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New Modified Weibull Distribution: Moment Properties and Bayes Analysis

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Abstract: The objective of the study is two fold. The first one is to give some recurrence relations satisfied by single and product moments of Generalized Record Values (GRV) from New Modified Weibull Distribution (NMWD). These relations are derived for moments of upper record values. Further, conditional expectation is used to characterize the New Modified Weibull Distribution. The second objective is to give complete Bayesian analysis of the New Modified Weibull Distribution for parameter estimation. We are proposing Markov chain Monte Carlo simulation with hybrid strategy for the same.

Keywords: Generalized Record Values (GRV), NMW Distribution, Single and Product Moment, Characterization, Bayes Estimation, Markov chain Monte Carlo method, Gibbs Sampler, Metropolis Algorithm.

1 Introduction

The statistical study of record values in a sequence of independent and identically-distributed continuous random variables was first carried out by Chandler (1952). Record values and associated statistics are of great importance in several real-life problems involving weather, consumer preference on products, economic studies, sports and so on. The prediction of a future record value is an interesting problem with many real-life applications. For example the predicted value of the amount of next record level of water that a dam will capture from rain and hold or discharge is helpful for future planning purposes, predicted intensity of the next strongest earthquake is essential for disaster management planning, prediction of next level of new record in athletic events is helpful for subjecting the prospective athletes to rigorous training, practice and so on. Several applications of $k-$th record values can be found in the literature, for instance, see the examples Kamps (1995) and Danielak and Raqab (2004) in reliability theory.

Let $\{X_n,\ n\geq 1\}$ be a sequence of independent and identically distributed (iid) random variables with $df\ F(x)$ and $pdf\ f(x)$. The $j-$th order statistics of $X_1,X_2,\ldots,X_n$ is denoted by $X_{j:n}$. For a fixed positive integer k , Dziubdziela and Kopociński (1986) define the sequences $\{U_n^{(k)},\ n\geq 1\}$ of $k-$th upper record times of $\{X_n,\ n\geq 1\}$ as follows:
$U_1^{(k)}=1$
$U_{n+1}^{(k)}=min\{j>U_n^{(k)}:X_{j:j+k-1}>X_{U_n^{(k)}:U_n^{(k)}+k-1}\}$.

The sequence $\{Y_n^{(k)},\ n\geq 1\}$, where $Y_n^{(k)}=X_{U_n^{(k)}}$ is called the sequence of $k-$th upper record values of $\{X_n,\ n\geq 1\}$.Note that for $k=1$ we have $Y_n^{(1)}=X_{U_n},n\geq 1$, which are the record values of $\{X_n,\ n\geq 1\}$ (Ahsanullah (1995)). Moreover we see that $Y_0^{(k)}=0$ and $Y_1^{(k)}=min(X_1,X_2,X_3,...,X_k)=X_{1:k}$.

Then the pdf of $Y_n^{(k)}$ and the joint pdf of $Y_m^{(k)}$ and $Y_n^{(k)}$ are given by (Dziubdziela and Kopcoiński (1986) , Grudzień (1982))

$$f_{Y_n^{(k)}}(x)=\frac{k^n}{(n-1)!}\,[-ln\bar{F}(x)]^{n-1}\,[\bar{F}(x)]^{k-1}\,f(x),\quad n\geq 1, \tag{1}$$

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$$f_{Y_m^{(k)},Y_n^{(k)}}(x,y) = \frac{k^n}{(n-1)!(n-m-1)!}[-ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}$$
$$\times[ln\bar{F}(x)-ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^{k-1}\ f(y),\ x<y,\ 1\leq m<n,\ n\geq 2. \quad (2)$$

where
$\bar{F}(x) = 1-F(x)$.
For some recent development on generalized record values with spcial reference to those arising from Pareto, generalizaed Pareto, Weibull, exponential-Weibull and modified Weibull distribution, see Pawlas and Syznal(1999, 2000), Khan and Khan (2016), Khan et. al (2017), respectively. Weibull distribution does not provide an appropriate parametric fit for phenomenon with non-monotone failure rates whereas modified Weibull distribution, discussed in the study, also gives bathtub-shaped failure rate which is very much useful in survival analysis. In addition this distribution may be applied in the process of constructing product demand forecasts for a business as well as consumer preference of brand and the most preferred brand .

Xie et al. (2002) addressed some properties of a modified Weibull distribution such as characterization of failure rate function, mean and variance of time to failure as well as the relationship to other distributions. Authors also discussed the method of maximum likelihood for parameter estimation where the equations cannot to solved analytically. The present study focuses on two different aspects. First, we focus on some moment properties of new modified Weibull distribution which are related to study of generalized record values arising from New Modified Weibull Distribution (NMWD). Second, we focus on Bayesian analysis of NMWD via Markov chain Monte Carlo (MCMC) method. We shall be discussing full posterior analysis of model under the assumed prior-likelihood setup and proposing hybrid strategy of parameter estimation that combines the Metropolis algorithm within the Gibbs sampler.

