quad +\rho + \mu)R(t) \\
\frac{dQ}{dt} &= \sigma_3 D(t) + \sigma_4 C(t) + \sigma_5 R(t) - \sigma_6 Q(t) - \mu Q(t),
\end{aligned} \tag{1}
$$

with initial conditions at $t = 0$:

$$S(0) = S_0, S_E(0) = S_{E_0}, I(0) = I_0, D(0) = D_0, \\ C(0) = C_0, R(0) = R_0, Q(0) = Q_0. \quad (2)$$

The total population is obtained by adding all the equations of (1) resulting to:

$$\frac{dN}{dt} \leq \pi - \mu N \quad (3)$$

Table 1. The variables and parameters of illicit drug use and terrorism [1]

	Description
Variable	
$S(t)$	Susceptible individuals
$I(t)$	Illicit drug user
$S_E(t)$	Suspected terrorist
$D(t)$	Individuals in detention
$C(t)$	Individuals in correction service
$R(t)$	Individuals in rehabilitating center
$Q(t)$	Quitters
Parameter	
Π	Recruitment rate into the susceptible population
β	Effective influence rate
α	Movement of rehabilitating individuals to illicit drug user
ρ	Movement of rehabilitating individuals to susceptible population
η	Modification parameter for terrorism
σ_1	Detention rate of illicit drug users
γ_1	Progression rate of illicit drug users to banditry
τ_1	Rehabilitation rate of illicit drug users
δ_1	Induce death rate of illicit drug users
δ_4	Induce death rate of those in correction service
μ	Natural rate
σ_2	Detention rate of terrorist
γ_2	Movement of terrorist to correction service
δ_2	Induced death rate of terrorist
τ_2	Rehabilitation rate of terrorist
σ_3	Quitting rate of detained individuals
γ_3	Progression rate of detained individual to correction service
τ_3	Rehabilitation rate of detained individual
σ_4	Quitting rate of individuals in the prison
τ_4	Rehabilitating rate of those in correction service
δ_4	Induced death rate of terrorist
σ_5	Treatment rate of those in rehabilitation center
σ_6	Movement rate of quitter to susceptible

2.1 Fundamental Properties

It is necessary to prove the non-negativity of the state variables for all times, given that all parameters are non-negativity since the formulated model (1) monitors human population.

2.1.1 Positivity solutions

Theorem 1.*The state variables, $S(t)$, $I(t)$, $S_E(t)$, $D(t)$, $C(t)$, $R(t)$ and $Q(t)$, of the illicit drug use and terrorism model* (1)*, with the non-negative initial data* (2)*, remain positive for all $t > 0$.*

*Proof.*From the first equation of (1)

$$\frac{dS}{dt} \geq -(\lambda + \mu)S(t),$$

it follows that,

$$\frac{d}{dt}\left(S(t)\exp\left(\mu t + \int_0^t \lambda(\Psi)d\Psi\right)\right) \geq 0,$$

then

$$S(t) \geq S(0)\exp\left(-\left(\mu t + \int_0^t \lambda(\Psi)d\Psi\right)\right) > 0, \forall\, t > 0.$$

The other state variables $I(t)$, $S_E(t)$, $D(t)$, $C(t)$, $R(t)$ and $Q(t)$, will non-negativity for all $t > 0$ when the same approach been used for $S(t)$ is employed.

2.1.2 Invariant region

The biologically feasible region, define by $\Gamma \subset \mathbb{R}^7_+$ was considered, such that $\Gamma = \left\{(S, I, S_E, D, C, R, Q) \in \mathbb{R}^7_+ : N \leq \frac{\Pi}{\mu}\right\}$. It was proved that Γ is positively invariant in that region.

Theorem 2.*The region Γ is positively invariant with respect to the model* (1)

*Proof.*Rate of change of the total population of model (1) is given by

$$\frac{dN}{dt} = \Pi - \mu N, \quad (4)$$

resulting into the solution $N(t) = N(0)\exp(-\mu t) + \frac{\Pi}{N}(1 - \exp(-\mu t))$. Following that $N(t) \longrightarrow \frac{\Pi}{\mu}$ as t $\longrightarrow \infty$ in particular, N(t) $\leqslant \frac{\Pi}{\mu}$ if $N(0) \leqslant \frac{\Pi}{\mu}$ with respect to the illicit drug use and terrorism model (1). Hence, it suffices to consider the dynamics of the model in Γ. In this region, the illicit drug use and terrorism model can be considered as being mathematically and biologically well-posed [17].

3 Existence and stability analysis of equilibria

In this section, the existence of steady-state solution of the model [1] is determined and the nature of bifurcation exhibited by the model is investigated.

3.1 Illicit drug use and terrorism-free Equilibrium

Illicit drug use and terrorism-free population means the point where there is no illicit drug user and terrorist in the population. Then at illicit drug use and terrorism-free equilibrium $I = S_E = 0$. Solving model (1) gives,

$$S = \frac{\Pi}{\mu}, D = 0, C = 0, R = 0.$$

Therefore, the existence of illicit drug use and terrorism-free equilibrium (IDTFE) is given by $\mathscr{D}_0$

$$\mathscr{D}_0 = (S_0, I_0, S_{E_0}, D_0, C_0, R_0) = \left(\frac{\Pi}{\mu}, 0, 0, 0, 0, 0\right) \quad (5)$$

The illicit drug use and terrorism reproduction number $\mathscr{R}_0$ is established next.

3.2 Illicit drug use and terrorism threshold ($\mathscr{R}_0$)

The illicit drug use and terrorism reproduction number also noted as Illicit drug use and terrorism threshold, $\mathscr{R}_0$, is a measure of the spread potential of illicit drug use and terrorism in a naive or susceptible population [1,2,19,20]. In other words, $\mathscr{R}_0$ represents the average number of secondary illicit drug use and terrorism cases influenced by a typical illicit drug and terrorist in a completely susceptible population.

