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Moments of Dual Generalized Order Statistics from Extended Erlang-Truncated Exponential Distribution

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Abstract: In this paper, we have derived explicit expressions and some recurrence relations for single and product moments of dual generalized order statistics from extended Erlang-truncated exponential distribution. These relations are used to discuss the special cases of dual generalized order statistics viz. order statistics and lower record values. Further, we have also characterized this distribution using conditional moments and recurrence relations for single moments of dual generalized order statistics.

Keywords: Dual generalized order statistics, order statistics, lower record values, single moments, product moments, recurrence relations, extended Erlang-truncated exponential distribution and characterization

1 Introduction

The concept of generalized order statistics was introduced by Kamps (1995) as the unified approach of models of ordered random variables. The model of *gos* contains special cases as order statistics, sequential order statistics and record values. Pawlas and Szynal (2001) introduced the concept of lower generalized order statistics (*lgos*), which was further studied by Burkschat *et al.* (2003) as a dual generalized order statistics (*dgos*). The *dgos* models enable us to study decreasingly ordered random variables such as reversed order statistics, $k-$th lower record values and lower Pfeiffer's records, through a common approach as follows:
Let $n \in N$, $k \geq 1$, then $\widetilde{m} = (m_1, m_2, \ldots, m_{n-1}) \in \Re^{n-1}$ be the parameters such that $\gamma_r = k + n - r + M_r$, $M_r = \sum_{j=r}^{n-1} m_j$, for $1 \leq r \leq n$. By the *dgos* from an absolutely continuous distribution function (*df*) $F()$ with the probability density function (*pdf*) $f()$ we mean random variables $X^*(1,n,m,k), X^*(2,n,m,k), \ldots, X^*(n,n,m,k)$, having a joint density function of the form

$$k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[F(x_i)]^{m_i} f(x_i)\right)[F(x_n)]^{k-1} f(x_n), \tag{1}$$

on the cone $F^{-1}(1) > x_1 \geq x_2 \geq \ldots . > F^{-1}(0)$.
There are two cases of *dgos* given as:
case I. $m_i = m_j = m, \quad i,j = 1,2,\ldots,n-1.$
case II. $\gamma_i \neq \gamma_j, \quad i \neq j,\ i,j = 1,2,\ldots,n-1.$
If $m_i = m_j = m$, the corresponding *dgos* is called *m-dgos*. In this paper, we have considered the case I.
The *pdf* of $r-$ th *dgos* $X^*(r,n,m,k), 1 \leq r \leq n$ is

$$f_{X^*(r,n,m,k)}(x) = \frac{C_{r-1}}{(r-1)!}[F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)). \tag{2}$$

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The joint *pdf* of $X^*(r,n,m,k)$ and $X^*(s,n,m,k), 1 \leq r < s \leq n$ is

$$f_{X^*(r,n,m,k),X^*(s,n,m,k)}(x,y) = \frac{C_{s-1}}{(r-1)!(s-r-1)!}[F(x)]^m f(x) g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1} [F(y)]^{\gamma_s - 1} f(y), \quad x > y. \tag{3}$$

where

$$C_{r-1} = \prod_{i=1}^{r} \gamma_i, \qquad h_m(x) = \begin{cases} -\frac{1}{m+1}(x)^{m+1}, & m \neq 1 \\ -\ln(x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(1), \quad x \in [0,1).$$

If $m = 0$ and $k = 1$, then $X^*(r,n,m,k)$ reduces to $(n-r+1)-$th order statistics, and when $m = -1$, $X^*(r,n,m,k)$ reduces to the $k-$th lower record values.
Many authors have utilized the concept of *dgos* in their work. References are Ahsanullah (2005), Mbah and Ahsanullah (2007), Khan *et al*. (2008), Khan and Kumar (2010, 2011), Khan *et al*. (2012), Khan and Khan (2015), Khan and Iqrar (2019) and many more.
A random variable X is said to have extended Erlang-truncated exponential distribution if its *pdf* is given by

$$f(x) = \alpha\beta \left(1 - e^{-\lambda}\right) e^{-\beta(1-e^{-\lambda})x} \left(1 - e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha-1}, \quad x > 0, \ \alpha, \ \beta, \ \lambda > 0 \tag{4}$$

with the *df*

$$F(x) = \left(1 - e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha}, \quad x > 0, \ \alpha, \ \beta, \ \lambda > 0. \tag{5}$$

Using (4) and (5), we get

$$\frac{f(x)}{F(x)} = \frac{\alpha\beta\left(1 - e^{-\lambda}\right)}{e^{\beta(1-e^{-\lambda})x} - 1}. \tag{6}$$

The extended Erlang-truncated exponential distribution has been introduced by Okorie *et al.* (2017) as a new life time distribution. This distribution is useful for analysing decreasing and unimodal data sets. The hazard rate function of this distribution could be increasing, decreasing or constant, depending upon the values of shape parameter (α).
The remaining part of the paper organized as follows. In section 2, we have presented the exact expressions and recurrence relations for single moments of *dgos* and also discussed the special cases for these relations. In section 3, exact expressions and recurrence relations are presented for product moments of *dgos*, and we also discuss the special cases for these relations. The relations presented in sections 2 and 3 generalized the results given by Khan and kumar (2011). In section 4, we have discussed the characterizing results using conditional moments and recurrence relation for single moments of *dgos*.