Thus, using the next generation matrix approach [18], this method has been employed in many literature such as [1,2,12,19,20] from there idea, we have :

$$F = \begin{pmatrix} \beta & \beta\eta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (6)$$

, let $k_1 = \sigma_1 + \tau_1 + \gamma_1 + \delta_1 + \mu$, $k_2 = \sigma_2 + \tau_2 + \gamma_2 + \delta_2 + \mu$, $k_3 = \sigma_3 + \tau_3 + \gamma_3 + \mu$, $k_4 = \sigma_4 + \tau_4 + \delta_4 + \mu$, $k_5 = \sigma_5 + \alpha + \rho + \mu$, $k_6 = \sigma_6 + \mu$,

and

$$V = \begin{pmatrix} k_1 & 0 & 0 & 0 & -\alpha \\ -\gamma_1 & k_2 & 0 & 0 & 0 \\ -\sigma_1 & -\sigma_2 & k_3 & 0 & 0 \\ 0 & -\gamma_2 & -\gamma_3 & k_4 & 0 \\ -\tau_1 & -\tau_2 & -\tau_3 & \tau_4 & k_5 \end{pmatrix} \quad (7)$$

. Therefore, the illicit drug use and terrorism reproduction number of the system (1), denoted by $\mathscr{R}_0 = \rho(FV^{-1})$, where ρ is the spectral radius of the product FV^{-1}, is obtained by:

$$\mathscr{R}_0 = \frac{\beta k_3 k_4 k_5 (k_2 + \eta\gamma_1)}{d_7}, \quad (8)$$

where;

$$d_7 = k_1k_2k_3k_4k_5 - \alpha(k_3(k_2k_4\tau_1 + k_4\gamma_1\tau_2 + \gamma_1\gamma_2\tau_4) + (k_4\tau_3 + \gamma_3\tau_4)(k_2\sigma_1 + \gamma_1\sigma_2)).$$

Algebraic simplification of d_7 shows that $k_1k_2k_3k_4k_5 > (k_2k_3k_4\tau_1\alpha + k_2k_4\tau_3\alpha\sigma_1 + k_4\sigma_2\alpha\gamma_1\tau_3 + k_2\tau_4\alpha\gamma_3\sigma_1 + k_3\gamma_2\alpha\gamma_1\tau_4 + \gamma_1\tau_4\alpha\gamma_3\sigma_2)$. To study the local asymptotically stability of $\mathscr{D}_0$ given by (5), the illicit drug use and terrorism reproduction number, $\mathscr{R}_0$, given by (8), is a major key needed for this analysis. The local asymptotically stability of $\mathscr{D}_0$ will be claimed by the following theorem.

Theorem 3. *The illicit drug use and terrorism-free equilibrium, $\mathscr{D}_0$, of the system (1) is locally asymptotically stable if $\mathscr{R}_0 < 1$ and unstable $\mathscr{R}_0 > 1$.*

The significance of Theorem 3 is that the problem of illicit drug use and terrorism model governed by (1) will be wiped out from the population, if the initial size of the illicit drug user and terrorist sub-populations are in the basin of attraction of the $\mathscr{D}_0$.

3.3 Illicit drug use and terrorism-present equilibrium

Illicit drug use and terrorism-present means the present of illicit drug user and terrorist in the population. The illicit drug use and terrorism-present equilibrium point (IDTPE) is referred to as the steady-state solution of model (1), this is the point when all the state variables are positive, which is denoted $\mathscr{D}^*$ and given by

$$\mathscr{D}^* = (S^*, I^*, S_E^*, D^*, C^*, R^*, Q^*). \quad (9)$$

Then, setting the right hand sides of (1) to zero, the following are obtained in terms of λ:

$$S_\lambda^* = \frac{\Pi b_1 k_6}{k_6(\lambda+\mu)b_1 + \sigma_6\lambda b_2},$$

$$I_\lambda^* = \frac{\Pi k_2k_3k_4k_5k_6\lambda}{k_6(\lambda+\mu)b_1 + \sigma_6\lambda b_2},$$

$$S_{E\lambda}^* = \frac{\gamma_1 \Pi k_3k_4k_5k_6\lambda}{k_6(\lambda+\mu)b_1 + \sigma_6\lambda b_2},$$

$$D_\lambda^* = \frac{B\Pi k_4k_5k_6\lambda}{k_6(\lambda+\mu)b_1 + \sigma_6\lambda b_2}, \quad (10)$$

$$C^*_{\lambda} = \frac{M\Pi k_5 k_6 \lambda}{k_6(\lambda+\mu)b_1 + \sigma_6 \lambda b_2},$$

$$R^*_{\lambda} = \frac{G\Pi k_6 \lambda}{k_6(\lambda+\mu)b_1 + \sigma_6 \lambda b_2},$$

$$Q^*_{\lambda} = \frac{b_3 \Pi \lambda}{k_6(\lambda+\mu)b_1 + \sigma_6 \lambda b_2},$$

where
$A = \tau_1 k_2 + \tau_2 \gamma_1$, $B = \sigma_1 k_2 + \sigma_2 \gamma_1$,
$F = \gamma_1 \gamma_2 k_3 + \gamma_1 B$, $G = Ak_3k_4 + Bk_4\tau_3 + \tau_4 F$,
$H = \sigma_3 Bk_4k_5 + \sigma_4 k_5 F$, $M = \gamma_1\gamma_2 k_2 + \gamma_3\sigma_1 k_2 + \gamma_1\gamma_3\sigma_2$,
$b_1 = k_1k_2k_3k_4k_5 - \alpha G$, $b_2 = H + \sigma_5 G$,
$b_3 = \sigma_3 Bk_4k_5 + \sigma_4 k5F + \sigma_5 G$.

$$\lambda^* = \frac{\beta(I^* + \eta S^*_E)}{N^*}, \quad (11)$$

and

$$N^* = S^* + I^* + S^*_E + D^* + C^* + R^* + Q^*. \quad (12)$$

Substituting (10) and (12) into equation (11) after algebraic manipulations, gives the following

$$\lambda = 0 \qquad or \qquad \lambda = \mu(\mathscr{R}_0 - 1). \quad (13)$$

When $\lambda = 0$ from system (10), it means illicit drug use and terrorism-present equilibrium does not exist, but we have illicit drug use and terrorism-free equilibrium, and when $\lambda = \mu(\mathscr{R}_0 - 1)$, we have a unique illicit drug use and terrorism-present equilibrium if $\mathscr{R}_0 > 1$. Simplifying (10), the following unique illicit drug use and terrorism-present equilibrium $\mathscr{D}^*$ are obtained in terms of $\mathscr{R}_0$ as follows

$$S^* = \frac{\Pi k_6}{k_6 \mu(\mathscr{R}_0 - 1)},$$

$$I^* = \frac{\Pi\beta(k_3k_4k_5)^2(\eta\gamma_1 + k_2)}{\mathscr{R}_0},$$

$$S^*_E = \frac{\gamma_1 \Pi\beta(k_3k_4k_5)^2(\eta\gamma_1 + k_2)}{\mathscr{R}_0},$$

$$D^* = \frac{\Pi\beta k_3(k_4k_5)^2(\eta\gamma_1 + k_2)}{\mathscr{R}_0}, \quad (14)$$

$$C^* = \frac{M\Pi\beta k_3k_4k_5^2(\eta\gamma_1 + k_2)}{\mathscr{R}_0},$$

$$R^* = \frac{G\Pi\beta k_3k_4k_5(\eta\gamma_1 + k_2)}{\mathscr{R}_0},$$

$$Q^* = \frac{b_3\Pi\beta k_3k_4k_5(\eta\gamma_1 + k_2)}{k_6\mathscr{R}_0}.$$

To further examine the nature of its stability when $\mathscr{R}_0 = 1$, the bifurcation analysis will be explore next.

3.4 Backward Bifurcation

To study the existence of bifurcation of the model (1), center manifold theory being described in Castillo-Chavez and Song [21] will be explored. This same theorem was used in some literatures like [12,22,23]. The illicit drug use and terrorism model (1) will be written in the vector form for this purpose:

$$\frac{dX}{dt} = F(X),$$

where $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T$ and $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T$ with $S(t) = x_1, I(t) = x_2, S_E(t) = x_3, D(t) = x_4, C(t) = x_5, R(t) = x_6, Q(t) = x_7$. Then, model (1) becomes

$$\begin{aligned}
f_1 &= \frac{dx_1}{dt} = \Pi - \frac{\beta\mu(x_2+\eta x_3)}{\Pi}x_1 + \sigma_6 x_7 + \rho x_6 - \mu x_1,\\
f_2 &= \frac{dx_2}{dt} = \frac{\beta\mu(x_2+\eta x_3)}{\Pi}x_1 + \alpha_1 x_6 - k_1 x_2,\\
f_3 &= \frac{dx_3}{dt} = \gamma_1 x_2 - k_2 x_3,\\
f_4 &= \frac{dx_4}{dt} = \sigma_1 x_2 + \sigma_2 x_3 - k_3 x_4,\\
f_5 &= \frac{dx_5}{dt} = \gamma_2 x_3 + \gamma_3 x_4 - k_4 x_5,\\
f_6 &= \frac{dx_6}{dt} = \tau_1 x_2 + \tau_2 x_3 + \tau_3 x_4 + \tau_4 x_5 - k_5 x_6,\\
f_7 &= \frac{dx_7}{dt} = \sigma_3 x_4 + \sigma_4 x_5 + \sigma_5 x_6 - k_6 x_7.
\end{aligned} \quad (15)$$

At $\mathscr{R}_0 = 1$ in (8), the bifurcation parameter β^* can be obtained as

$$\beta^* = \frac{d_7}{k_3k_4k_5(k_2 + \eta\gamma_1)}. \quad (16)$$

Recall;

$$d_7 = k_1k_2k_3k_4k_5 - \alpha(k_3(k_2k_4\tau_1 + k_4\gamma_1\tau_2 + \gamma_1\gamma_2\tau_4) + (k_4\tau_3 + \gamma_3\tau_4)(k_2\sigma_1 + \gamma_1\sigma_2)).$$

The linearized matrix of the system (15) around the illicit drug use and terrorism-free equilibrium is obtained $\mathscr{D}_0$ and evaluated at β^* to derived the eigenvalues λ of $\mathscr{J}_{(\mathscr{D}_0,\beta^*)}$ given by the obtained matrix, we have the roots of the characteristic equation of it in the form:

$$(\lambda + \mu)(\lambda + k_6)P(\lambda) = 0, \quad (17)$$

where $P(\lambda)$ is a polynomial of degree five whose roots are all negative except one zero eigenvalue. Further, the right eigenvector, $w = (w_1, w_2, \ldots, w_6)^T$, associated with this simple zero eigenvalue can be obtained from $w\mathscr{J}_{(\mathscr{D}_0,\beta^*)}=0$. As a result, we have

$$w_1=\frac{-(\beta^*k_3k_4k_5k_6(k_2+\eta\gamma_1)-\sigma_6p_3)}{k_2k_3k_4k_5k_6\mu}w_2,\qquad w_3=\frac{\gamma_1}{k_2}w_2,$$

$$w_4=\frac{p_0}{k_2k_3}w_2,\qquad w_6=\frac{p_0p_1+k_3\gamma_1p_2+\tau_1k_2k_3k_4}{k_2k_3k_4k_5}w_2,$$

$$w_5=\frac{\gamma_1\gamma_2k_3+\gamma_3p_0}{k_2k_3k_4}w_2,\qquad w_7=\frac{k_4k_5\sigma_3p_0+\sigma_4q_0k_5\sigma_5q_1}{k_2k_3k_4k_5k_6}w_2. \tag{18}$$

where
$p_0=\sigma_1k_2+\sigma_2\gamma_1,\qquad p_1=k_4\tau_3+\tau_4\gamma_3,$
$p_2=\tau_2k_4+\tau_4\gamma_2,\qquad p_3=k_4k_5\sigma_3p_0+\sigma_4k_5q_0+\sigma_5q_1$
$q_0=\gamma_1\gamma_2k_3+\gamma_3p_0,\qquad q_1=p_0p_1+k_3\gamma_1p_2+\tau_1k_2k_3k_4,$

Similarly, the Jacobian, $\mathscr{J}_{(\mathscr{D}_0,\beta^*)}$, has a left eigenvector, $v=(v_1,v_2,...,v_7)$, satisfying $v.w=1$,
where

$$v_1=v_7=0,\qquad v_4=\frac{\gamma_3\tau_2\alpha+k_4\tau_3\alpha}{k_3k_4k_5}v_2,\qquad v_5=\frac{\tau_2\alpha}{k_4k_5}v_2,$$

$$v_3=\frac{\beta^*\eta k_3k_4k_5+\sigma_2\alpha(\gamma_3\tau_2+k_4\tau_3)+k_3\tau_2\alpha(\gamma_2+k_4)}{k_2k_3k_4k_5}v_2, \tag{19}$$

$$v_6=\frac{\alpha}{k_5}v_2.$$

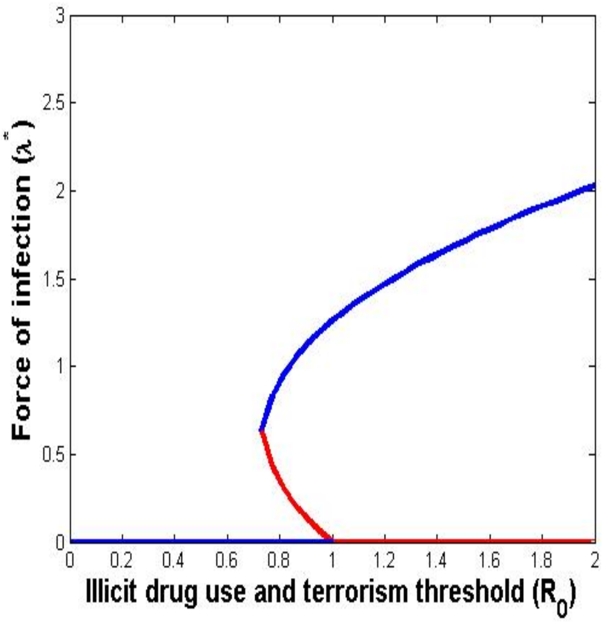
It should be noted that the components of w and v are obtained so that $v.w=1$ as required in [21]. All the second-order partial derivatives of $f_i, i=i=1,2...,7$, from the system (15) are zero at point $(\mathscr{D}_0,\beta^*)$ except the following :