2 Relations for Single Moments

In this section, the exact expressions and recurrence relations for single moments of *dgos* are deduced.
Theorem 2.1. For the distribution given in (4) and $1 \leq r \leq n, k = 1,2,\ldots, m \neq -1$,

$$E\left[X^{*j}(r,n,m,k)\right] = \frac{C_{r-1}}{\left[\beta(1-e^{-\lambda})\right]^j (r-1)!(m+1)^{r-1}} \sum_{p=0}^{\infty} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{\alpha_p(j)}{\gamma_{r-u} + (j+p)/\alpha}. \tag{7}$$

Proof. Using (2), we have

$$E\left[X^{*j}(r,n,m,k)\right] = \frac{C_{r-1}}{(r-1)!} \int_0^{\infty} x^j [F(x)]^{\gamma_r - 1} f(x) g_m^{r-1}(F(x)) dx.$$

On expanding $g_m^{r-1}(F(x)) = \left\{\frac{1}{m+1}(1-(F(x))^{m+1})\right\}^{r-1}$ binomially, we get

$$E\left[X^{*j}(r,n,m,k)\right] = \frac{C_{r-1}}{(r-1)!(m+1)^{r-1}}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\int_0^{\infty} x^j\left[F(x)\right]^{\gamma_{r-u}-1} f(x)dx. \tag{8}$$

Substituting $[F(x)]^{1/\alpha} = z$ in (8), we get

$$E\left[X^{*j}(r,n,m,k)\right] = \frac{\alpha C_{r-1}}{\left[\beta\left(1-e^{-\lambda}\right)\right]^j(r-1)!(m+1)^{r-1}}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\int_0^1\left[-\log(1-z)\right]^j z^{\alpha\gamma_{r-u}-1}dz. \tag{9}$$

we have the logarithmic expansion

$$[-\ln(1-t)]^j = \left(\sum_{p=1}^{\infty}\frac{t^p}{p}\right)^j = \sum_{p=0}^{\infty}\alpha_p(j)t^{j+p}, \tag{10}$$

where $\alpha_p(j)$ is the coefficient of t^{j+p} in the above expansion (see Balakrishnan and Cohen (1991)).
Using (10), (9) can be expressed as

$$E\left[X^{*j}(r,n,m,k)\right] = \frac{\alpha C_{r-1}}{\left[\beta\left(1-e^{-\lambda}\right)\right]^j(r-1)!(m+1)^{r-1}}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u\alpha_p(j)\binom{r-1}{u}\int_0^1 z^{\alpha\gamma_{r-u}+j+p-1}dz. \tag{11}$$

On simplifying (11), we obtain the required result.
Identity 2.1. For $\gamma_r \geq 1$, $k \geq 1$ and $m \neq -1$,

$$\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\frac{1}{\gamma_{r-u}} = \frac{(r-1)!(m+1)^{r-1}}{\prod_{t=1}^{r}\gamma_t}. \tag{12}$$

Proof. This identity can be proved by setting $j = 0$ and $p = 0$ in (7).
Special cases:
(i). Putting $m = 0$ and $k = 1$ in (7), we get the explicit expression for the moments of order statistics from extended Erlang-truncated exponential distribution as given below

$$E(X_{n-r+1:n}^j) = \frac{C_{r:n}}{\left[\beta\left(1-e^{-\lambda}\right)\right]^j}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\frac{\alpha_p(j)}{(n-r+u+1)+(j+p)/\alpha},$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$.
(ii). For $m = -1$, we see that (7) is in indeterminate form as

$$\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u} = 0 \text{ (see Balakrishnan and Cohen (1991)).}$$

From (7),

$$E\left[X^{*j}(r,n,m,k)\right] = \frac{C_{r-1}}{\left[\beta\left(1-e^{-\lambda}\right)\right]^j(r-1)!}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}$$
$$\times\frac{\alpha_p(j)\left((k+(n-r+u)(m+1))+(j+p)/\alpha\right)^{-1}}{(m+1)^{r-1}}. \tag{13}$$

Since (7) is in the indeterminate form, when applying L' Hospital rule on (13) and differentiating the numerator and denominator of (13) by $(r-1)$ times with respect to m, we get

$$E\left[X^{*j}(r,n,m,k)\right] = \frac{C_{r-1}}{\left[\beta\left(1-e^{-\lambda}\right)\right]^j}\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}(-1)^{u+r-1}\binom{r-1}{u}$$
$$\times\frac{\alpha_p(j)(n-r+u)^{r-1}}{(r-1)!\left((k+(n-r+u)(m+1))+(j+p)/\alpha\right)^r}.$$

Taking limit $m \to -1$ on both sides, we get

$$E(X_{L(r)}^{(k)})^j = \frac{k^r}{\left[\beta\left(1-e^{-\lambda}\right)\right]^j (r-1)!} \sum_{p=0}^{\infty} \sum_{u=0}^{r-1} (-1)^{u+r-1} \binom{r-1}{u} \frac{(n-r+u)^{r-1} \alpha_p(j)}{(k+(j+p)/\alpha)^r}. \tag{14}$$

However, for all integers $n \geq 0$ and for all real numbers x, from Ruiz (1996), we have

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (x-i)^n = n!. \tag{15}$$

Using (14) and (15), we obtain the exact expression for the moments of $k-$ th lower record values from the extended Erlang-truncated exponential distribution as given bellow

$$E(X_{L(r)}^{(k)})^j = \frac{k^r}{\left[\beta\left(1-e^{-\lambda}\right)\right]^j} \sum_{p=0}^{\infty} \binom{r-1}{u} \frac{\alpha_p(j)}{(k+(j+p)/\alpha)^r},$$

as obtained by Singh and Khan (2018).
Remarks: When $\lambda \to \infty$ in (2.1), we obtain the exact expression for the moments of *dgos* from generalized exponential distribution as (see Khan and kumar (2011))

$$E\left[X^{*j}(r,n,m,k)\right] = \frac{C_{r-1}}{\beta^j (r-1)!(m+1)^{r-1}} \sum_{p=0}^{\infty} \sum_{u=0}^{r-1} (-1)^u \binom{r-1}{u} \frac{\alpha_p(j)}{\gamma_{r-u}+(j+p)/\alpha}.$$

Numerical computation
We have computed the means of order statistics for the arbitrary values of parameters α, β and λ for various sample sizes $n = 1, 2, \ldots, 10$ by using the results of special case (i).

Table 1
$\alpha = 0.5, \beta = 1, \lambda = 1$

n	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
1	0.97087									
2	0.35975	1.58198								
3	0.19219	0.694892	2.02552							
4	0.12072	0.40660	0.98318	2.37297						
5	0.08325	0.27062	0.61057	1.23159	2.65831					
6	0.06103	0.19433	0.42320	0.79795	1.44841	2.90029				
7	0.04673	0.14683	0.31308	0.57003	0.96888	1.64022	3.11030			
8	0.03697	0.11509	0.24204	0.43148	0.70858	1.12507	1.81193	3.29578		
9	0.02999	0.09276	0.19323	0.33968	0.54624	0.83845	1.26837	1.96723	3.46185	
10	0.02483	0.07643	0.15809	0.27520	0.43639	0.65607	0.96003	1.40052	2.10891	3.61218

Table 2
$\alpha = 0.5, \beta = 1, \lambda = 2$

n	r=1	r=2	r=3	r=4	r=5	r=6	r=7	r=8	r=9	r=10
1	0.97087									
2	0.35975	1.58198								
3	0.19219	0.694892	2.02552							
4	0.12072	0.40660	0.98318	2.37297						
5	0.08325	0.27062	0.61057	1.23159	2.65831					
6	0.06103	0.19433	0.42320	0.79795	1.44841	2.90029				
7	0.04673	0.14683	0.31308	0.57003	0.96888	1.64022	3.11030			
8	0.03697	0.11509	0.24204	0.43148	0.70858	1.12507	1.81193	3.29578		
9	0.02999	0.09276	0.19323	0.33968	0.54624	0.83845	1.26837	1.96723	3.46185	
10	0.02483	0.07643	0.15809	0.27520	0.43639	0.65607	0.96003	1.40052	2.10891	3.61218

By using the fact $\sum_{i=1}^{n} X_{i:n}^j = nE(X)^j$ (David and Nagaraja (2003)), we can test the validity of the calculated results for the moments of order statistics.

Recurrence relations for single moments:
Theorem 2.2. For the distribution as given in (4) and $m \in \mathbb{R}$, $n \in \mathbb{N}$, $2 \le r \le n$, $n \ge 2$, $k = 1,2,\ldots$

$$E\left[X^{*j}(r,n,m,k)\right] - E\left[X^{*j}(r-1,n,m,k)\right] = \frac{j}{\alpha\beta\gamma_r(1-e^{-\lambda})}\left\{E\left[X^{*j-1}(r,n,m,k)\right] - E\left[\phi(X^*(r,n,m,k))\right]\right\}, \quad (16)$$

where $\phi(x) = x^{j-1}e^{\beta(1-e^{-\lambda})x}$.
Proof: As viewed by khan *et al*. (2008), note that

$$E\left[X^{*j}(r,n,m,k)\right] - E\left[X^{*j}(r-1,n,m,k)\right] = -\frac{jC_{r-1}}{\gamma_r(r-1)!}\int_0^\infty x^{j-1}\left[\bar{F}(x)\right]^{\gamma_r} g_m^{r-1}(F(x))dx, \quad (17)$$

on substituting (6) in (17); after simplification, we get the required result.
Special cases:
(i). Setting $m = 0$ in (16), we get the recurrence relation for single moments of order statistics from extended Erlang-truncated exponential distribution given by

$$E(X^j_{n-r+1:n}) - E(X^j_{n-r+2:n}) = \frac{j}{\alpha\beta(n-r+1)(1-e^{-\lambda})}\left\{E(X^{j-1}_{n-r+1:n}) - E\left(\phi(X_{n-r+1:n})\right)\right\}. \quad (18)$$