$$\begin{aligned}\frac{\partial^2 f_1}{\partial x_1\partial x_2}&=\frac{\partial^2 f_1}{\partial x_2\partial x_1}=\frac{-\beta^*\mu}{\pi},\\ \frac{\partial^2 f_1}{\partial x_1\partial x_3}&=\frac{\partial^2 f_1}{\partial x_3\partial x_1}=\frac{-\beta^*\mu\eta}{\pi},\\ \frac{\partial^2 f_2}{\partial x_1\partial x_2}&=\frac{\partial^2 f_2}{\partial x_2\partial x_1}=\frac{\beta^*\mu}{\pi},\\ \frac{\partial^2 f_2}{\partial x_1\partial x_3}&=\frac{\partial^2 f_2}{\partial x_3\partial x_1}=\frac{\beta^*\mu\eta}{\pi},\end{aligned} \tag{20}$$

with

$$\begin{aligned}\frac{\partial^2 f_1}{\partial x_2\partial\beta}&=-1,\qquad \frac{\partial^2 f_1}{\partial x_3\partial\beta}=-\eta,\\ \frac{\partial^2 f_2}{\partial x_2\partial\beta}&=1,\qquad \frac{\partial^2 f_2}{\partial x_3\partial\beta}=\eta.\end{aligned} \tag{21}$$

The direction of the bifurcation at $\mathscr{R}_0=1$ is determined by the signs of the bifurcation coefficients **a** and **b**, which can be obtained by using the (20) and (21). Then, **a** and **b** are defined as follow:-



(a)

Fig. 2: Backward bifurcation diagram for model (1)

$$a=\sum_{k,i,j=1}^{6}v_kw_iw_j\frac{\partial^2 f_k}{\partial x_i\partial x_j}(\mathscr{D}_0,\beta^*) \tag{22}$$

since $v_1=v_7=0$, then (22) can be written as

$$\begin{aligned}a=&v_2w_1w_2\frac{\partial^2 f_2}{\partial x_1\partial x_2}+v_2w_2w_1\frac{\partial^2 f_2}{\partial x_2\partial x_1}\\ &+v_2w_1w_3\frac{\partial^2 f_2}{\partial x_1\partial x_3}+v_2w_3w_1\frac{\partial^2 f_2}{\partial x_3\partial x_1}\end{aligned} \tag{23}$$

and

$$b=v_2w_2\frac{\partial^2 f_2}{\partial x_2\partial\beta}+v_2w_3\frac{\partial^2 f_2}{\partial x_3\partial\beta}. \tag{24}$$

Substituting (16), (19), (18), (20) and (21) into (23) and (24) gives the following

$$a=2\frac{k_{10}(\sigma_6p_3+k_6k_{10})}{k_2k_3^2k_4^2k_5^2k_6^2(\eta\gamma_1+k_2)\Pi}\frac{k_2+\eta\gamma_1}{k_2}w_2^2v_2 \tag{25}$$

where

$$b=\sum_{k,i=1}^{6}v_kw_i\frac{\partial^2 f_k}{\partial x_i\partial\beta}(\mathscr{D}_0,\beta^*). \tag{26}$$

and
$$b=\frac{k_2+\eta\gamma_1}{k_2}w_2v_2. \qquad (27)$$
where $k_{10}=k_1k_2k_3k_4k_5-\alpha(k_3(k_2k_4\tau_1+k_4\gamma_1\tau_2+\gamma_1\gamma_2\tau_4)+(k_4\tau_3+\gamma_3\tau_4)(k_2\sigma_1+\gamma_1\sigma_2)$
It is important to note that $a>0$ and $b>0$, then illicit drug use and terrorism model (1) exhibits a backward bifurcation and the illicit drug use and terrorism-present equilibrium $\mathscr{D}^*$ is locally unstable. This is shown graphically in Figure 4 for better understanding. This result is claimed as follows.

Theorem 4. *The illicit drug use and terrorism model governed by (1) exhibits a backward bifurcation at the threshold $\mathscr{R}_0=1$ (or, equivalently, there exists a point where illicit drug use and terrorism-present equilibrium, $\mathscr{D}^*$, co-exist with illicit drug use and terrorism-free equilibrium, $\mathscr{D}_0$, .*

The implication, from epidemiological point, of the existence of backward bifurcation is that the classical condition needed of $\mathscr{R}_0<1$ is not sufficient to effectively control the problem of illicit drug use and terrorism spread in the population. The occurrence of backward bifurcation, the model exhibited by illicit drug use and terrorism model (1), is caused by movement rate of quitter to susceptible σ_6. Hence, with $\sigma_6\neq 0$, it is meaningless to investigate the global asymptotic dynamics of D_0 of model (1), because it will not be stable, rather the D^* of model (1) will investigate next.

3.5 Global stability of IDTPE

Theorem 5. *The IDTPE of the illlicit drug use and terrorism model (1), given by (14), is globally asymptotically stable in $\Gamma\setminus\Gamma_0$ if $\mathscr{R}_0>1$, $RI^{**}\le R^{**}I, S_ER^{**}\le S_E^{**}R$ and $IS_E^{**}\le I^{**}S_E$.*