(ii) For $m = -1$ in (16), we get the recurrence relation for moments of k-th lower record values from extended Erlang-truncated exponential distribution given by

$$E(X^{(k)}_{L(r)})^j - E(X^{(k)}_{L(r-1)})^j = \frac{j}{\alpha\beta k(1-e^{-\lambda})}\left\{E(X^{(k)}_{L(r)})^{j-1} - E(\phi(X^{(k)}_{L(r)}))\right\}, \quad (19)$$

as obtained by Singh and Khan (2018).
Remarks.
(i) When $\lambda \to \infty$ in (17), we get the recurrence relation for single moments of *dgos* from generalized exponential distribution as

$$E\left[X^{*j}(r,n,m,k)\right] - E\left[X^{*j}(r-1,n,m,k)\right] = \frac{j}{\alpha\beta\gamma_r}\left\{E\left[X^{*j-1}(r,n,m,k)\right] - E\left[\phi(X^*(r,n,m,k))\right]\right\}.$$

as obtained by Khan and kumar (2011).
(ii) When $\lambda \to \infty$ in (18) and (19), we get the recurrence relations for single moments of order statistics and $k-$ th lower record values from generalized exponential distribution, respectively, as

$$E(X^j_{n-r+1:n}) - E(X^j_{n-r+2:n}) = \frac{j}{\alpha\beta(n-r+1)}\left\{E(X^{j-1}_{n-r+1:n}) - E\left(\phi(X_{n-r+1:n})\right)\right\},$$

$$E(X^{(k)}_{L(r)})^j - E(X^{(k)}_{L(r-1)})^j = \frac{j}{\alpha\beta k}\left\{E(X^{(k)}_{L(r)})^{j-1} - E(\phi(X^{(k)}_{L(r)}))\right\}.$$

3 Relations for Product Moments

In this section, the exact expressions and recurrence relations for product moments of *dgos* are deduced.
Theorem 3.1. For the distribution as given in (4) and $1 \le r < s \le n$, $k = 1,2,\ldots$, $m \ne -1$

$$E\left[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)\right] = \frac{C_{s-1}}{\left[\beta(1-e^{-\lambda})\right]^{i+j}(r-1)!(s-r-1)!(m+1)^{s-2}}$$
$$\times \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}\frac{\alpha_p(j)\alpha_q(i)}{[\gamma_{s-v}+(p+j)/\alpha][\gamma_{r-u}+(p+q+i+j)/\alpha]}. \quad (20)$$

Proof. On using (3), we have

$$E\left[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)\right] = \frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$
$$\times \int_0^\infty x^i\left[\bar{F}(x)\right]^m f(x)\left[1-(\bar{F}(x))^{m+1}\right]^{r-1} I(x)dx, \quad (21)$$

where

$$I(x)=\int_0^x y^j\left[(F(x))^{m+1}-(F(y))^{m+1}\right]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dy.$$

On expanding $\left[(F(x))^{m+1}-(F(y))^{m+1}\right]^{s-r-1}$ binomially and simplifying, we get

$$I(x)=\sum_{v=0}^{s-r-1}(-1)^v\binom{s-r-1}{v}[F(x)]^{(s-r-1-v)(m+1)}\int_0^x y^j[F(y)]^{\gamma_{s-v}-1}f(y)dy. \tag{22}$$

On Substituting $[F(y)]^{1/\alpha}=t$ in (22) and simplifying, we get

$$I(x)=\alpha\sum_{p=0}^{\infty}\sum_{v=0}^{s-r-1}(-1)^v\binom{s-r-1}{v}\frac{\alpha_p(j)\,[F(x)]^{\gamma_{r+1}+(p+j)/\alpha}}{\left[\beta(1-e^{-\lambda})\right]^j[\alpha\gamma_{s-v}+p+j]}. \tag{23}$$

On substituting (23) in (21), we get

$$E\left[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)\right]=\frac{\alpha C_{s-1}}{\left[\beta(1-e^{-\lambda})\right]^j(r-1)!(s-r-1)!(m+1)^{s-2}}$$
$$\times\sum_{p=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}\frac{\alpha_p(j)}{[\alpha\gamma_{s-v}+p+j]}\int_0^{\infty}x^i[F(x)]^{\gamma_{r-u}+(p+j)/\alpha-1}dx. \tag{24}$$

Again, substituting $[F(x)]^{1/\alpha}=z$ in (24); and after simplifying, we get the required result.
Identity 3.1. For $\gamma_r,\ \gamma_s\geq 1,\ k\geq 1,\ 1\leq r<s\leq n$ and $m\neq -1$

$$\sum_{v=0}^{s-r-1}(-1)^v\binom{s-r-1}{v}\frac{1}{\gamma_{s-v}}=\frac{(s-r-1)!(m+1)^{s-r-1}}{\prod_{t=r+1}^{s}\gamma_t}. \tag{25}$$

Proof. At $i=j=0$ in (20), we have

$$1=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}\frac{\alpha_p(0)\alpha_q(0)}{[\gamma_{s-v}+p/\alpha][\gamma_{r-u}+(p+q)/\alpha]}.$$

In view of Bakoban and Ibrahim (2008), for $i=j=0$,

$$\begin{cases}\alpha_p(0)=\alpha_q(0)=1, & p=q=0\\ \alpha_p(0)=\alpha_q(0)=0, & p,q>0\end{cases}$$

Therefore,

$$\sum_{v=0}^{s-r-1}(-1)^v\binom{s-r-1}{v}\frac{1}{\gamma_{s-v}}=\frac{(r-1)!(s-r-1)!(m+1)^{s-2}}{C_{s-1}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\frac{1}{\gamma_{r-u}}}$$

Now, using (12), we obtain the result given in (25).
Special cases
(i) Putting $m=0,\ k=1$ in (20), we obtain the exact expression for product moments of order statistics from extended Erlang-truncated exponential distribution as given by

$$E[X^i_{n-r+1}X^j_{n-s+1}]=\frac{C_{r:s,n}}{\left[\beta(1-e^{-\lambda})\right]^{i+j}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$
$$\times\frac{\alpha_p(j)\alpha_q(i)}{[(n-s+v+1)+(p+j)/\alpha][(n-r+u+1)+(p+q+i+j)/\alpha]}, \tag{26}$$

where

$$C_{r:s,n}=\frac{n!}{(r-1)!(s-r-1)!(n-s)!}.$$

(ii) For $m=-1$, (20) is in indeterminate form

$$\text{as } \sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}=0 \quad \text{and} \quad \sum_{v=0}^{s-r-1}(-1)^v\binom{s-r-1}{v}=0.$$

Consider

$$E\left[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)\right]=\frac{1}{\left[\beta(1-e^{-\lambda})\right]^{i+j}(r-1)!(s-r-1)!}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\alpha_p(j)\alpha_q(i)$$
$$\times\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\frac{[\gamma_{r-u}+(p+q+i+j)/\alpha]^{-1}}{(m+1)^{r-1}(C_{s-1})^{-1}}\sum_{v=0}^{s-r-1}(-1)^v\binom{s-r-1}{v}\frac{[\gamma_{s-v}+(p+j)/\alpha]^{-1}}{(m+1)^{s-r-1}}.$$

Now, applying the L'Hospital rule independently on the series in the above expression and using the result given in (15), we obtain the exact expression for product moments of $k-$ th lower record values from extended Erlang-truncated exponential distribution as given by

$$E[(X_{L(r)}^{(k)})^i(X_{L(s)}^{(k)})^j]=\frac{k^s}{\left[\beta(1-e^{-\lambda})\right]^{i+j}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\alpha_p(j)\alpha_q(i)}{[k+(p+q+i+j)/\alpha]^r\,[k+(p+j)/\alpha]^{s-r}}, \tag{27}$$

as obtained by Singh and Khan (2018).

Remarks

(i) When $\lambda\to\infty$ in (20), this expression reduces to the product moment of *lgos* from generalized exponential distribution as

$$E\left[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)\right]=\frac{C_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}\beta^{i+j}}$$
$$\times\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}\frac{\alpha_p(j)\alpha_q(i)}{[\gamma_{s-v}+(p+j)/\alpha]\,[\gamma_{r-u}+(p+q+i+j)/\alpha]}, \tag{28}$$

as obtained by Khan and kumar (2011).

(ii). When $\lambda\to\infty$ in (26) and (27), we get the exact expressions for product moments of order statistics and $k-$ th lower record values from generalized exponential distribution respectively as

$$E[X_{n-r+1}^iX_{n-s+1}^j]=\frac{C_{r:s,n}}{\beta^{i+j}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}$$
$$\times\frac{\alpha_p(j)\alpha_q(i)}{[(n-s+v+1)+(p+j)/\alpha][(n-r+u+1)+(p+q+i+j)/\alpha]},$$

$$E[(X_{L(r)}^{(k)})^i(X_{L(s)}^{(k)})^j]=\frac{k^s}{\beta^{i+j}}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\alpha_p(j)\alpha_q(i)}{[k+(p+q+i+j)/\alpha]^r\,[k+(p+j)/\alpha]^{s-r}}.$$

as obtained by Khan and kumar (2011).

(iii) At $j=0$ and $p=0$ in (20), we have

$$E\left[X^{*i}(r,n,m,k)\right]=\frac{C_{s-1}}{\left[\beta(1-e^{-\lambda})\right]^i(r-1)!(s-r-1)!(m+1)^{s-2}}$$
$$\times\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}\sum_{v=0}^{s-r-1}(-1)^{u+v}\binom{r-1}{u}\binom{s-r-1}{v}\frac{\alpha_q(i)}{\gamma_{s-v}\,[\gamma_{r-u}+(q+i)/\alpha]}. \tag{29}$$

Making the use of identity 3.1. in (29) and simplifying the resulting expression, we get

$$E\left[X^{*i}(r,n,m,k)\right]=\frac{C_{r-1}}{\left[\beta\left(1-e^{-\lambda}\right)\right]^i(r-1)!(m+1)^{r-1}}\sum_{q=0}^{\infty}\sum_{u=0}^{r-1}(-1)^u\binom{r-1}{u}\frac{\alpha_q(i)}{\gamma_{r-u}+(q+i)/\alpha},$$