Proof. Consider the non-linear Lyapunov function $\mathfrak{F}:\Gamma\setminus\Gamma_0\to\mathbb{R}$ defined by
$$\begin{aligned}\mathfrak{F}&=S-S^{**}-S^{**}\ln\frac{S}{S^{**}}+\left(I-I^{**}-I^{**}\ln\frac{I}{I^{**}}\right)\\&+\frac{\alpha\tau_3(\gamma_3+k_4)}{k_3k_4k_5}\left(D-D^{**}-D^{**}\ln\frac{D}{D^{**}}\right)\\&+\frac{\tilde{\beta}S(I+\eta S_E)k_3k_4k_5+p_5}{k_2k_3k_4k_5}\left(S_E-S_E^{**}-S_E^{*}\ln\frac{S_E}{S_E^{**}}\right)\\&+\frac{\alpha\tau_3}{k_4k_5}\left(C-C^{**}-C^{**}\ln\frac{C}{C^{**}}\right)\\&+\frac{\alpha}{k_5}\left(R-R^{**}-R^{**}\ln\frac{R}{R^{**}}\right),\end{aligned} \qquad (28)$$

let $p_5=\alpha(\tau_3(\sigma_2(\gamma_3+k_4)+\gamma_2k_3)+k_3k_4\tau_2)$. Equation (28) is of Goh-Volterra type (see, [13,20,24]). The Lyapunov derivative is given by

$$\begin{aligned}\frac{d\mathfrak{F}}{dt}&=\frac{dS}{dt}-\frac{S^{**}}{S}\frac{dS}{dt}+\left(\frac{dI}{dt}-\frac{I^{**}}{I}\frac{dI}{dt}\right)+\frac{\alpha\tau_3(\gamma_3+k_4)}{k_3k_4k_5}\\&\left(\frac{dD}{dt}-\frac{D^{**}}{D}\frac{dD}{dt}\right)+\frac{\alpha}{k_5}\left(\frac{dR}{dt}-\frac{R^{**}}{R}\frac{dR}{dt}\right)+\frac{\alpha\tau_3}{k_4k_5}\\&\left(\frac{dC}{dt}-\frac{C^{**}}{C}\frac{dC}{dt}\right)+\frac{\tilde{\beta}S(I+\eta S_E)k_3k_4k_5+p_5}{k_2k_3k_4k_5}\\&\left(\frac{dS_E}{dt}-\frac{S_E^{**}}{S_E}\frac{dS_E}{dt}\right)\end{aligned} \qquad (29)$$

Let $\frac{\beta(I(t)+\eta S_E(t))}{N(t)}=\tilde{\beta}(I(t)+\eta S_E(t))$, then putting the appropriate equations of the system (1) into (29), one obtains

$$\begin{aligned}\frac{d\mathfrak{F}}{dt}&=\left(1-\frac{S^{**}}{S}\right)(\Pi-\tilde{\beta}(I(t)+\eta S_E(t))S(t)+\rho R(t)\\&-\mu S(t))+\left(1-\frac{I^{**}}{I}\right)(\tilde{\beta}(I(t)+\eta S_E(t))S(t)\\&+\alpha R(t)-k_1I(t))+\frac{\tilde{\beta}S(I+\eta S_E)k_3k_4k_5+p_5}{k_2k_3k_4k_5}\\&\left(1-\frac{S_E^{**}}{S_E}\right)(\gamma_1I(t)-k_2S_E(t))+\frac{\alpha\tau_3(\gamma_3+k_4)}{k_3k_4k_5}\\&\left(1-\frac{D^{**}}{D}\right)(\sigma_1I(t)+\sigma_2S_E(t)-k_3D(t))+\frac{\alpha\tau_3}{k_4k_5}\\&\left(1-\frac{C^{**}}{C}\right)(\gamma_2S_E(t)+\gamma_3D(t)-k_4)C(t))+\frac{\alpha}{k_5}\\&\left(1-\frac{R^{**}}{R}\right)(\tau_1I(t)+\tau_2S_E(t)+\tau_3D(t)+\tau_4C(t)\\&-k_5R(t)).\end{aligned} \qquad (30)$$

At the illicit drug use and terrorism-present equilibrium, the following relations hold from the system (1):

$$
\begin{aligned}
\Pi &= \tilde{\beta}(I^{**}+\eta S_E^{**})S^{**} - \rho R^{**} + \mu S^{**},\\
k_1 &= \tfrac{\tilde{\beta}(I^{**}+\eta S_E^{**})}{I^{**}} + \tfrac{\alpha_1 R^{**}}{I^{**}},\\
k_2 &= \tfrac{\gamma_1 I^{**}}{S_E^{**}},\\
k_3 &= \tfrac{\sigma_1 I^{**}+\sigma_2 S_E^{**}}{D^{**}},\\
k_4 &= \tfrac{\gamma_2 S_E^{**}+\gamma_3 D^{**}}{C^{**}},\\
k_5 &= \tfrac{\tau_1 I^{**}+\tau_2 S_E^{**}+\tau_3 D^{**}+\tau_4 C^{**}}{R^{**}}.
\end{aligned} \tag{31}
$$

Using the relations (31) in (30) and simplifying yields

$$
\begin{aligned}
\frac{d\mathfrak{F}}{dt} &= \tilde{\beta}S^{**}(I^{**}+\eta S_E^{**})\left(2-\frac{S^{**}}{S}-\frac{S}{S^{**}}\right)+\\
&\mu S^{**}\left(2-\frac{S}{S^{**}}-\frac{S^{**}}{S}\right)+\alpha\rho R^{**}\left(1-\frac{RI^{**}}{R^{**}I}\right)\\
&+\tfrac{\sigma_2\alpha\tau_3(\gamma_3+k_4)}{k_3k_4k_5}S_E^{**}\left(2-\frac{ID^{**}}{I^{**}D}-\frac{C^{**}D}{CD^{**}}\right)\\
&+\tfrac{\alpha\tau_3\gamma_2}{k_4k_5}S_E^{**}\left(2-\frac{S_EC^{**}}{S_E^{**}C}-\frac{D^{**}S_E}{DS_E^{**}}\right)\\
&+\tfrac{\alpha\tau_2}{k_5}S_E^{**}\left(1-\frac{S_ER^{**}}{S_E^{**}R}\right)+k_1I^{**}\left(1-\frac{IS_E^{**}}{I^{**}S_E}\right)\\
&+\tfrac{\alpha\tau_3}{k_5}C^{**}\left(2-\frac{CR^{**}}{C^{**}R}-\frac{CD^{**}}{C^{**}D}\right)
\end{aligned} \tag{32}
$$