This is the exact expression for single moments as given in (7).
Recurrence relations for product moments
Theorem 3.2. For the distribution given in (4) and $2 \leq r < s \leq n$, $n \geq 2$, $k = 1,2,\ldots$

$$E\left[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)\right] - E\left[X^{*i}(r,n,m,k)X^{*j}(s-1,n,m,k)\right]$$
$$= \frac{j}{\alpha\beta\gamma_s(1-e^{-\lambda})}\left\{E\left[X^{*i}(r,n,m,k)X^{*j-1}(s,n,m,k)\right] - E\left[\phi(X^*(r,n,m,k),X^*(s,n,m,k))\right]\right\}, \quad (30)$$

where
$$\phi(x,y) = x^i y^{j-1} e^{\beta(1-e^{-\lambda})y}.$$

Proof: Khan *et al*. (2008) have shown that for $1 \leq r < s \leq n$, $n \geq 2$ and $k = 1,2,\ldots$

$$E\left[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)\right] - E\left[X^{*i}(r,n,m,k)X^{*j}(s-1,n,m,k)\right] = -\frac{jC_{s-1}}{\gamma_s(r-1)!(s-r-1)!}$$
$$\times \int_0^\infty \int_0^x x^i y^{j-1}\left[F(x)\right]^m f(x) g_m^{r-1}(F(x))\left[h_m(F(y)) - h_m(F(x))\right]^{s-r-1}\left[F(y)\right]^{\gamma_s} dy dx. \quad (31)$$

On substituting (6) in (31) and simplifying, we get the required result.
Special cases
(i) Putting $m = 0, k = 1$ in (30), we obtain the recurrence relation for product moments of order statistics from extended Erlang-truncated exponential distribution as given by

$$E[X^i_{n-r+1:n}X^j_{n-s+1:n}] - E[X^i_{n-r+1:n}X^j_{n-s+2:n}]$$
$$= \frac{j}{\alpha\beta(1-e^{-\lambda})(n-s+1)}\left\{E[X^i_{n-r+1:n}X^{j-1}_{n-s+1:n}] - E[\phi(X_{n-r+1:n}, X_{n-s+1:n})]\right\}. \quad (32)$$

(ii) For $m = -1$ in (30), we obtain the recurrence relation for product moments of $k-$ th lower record values from extended Erlang-truncated exponential distribution as given by

$$E[(X^{(k)}_{L(r)})^i(X^{(k)}_{L(s)})^j] - E[(X^{(k)}_{L(r)})^i(X^{(k)}_{L(s-1)})^j]$$
$$= \frac{j}{\alpha\beta k(1-e^{-\lambda})}\left\{E[(X^{(k)}_{L(r)})^i(X^{(k)}_{L(s)})^{j-1}] - E[\phi(X^{(k)}_{L(r)}, X^{(k)}_{L(s)})]\right\}. \quad (33)$$

Remark. (i). When $\lambda \to \infty$ in (30), we get the recurrence relation for product moments of *lgos* from generalized exponential distribution as

$$E\left[X^{*i}(r,n,m,k)X^{*j}(s,n,m,k)\right] - E\left[X^{*i}(r,n,m,k)X^{*j}(s-1,n,m,k)\right]$$
$$= \frac{j}{\alpha\beta\gamma_s}\left\{E\left[X^{*i}(r,n,m,k)X^{*j-1}(s,n,m,k)\right] - E\left[\phi(X^*(r,n,m,k),X^*(s,n,m,k))\right]\right\},$$

as obtained by Khan and kumar (2011).
(ii). When $\lambda \to \infty$ in (32) and (33)), we get the recurrence relations for product moments of order statistics and $k-$ th lower record values from generalized exponential distribution, respectively, as

$$E[X^i_{n-r+1:n}X^j_{n-s+1:n}] - E[X^i_{n-r+1:n}X^j_{n-s+2:n}]$$
$$= \frac{j}{\alpha\beta(n-s+1)}\left\{E[X^i_{n-r+1:n}X^{j-1}_{n-s+1:n}] - E[\phi(X_{n-r+1:n}, X_{n-s+1:n})]\right\},$$

$$E[(X^{(k)}_{L(r)})^i(X^{(k)}_{L(s)})^j] - E[(X^{(k)}_{L(r)})^i(X^{(k)}_{L(s-1)})^j]$$
$$= \frac{j}{\alpha\beta k}\left\{E[(X^{(k)}_{L(r)})^i(X^{(k)}_{L(s)})^{j-1}] - E[\phi(X^{(k)}_{L(r)}, X^{(k)}_{L(s)})]\right\},$$

as obtained by Khan and kumar (2011).

4 Characterization

In this section, we have characterized the extended Erlang-truncated exponential distribution using conditional moments and recurrence relation for single moments of *dgos*.
Let $X^*(r,n,m,k), r=1,2,\dots,n$ be *dgos* from a continuous population with *pdf* $f(x)$ and *cdf* $F(x)$ then the conditional distribution of $X^*(s,n,m,k)$ given $X^*(r,n,m,k)=x$, for $1 \le r < s \le n$ is given by

$$f_{X^*(s,n,m,k)|X^*(r,n,m,k)}(y|x) = \frac{C_{s-1}}{(s-r-1)!C_{r-1}} \frac{[h_m(F(y))-h_m(F(x))]^{s-r-1}[F(y)]^{\gamma_s-1}}{[F(x)]^{\gamma_{r+1}}} f(y), \quad x>y,\ m \neq -1. \tag{34}$$

$$f_{X_{L(s)}^{(k)}|X_{L(r)}^{(k)}}(y|x) = \frac{k^{s-r}}{(s-r-1)!}[\ln F(x) - \ln F(y)]^{s-r-1}\left(\frac{F(y)}{F(x)}\right)^{k-1}\frac{f(y)}{F(x)}, \quad x>y,\ m=-1. \tag{35}$$