Since $RI^{**}\leq R^{**}I, S_ER^{**}\leq S_E^{**}R$ and $IS_E^{**}\leq I^{**}S_E$, then $1-\dfrac{R^{**}I}{RI^{**}}\leq 0,\ 1-\dfrac{S_E^{**}R}{S_ER^{**}}\leq 0$ and $1-\dfrac{I^{**}S_E}{IS_E^{**}}\leq 0$ with equality if $RI^{**}=R^{**}I,\ S_ER^{**}=S_E^{**}R$ and $IS_E^{**}=I^{**}S_E$. Consequently, (32) becomes

$$
\begin{aligned}
\frac{d\mathfrak{F}}{dt} &= \tilde{\beta}S^{**}(I^{**}+\eta S_E^{**})\left(2-\frac{S^{**}}{S}-\frac{S}{S^{**}}\right)\\
&+\mu S^{**}\left(2-\frac{S}{S^{**}}-\frac{S^{**}}{S}\right)\\
&+\tfrac{\sigma_2\alpha\tau_3(\gamma_3+k_4)}{k_3k_4k_5}S_E^{**}\left(2-\frac{ID^{**}}{I^{**}D}-\frac{C^{**}D}{CD^{**}}\right)\\
&+\tfrac{\alpha\tau_3\gamma_2}{k_4k_5}S_E^{**}\left(2-\frac{S_EC^{**}}{S_E^{**}C}-\frac{D^{**}S_E}{DS_E^{**}}\right)\\
&+\tfrac{\alpha\tau_3}{k_5}C^{**}\left(2-\frac{CR^{**}}{C^{**}R}-\frac{CD^{**}}{C^{**}D}\right)
\end{aligned} \tag{33}
$$

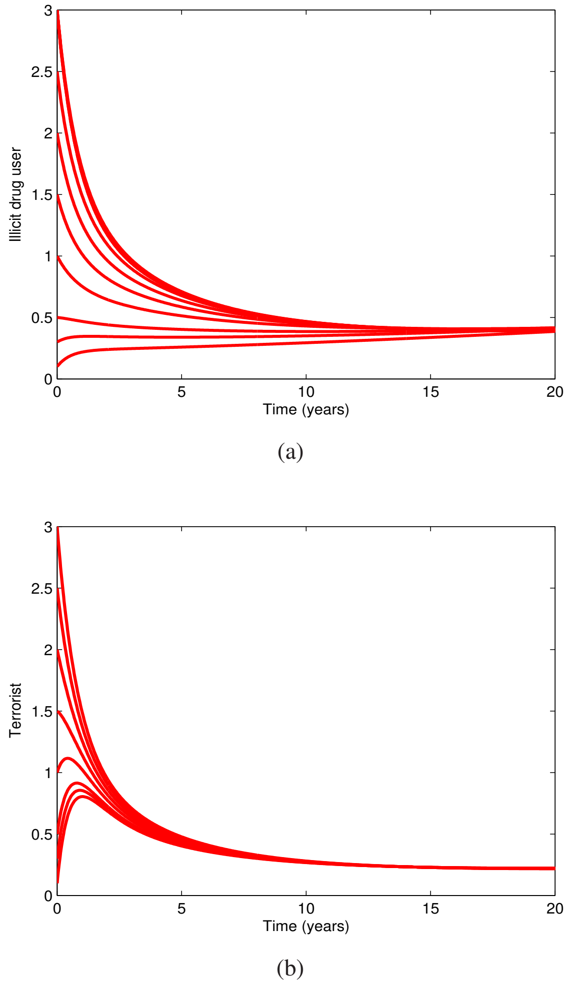


Fig. 3: Behaviour of the illicit drug use and terrorism model (1) showing the trajectories solution of different initial values of the illicit drug use and terrorism-present.That is when $\mathscr{R}_0 > 1$, Parameter values used are given in Table 4.

Using Arithmetic Mean -Geometric Mean inequality [25], the following ineqaulities hold:

$$
\left(2-\frac{S}{S^{**}}-\frac{S^{**}}{S}\right)\leq 0,\quad \left(2-\frac{ID^{**}}{I^{**}D}-\frac{C^{**}D}{CD^{**}}\right)\leq 0,
$$

$$
\left(2-\frac{S_EC^{**}}{S_E^{**}C}-\frac{D^{**}S_E}{DS_E^{**}}\right)\leq 0,\quad \left(2-\frac{CR^{**}}{C^{**}R}-\frac{CD^{**}}{C^{**}D}\right)\leq 0,
$$

Moreover, since all the model parameters are non-negative, it follows from (33) that $\frac{d\mathfrak{F}}{dt}\leq 0$ with equality if and only if $S=S^{**}$, $I=I^{**}$, $S_E=S_E^{**}$, $D=D^{**}$, $C=C^{**}$, $R=R^{**}$, $Q=Q^{**}$. Hence, by LaSalle's invariance principle [26],

$(S,I,S_E,D,C,R,Q) \to \mathscr{D}_0$ as $t \to \infty$. This completes the proof.

The implication of Theorem 5 is that illicit drug use and terrorism will persist, regardless of the initial sizes of illicit drug users and terrorists in the population, whenever $\mathscr{R}_0 > 1$. This stability property is shown in Figure 3, where all the solutions tend to illicit drug use and terrorism-present equilibrium (D^*).

4 Sensitivity Analysis

The idea in the following literature [1,12,23,24,28], were used extensive to perform the sensitivity analysis of the model (1),this was done in order to determine the contributory effects of the model parameters on the transmission and spread of the illicit drug use and terrorism menace in a population.
Normalized forward-sensitivity index of a variable was employed, v, that depends differentiably on a parameter, p, is defined as

$$\Upsilon_p^v = \frac{\partial v}{\partial p} \times \frac{p}{v}. \tag{34}$$

From the R_0 obtained at the equation (8)

$$\mathscr{R}_0 = \frac{\beta k_3 k_4 k_5 (k_2 + \eta\gamma_1)}{d_7}, \tag{35}$$

More so, sensitivity indices's of the illicit drug use and terrorism threshold, $\mathscr{R}_0$, with respect to the model parameters are obtained below and the summary is presented in the in Table 3 and figure 4.1.below