Theorem 4.1. For a non-negative random variable having an absolutely continuous *df F(x)*

$$E\left[X^*(s,n,m,k)|X^*(r,n,m,k)=x\right]$$

$$= \frac{1}{\beta(1-e^{-\lambda})}\sum_{p=1}^{\infty}\frac{(1-e^{-\beta(1-e^{-\lambda})x})^p}{p}\prod_{j=1}^{s-r}\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\alpha}, \quad m\neq -1 \tag{36}$$

$$= \frac{1}{\beta(1-e^{-\lambda})}\sum_{p=1}^{\infty}\frac{(1-e^{-\beta(1-e^{-\lambda})x})^p}{p}\prod_{j=1}^{s-r}\frac{k}{k+p/\alpha}, \quad m=-1. \tag{37}$$

if and only if

$$F(x) = \left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha}, \quad x>0,\ \alpha,\ \beta,\ \lambda>0.$$

Proof. Using (34), we have

$$E\left[X^*(s,n,m,k)|X^*(r,n,m,k)=x\right] = \frac{C_{s-1}}{(s-r-1)!C_{r-1}}\int_0^x y\left(1-\left(\frac{F(y)}{F(x)}\right)^{m+1}\right)^{s-r-1}\left(\frac{F(y)}{F(x)}\right)^{\gamma_s-1}\frac{f(y)}{F(x)}dy. \tag{38}$$

Substituting $\frac{F(y)}{F(x)}=z$ in (38), we obtain

$$E\left[X^*(s,n,m,k)|X^*(r,n,m,k)=x\right] = \frac{C_{s-1}}{\beta(1-e^{-\lambda})(s-r-1)!C_{r-1}(m+1)^{s-r-1}} \times \sum_{p=1}^{\infty}\frac{\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^p}{p}\int_0^1 z^{\gamma_s+p/\alpha-1}(1-z^{m+1})^{s-r-1}dz. \tag{39}$$

Setting $z^{m+1}=t$ in (39) , we get

$$E\left[X^*(s,n,m,k)|X^*(r,n,m,k)=x\right] = \frac{C_{s-1}}{\beta(1-e^{-\lambda})(s-r-1)!C_{r-1}(m+1)^{s-r}} \times \sum_{p=1}^{\infty}\frac{\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^p}{p}\int_0^1 t^{\frac{(k+p/\alpha)}{m+1}+n-s-1}(1-t)^{s-r-1}dt.$$

$$E\left[X^*(s,n,m,k)|X^*(r,n,m,k)=x\right] = \frac{C_{s-1}}{\beta(1-e^{-\lambda})(s-r-1)!C_{r-1}(m+1)^{s-r}} \times \sum_{p=1}^{\infty}\frac{\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^p}{p}\frac{\Gamma\left(\frac{k+p/\alpha}{m+1}+n-s\right)\Gamma(s-r)}{\Gamma\left(\frac{k+p/\alpha}{m+1}+n-r\right)}. \tag{40}$$

After simplifying (40), we obtain the required result in (36).
To prove the sufficient part using (36) and (38), we get

$$\frac{C_{s-1}}{(s-r-1)!C_{r-1}(m+1)^{s-r-1}}\int_0^x y\left[(F(x))^{m+1}-(F(y))^{m+1}\right]^{s-r-1}[F(y)]^{\gamma_s-1}f(y)dy = [F(x)]^{\gamma_{r+1}}H_{s|r}(x), \tag{41}$$

where

$$H_{s|r}(x)=\frac{1}{\beta(1-e^{-\lambda})}\sum_{p=1}^{\infty}\frac{(1-e^{-\beta(1-e^{-\lambda})x})^{p}}{p}\prod_{j=1}^{s-r}\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\alpha}.$$

Differentiating (41) on both sides with respect to x, we get

$$\frac{C_{s-1}[F(x)]^{m}f(x)}{(s-r-2)!C_{r-1}(m+1)^{s-r-2}}\int_{0}^{x}y\left[(F(x))^{m+1}-(F(y))^{m+1}\right]^{s-r-2}[F(y)]^{\gamma_s-1}f(y)dy$$
$$=\gamma_{r+1}[F(x)]^{\gamma_{r+1}-1}f(x)H_{s|r}(x)+[F(x)]^{\gamma_{r+1}}H'_{s|r}(x)$$

or

$$\gamma_{r+1}[F(x)]^{\gamma_{r+2}+m}f(x)H_{s|r+1}(x)=\gamma_{r+1}[F(x)]^{\gamma_{r+1}-1}f(x)H_{s|r}(x)+[F(x)]^{\gamma_{r+1}}H'_{s|r}(x). \tag{42}$$

After simplifying (42), we get

$$\frac{f(x)}{F(x)}=\frac{H'_{s|r}(x)}{\gamma_{r+1}\left[H_{s|r+1}(x)-H_{s|r}(x)\right]},$$

where

$$H'_{s|r}(x)=e^{-\beta(1-e^{-\lambda})x}\sum_{p=1}^{\infty}(1-e^{-\beta(1-e^{-\lambda})x})^{p-1}\prod_{j=1}^{s-r}\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\alpha}$$

$$H_{s|r+1}(x)-H_{s|r}(x)=\frac{1}{\beta(1-e^{-\lambda})\gamma_{r+1}}\sum_{p=1}^{\infty}\left(1-e^{-\beta(1-e^{-\lambda})}\right)^{p}\prod_{j=1}^{s-r}\frac{\gamma_{r+j}}{\gamma_{r+j}+p/\alpha}.$$