$$
\begin{aligned}
1.\sigma_1 &= \frac{-k_2k_3k_5\sigma_1}{d_7}\\
2.\gamma_1 &= \frac{\gamma_1[k_2k_3k_5A_0 + \alpha(A_1 - A_2)]}{(k_2+\eta\gamma_1)d_7}\\
3.\delta_1 &= \frac{-k_2k_3k_5\delta_1}{d_7}\\
4.\sigma_2 &= \frac{\sigma_2[\alpha(A_3 - A_4) - k_1k_3k_5\eta\gamma_1]}{(k_2+\eta\gamma_1)d_7}\\
5.\tau_1 &= \frac{-k_2k_3k_5\tau_1}{d_7}\\
6.\tau_2 &= \frac{\tau_2[\alpha(A_5 - A_4) - k_1k_3k_5\eta\gamma_1]}{(k_2+\eta\gamma_1)d_7}\\
7.\gamma_2 &= \frac{\gamma_2[\alpha(A_6 - A_4) - k_1k_3k_5\eta\gamma_1]}{(k_2+\eta\gamma_1)d_7}\\
8.\delta_2 &= \frac{\delta_2[\alpha(A_7 - A_4) - k_1k_3k_5\eta\gamma_1]}{(k_2+\eta\gamma_1)d_7}\\
9.\tau_3 &= \frac{\alpha\tau_3A_8}{k_3d_7}\\
10.\sigma_3 &= \frac{-\alpha\sigma_3(k_4\tau_3+\gamma_3\tau_4)(k_2\sigma_1+\gamma_1\sigma_2)}{k_3d_7}\\
11.\gamma_3 &= \frac{\alpha\gamma_3A_9}{k_3d_7}\\
12.\sigma_4 &= \frac{\sigma_4[k_1k_2k_3k_5 - \alpha A_{10}]}{k_4d_7}\\
13.\tau_4 &= \frac{\tau_4(k_1k_2k_3k_5 - \alpha(A_{11}+A_{12}))}{k_4d_7}\\
14.\delta_4 &= \frac{\delta_4(k_1k_2k_3k_5 - \alpha A_{13})}{k_4d_7}\\
15.\sigma_5 &= \frac{\sigma_5(k_1k_2k_3(k_5-1) - \alpha A_{14})}{k_5d_7}\\
16.\alpha &= \frac{\alpha(\sigma_5+\mu)A_{14}}{k_5d_7}\\
17.\beta &= 1 \qquad 18.\eta = \frac{\gamma_1\eta}{k_2+eta\gamma_1}\\
19.\rho &= \frac{\rho(\sigma_5+\mu)A_{14}}{k_5d_7}
\end{aligned}
\tag{36}
$$

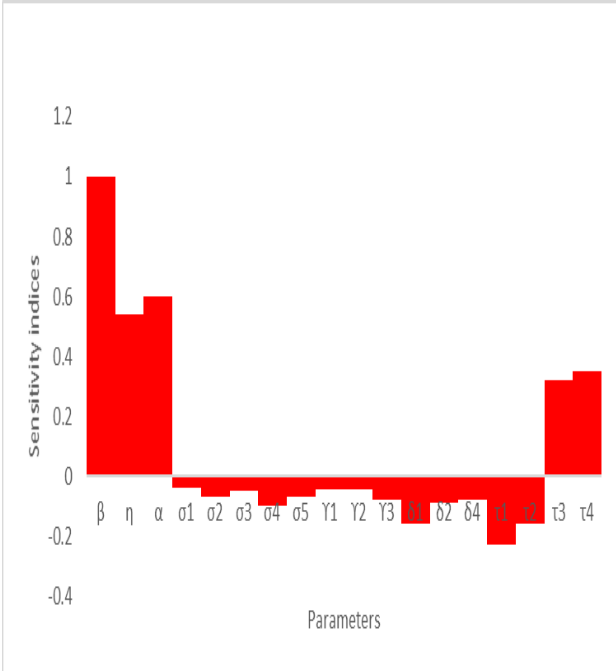
where

$$
\begin{aligned}
A_0 &= (\eta k_1 - k_2 - \eta \gamma_1) \\
A_1 &= (k_2 + \eta \gamma_1)(k_3 k_4 \tau_2 + \gamma_2 k_3 k_4 + (k_4 \tau_3 + \gamma_3 \tau_4)\sigma_2) \\
A_2 &= \eta((k_2 k_3 k_4 \tau_1 + k_3 k_4 \gamma_1 \tau_2 + k_3 \gamma_1 \gamma_2 \tau_4) \\
&\quad + (k_4 \tau_3 + \gamma_3 \tau_4)(k_2 \sigma_1 + \gamma_1 \sigma_2)) \\
A_3 &= (k_2 + \eta \gamma_1)(k_3 k_4 \tau_1 + (k_4 \tau_2 + \gamma_3 \tau_4)(\sigma_1 + \gamma_1) \\
A_4 &= k_3(k_2 k_4 \tau_1 + k_4 \gamma_1 \tau_2 + \gamma_1 \gamma_2 \tau_4) \\
&\quad + (k_4 \tau_3 + \gamma_3 \tau_4)(k_2 \sigma_1 + \gamma_1 \sigma_2) \\
A_5 &= (k_2 + \eta \gamma_1)(k_3 k_4 (\tau_1 + \gamma_1) + (k_4 \tau_2 + \gamma_3 \tau_4)\sigma_1 \\
A_6 &= (k_2 + \eta \gamma_1)(k_3 (k_4 \tau_1 + \gamma_1 \tau_4) + (k_4 \tau_2 + \gamma_3 \tau_4)\sigma_1 \\
A_7 &= (k_2 + \eta \gamma_1)(k_3 k_4 \tau_1 + (k_4 \tau_2 + \gamma_3 \tau_4)\sigma_1) \\
A_8 &= (k_2 \sigma_1 + \gamma_1 \sigma_2)[(\gamma_3 + \sigma_3 + \mu)(\sigma_4 + \delta_4 + \mu) \\
&\quad + \tau_4(\sigma_3 + \mu)] \\
A_9 &= (k_2 \sigma_1 + \gamma_1 \sigma_2)[\tau_4(\sigma_3 + \mu) - \tau_3(\sigma_4 + \delta_4 + \mu)] \\
A_{10} &= (\gamma_1 \gamma_2 \tau_4 k_3 + \gamma_3 \tau_4 (k_2 \sigma_1 + \gamma_1 \sigma_2) \\
A_{11} &= k_3[(k_2 \tau_1 + \gamma_1 \tau_2)(1 - k_4) + \gamma_1 \gamma_2 (1 - \tau_4)] \\
A_{12} &= (k_2 \sigma_1 + \gamma_1 \sigma_2)(\tau_3 (1 - k_4) + \gamma_3 (1 - \tau_4)) \\
A_{13} &= \gamma_1 \gamma_2 \tau_4 k_3 + \gamma_3 \tau_4 (k_2 \sigma_1 + \gamma_1 \tau_2) \\
A_{14} &= k_3(k_2 k_4 \tau_1 + k_4 \gamma_1 \tau_2 + \gamma_1 \gamma_2 \tau_4) \\
&\quad + (k_3 \tau_3 + \gamma_3 \tau_4)(k_2 \sigma_1 + \gamma_1 \sigma_2)
\end{aligned}
$$

Table 3. Sensitivity sign of each parameters of the model

Parameter	Sign	Parameter	Sign
β	Positive	η	Positive
γ_1	Negative	α	Positive
τ_1	Negative	τ_2	Negative
τ_3	Positive	τ_4	Positive
γ_2	Negative	γ_3	Negative
σ_1	Negative	σ_2	Negative
σ_3	Negative	σ_4	Negative
σ_5	Negative	δ_1	Negative
δ_2	Negative	δ_3	Negative
δ_4	Negative	ρ	Positive

The sign of the sensitivity index plays a key role in determining how the parameters of the model relate to the illicit drug use and terrorism threshold, $\mathscr{R}_0$, of the model.



(a)

Fig. 4: Sensitivity of illicit drug use and terrorism model (1) parameters

In Table 3, the parameters with positive sign have a direct relation on $\mathscr{R}_0$ while the ones with negative sensitivity sign as an inverse relation on $\mathscr{R}_0$. This means that increasing any parameter with positive sign will in turn increase the value of $\mathscr{R}_0$ (i.e the menace will persist) and the vice versa, while increasing any parameter with negative sign will in turn decrease the value of $\mathscr{R}_0$ (i.e the menace will fade out) and the vice versa. With sensitivity analysis, one can get insight on the appropriate intervention strategies to prevent and control the effect and spread of the illicit drug use and terrorism in the population. This means that effect should be made to reduce the value of parameters with positive sign, in like manner, the value of parameters with negative sign must be increase at all cost.

In addition, the simulations of the illicit drug use and terrorism model (1) are conducted, using the same parameter values in Table 4.

Table 4. Values of parameters of the model

Parameter	Range	Baseline value	Source
Π	40 - 60	50	[31,32]
β	0.2 - 0.5	0.64	[14,30]
η	0.6 - 0.9	0.7	[29]
μ	0.009 - 0.04	0.02	[29]
α	0.6 - 0.9	0.75	[31,32]
σ_1	0.5 - 0.8	0.65	[29]
σ_2	0.5 - 0.8	0.6	[31,29]
σ_3	0.6 - 0.9	0.7	[30]
σ_4	0.1 - 0.3	0.2	[30]
σ_5	0.4 - 0.9	0.55	Assumed
σ_6	0.4 - 0.9	0.55	Assumed
γ_1	0.5 - 0.8	0.6	Assumed
γ_2	0.5 - 0.8	0.65	[30]
γ_3	0.6 - 0.9	0.7	[30]
δ_1	0.09 - 0.2	0.14	[31]
δ_2	0.09 - 0.2	0.14	[31]
δ_3	0.08 - 0.2	0.14	[31]
ρ	0.6 - 0.8	0.77	[31]
τ_1	0.7 - 0.95	0.85	[31]
τ_2	0.05 - 0.2	0.1	[4,31]
τ_3	0.7 - 0.95	0.85	[31]
τ_4	0.05 - 0.2	0.1	[4,31]

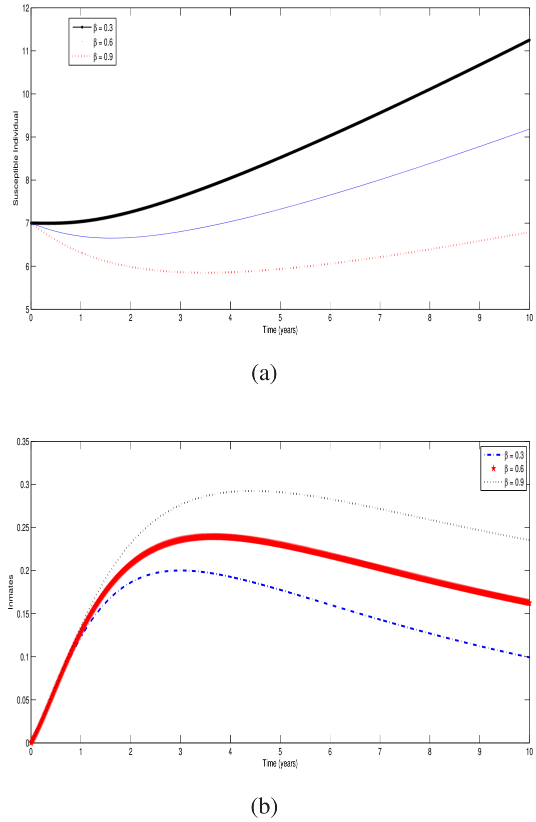


Fig. 5: Simulation results of model (1) with the effect of β with susceptible (a) and illicit drug user (b)

5 Numerical Simulation

Numerical Simulation of the model (1), It can be observed in Figure 5 that as the effective influence rate raises (reduces), the problem of illicit drug use and terrorism raises (reduces) in the population. Similarly, any raising (reducing) in the movement of rehabilitating individuals back to susceptible population will raise (reduce) the problem of illicit drug use and terrorism in the population. This is buttressing the fact that both effective influence rate and movement of rehabilitating individuals back to susceptible population has a positive sensitivity sign. It is important to note that movement of rehabilitating individuals back to susceptible population must be monitored closely, because of it significant in this model (1). The physical meaning of this is that the higher the movement of rehabilitating individuals back to susceptible population, the higher the number of the illicit drug users and terrorist in the population.

Thus, movement of rehabilitating individuals back to susceptible population, is a serious parameter that must be targeted with a control if the spread of illicit drug use and terrorism in the population will be reduce.

6 Conclusion

A mathematical model approach was used to studied the co-dynamics between illicit drug use and terrorism in a population, this was done with the help of nonlinear system of ordinary differential equations. The model was studied the to know the aid that illicit drug provides for those who are involved in terrorism within a population. Basic properties of the model was established. The equilibrium points of the model were obtained, which are illicit drug use and terrorism-free (IDTFE) and illicit drug use and terrorism -present (IDTPE) equilibria. The illicit drug use and terrorism basic reproduction number, $\mathscr{R}_0$ was derived, this tells us the probability whether the illicit drug use and terrorism will be wipe out or persist in the population. Because of the nature of the illicit drug use and terrorism -present (IDTPE) equilibrium, bifurcation analysis of the model was carried out, the model exhibits backward bifurcation properties, which shows that the model is not globally asymptotically stable whenever $\mathscr{R}_0 < 1$. A rigorous analysed on the model shows the long-time behaviours of solution trajectories about IDTPE. Using Lyapunov stability methods, the model was shown to be globally asymptotically stable $\mathscr{R}_0 1$. Sensitivity analysis of the model was carried out to access the impact of each parameter on the dynamic spread of illicit drug use and terrorism in the population.

Furthermore, the numerical simulations, was performed to validate all the theoretical analyses It can be deduce from the analysis that the menace of illicit drug use and terrorism can be curbed if measures are in place to prevent the increase in the influence rate. It is also shows that increasing the rates of rehabilitating drug users and terrorists have positive effect in reducing the burden of illicit drug use and terrorism in the population. Therefore, this study provides a lasting solution to the problem of illicit drug use and terrorism in a population.

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Conflict of interest

The authors declare that there is no conflict regarding the publication of this paper.

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