Therefore,

$$\frac{f(x)}{F(x)}=\frac{\alpha\beta(1-e^{-\lambda})}{e^{\beta(1-e^{-\lambda})x}-1}.$$

This proves that

$$F(x)=\left(1-e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha},\quad x>0,\ \alpha,\ \beta,\ \lambda>0.$$

For the case $m=-1$ from (35), we have

$$E(X^{(k)}_{L(s)}|X^{(k)}_{L(r)}=x)=\frac{k^{s-r}}{(s-r-1)!}\int_{0}^{x}y[\ln F(x)-\ln F(y)]^{s-r-1}\left(\frac{F(y)}{F(x)}\right)^{k-1}\frac{f(y)}{F(x)}dy.$$

By using the transformation

$$u=\frac{F(y)}{F(x)}=\left(\frac{1-e^{-\beta(1-e^{-\lambda})y}}{1-e^{-\beta(1-e^{-\lambda})x}}\right)^{\alpha}$$

we obtain

$$E(X^{(k)}_{L(s)}|X^{(k)}_{L(r)}=x)=k^{s-r}\sum_{p=0}^{\infty}\frac{(1-e^{-\beta(1-e^{-\lambda})x})^{p}}{p}\int_{0}^{1}u^{k+p/\alpha-1}(\ln u)^{s-r-1}du. \tag{43}$$

We have Gradeshteyn and Ryzhik (2007, p-551),

$$\int_{0}^{1}(-\ln w)^{\delta-1}w^{\theta-1}dw=\frac{\Gamma(\delta)}{\theta^{\delta}},\quad \delta,\ \theta>0. \tag{44}$$

By using (44) in (43), we obtain the required result in (37).
The sufficiency part can be proved on the lines of case $m\neq-1$.
Remark: For $\lambda\to\infty$, we have obtained the characterization result for generalized exponential distribution.
Theorem 4.2. For a non-negative random variable having an absolutely continuous *df* $F(x)$,

$$E\left[X^{*j}(r,n,m,k)\right]-E\left[X^{*j}(r-1,n,m,k)\right]=\frac{j}{\alpha\beta\gamma_r(1-e^{-\lambda})}\left\{E\left[X^{*j-1}(r,n,m,k)\right]-E\left[\phi(X^{*}(r,n,m,k))\right]\right\}, \tag{45}$$

if and only if

$$F(x) = \left(1 - e^{-\beta(1-e^{-\lambda})x}\right)^{\alpha}, \quad x > 0,\ \alpha,\ \beta,\ \lambda > 0.$$

Proof. The necessary part follows immediately from the theorem 2.2. Now, from (45), we have

$$\begin{aligned}
&\frac{C_{r-1}}{(r-1)!}\int_0^{\infty} x^j\,[F(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x))\,dx \\
&= \frac{C_{r-2}}{(r-2)!}\int_0^{\infty} x^j\,[F(x)]^{\gamma_r+m} f(x) g_m^{r-2}(F(x))\,dx \\
&\quad + \frac{jC_{r-1}}{\alpha\beta(1-e^{-\lambda})\gamma_r(r-1)!}\int_0^{\infty} x^{j-1}\,[F(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x))\,dx \\
&\quad - \frac{jC_{r-1}}{\alpha\beta(1-e^{-\lambda})\gamma_r(r-1)!}\int_0^{\infty} x^{j-1} e^{\beta(1-e^{-\lambda})x}\,[F(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x))\,dx. \qquad (46)
\end{aligned}$$

Now, integrating by parts the first integral on the right hand side of (46) and simplifying the resulting expression, we get

$$\frac{C_{r-2}}{(r-1)!}\int_0^{\infty} x^{j-1}\,[F(x)]^{\gamma_r-1} f(x) g_m^{r-1}(F(x))\left\{\frac{1}{\alpha\beta(1-e^{-\lambda})} - \frac{e^{\beta(1-e^{-\lambda})x}}{\alpha\beta(1-e^{-\lambda})} + \frac{F(x)}{f(x)}\right\}dx = 0. \qquad (47)$$

Now, applying a generalization of the Muntz-Szasz theorem (Hwang and Lin (1984)) to (47), we get

$$\frac{1}{\alpha\beta(1-e^{-\lambda})} - \frac{e^{\beta(1-e^{-\lambda})x}}{\alpha\beta(1-e^{-\lambda})} + \frac{F(x)}{f(x)} = 0.$$

$$\implies \frac{f(x)}{F(x)} = \frac{\alpha\beta(1-e^{-\lambda})}{e^{\beta(1-e^{-\lambda})x} - 1},$$

which proves that

$$F(x) = (1 - e^{-\beta(1-e^{-\lambda})x})^{\alpha}, \quad x > 0,\ \alpha,\ \beta,\ \lambda > 0.$$

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Conflict of Interest

The authors declare that they have no conflict of interest.

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