This paper is organized as follow. Next section gives a brief description of NMWD. Recurrence relations satisfied by single and product moments and some characterization are studied in section 3. In the section 3, relationship between NMWD and other models are also discussed. Section 4 is devoted to parameter estimation. This section provides details of Bayesian model formulation along with discussion on MCMC method. Section 5 provides a brief conclusion.

2 A New Modified Weibull Distribution

A random variable X is said to have a NMWD (Xie et al. (2002)) if its pdf is of the form

$$f(x) = \lambda\beta(\frac{x}{\alpha})^{\beta-1}exp(\frac{x}{\alpha})^{\beta}exp[\lambda\alpha(1-exp(\frac{x}{\alpha})^{\beta})],\ \ x\geq 0,\ \ \alpha,\ \lambda,\ \beta>0 \quad (3)$$

and df is of the form

$$F(x) = 1-exp[\lambda\alpha(1-exp(\frac{x}{\alpha})^{\beta})],\ \ x\geq 0,\ \ \alpha,\ \lambda,\ \beta>0 \quad (4)$$

and its easy to see that

$$f(x) = \frac{\beta}{\alpha^{\beta}}x^{\beta-1}[\lambda\alpha+(-ln\bar{F}(x))]\bar{F}(x), \quad (5)$$

NMWD is related to the Weibull distribution in an interesting way, when the scale parameter α becomes very large or approaches infinity. Chen distribution and Gompertz distribution are also the special case of NMWD at $\alpha = 1$ and $\alpha = 1/\alpha, \beta = 1$ respectively. The hazard rate and reliabilty function of NMWD respectively are as follows:

$$h(x) = \lambda\beta(\frac{x}{\alpha})^{\beta-1}exp(\frac{x}{\alpha})^{\beta},$$

$$R(x) = exp[\lambda\alpha(1-exp(\frac{x}{\alpha})^{\beta})].$$

3 Moment properties GRV of NMWD

3.1 Relations for Single Moments

Theorem 3.1. For distribution given in (4), Fix a positive integer $k\geq 1$, for $n\geq 1$ and $j = 0,1,...$

$$E(Y_n^{(k)})^j = \frac{\lambda\beta k}{\alpha^{\beta-1}(j+\beta)}\{E(Y_n^{(k)})^{j+\beta} - E(Y_{n-1}^{(k)})^{j+\beta}\}$$

$$+\frac{n\beta}{\alpha^{\beta}(j+\beta)}\{E(Y_{n+1}^{(k)})^{j+\beta} - E(Y_n^{(k)})^{j+\beta}\}. \quad (6)$$

Proof. From (1) and (5), we have

$$E(Y_n^{(k)})^j = \frac{k^n}{(n-1)!}\frac{\lambda\beta}{\alpha^{\beta-1}}\int_0^\infty x^{j+\beta-1}[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^k dx$$

$$+\frac{k^n}{(n-1)!}\frac{\beta}{\alpha^{\beta}}\int_0^\infty x^{j+\beta-1}[-ln\bar{F}(x)]^{n}[\bar{F}(x)]^k dx.$$

Now (6)can be seen by noting that in view of Khan *et al.(2017)*

$$E(Y_n^{(k)})^j - E(Y_{n-1}^{(k)})^j = \frac{jk^{n-1}}{(n-1)!}\int_\alpha^\beta x^{j-1}[-ln\bar{F}(x)]^{n-1}[\bar{F}(x)]^k dx.$$

Remarks
i)Setting $\alpha = 1$ in (6), we deduce the recurrence relation for single moments of k-th upper record values from the Chen distribution established by Kumari and Pathak (2013).
ii)Setting $\beta = 1$ and $\alpha = 1/\alpha$ in (6), we get the recurrence relation for single moments of k-th upper record values from the Gompertz distribution, obtained by Minimol and Thomas (2014).

Corollary 3.1. *The recurrence relation for single moments of upper record values from the NMWD has the form*

$$E(X_{U_n}^j) = \frac{\lambda\beta}{\alpha^{\beta-1}(j+\beta)}(EX_{U_n}^{j+\beta} - EX_{U_{n-1}}^{j+\beta})$$

$$+\frac{n\beta}{\alpha^{\beta}(j+\beta)}(EX_{U_{n+1}}^{j+\beta} - EX_{U_n}^{j+\beta}). \quad (7)$$

Remarks
i) Setting $\beta = 1$ and $\alpha = 1/\alpha$ in (7), we get the recurrence relation for single moments of upper record values from the Gompertz distribution,as obtained by Khan and Zia (2009).

3.2 Relations for Product Moment

Theorem 3.2. *For distribution given in (4) and $m \geq 1$, $i, j = 0, 1, ...$*

$$E[(Y_m^{(k)})^i(Y_{m+1}^{(k)})^j] = \frac{\lambda\beta k}{\alpha^{\beta-1}(i+\beta)}\left\{E[(Y_m^{(k)})^{i+j+\beta}] - E[(Y_{m-1}^{(k)})^{i+\beta}(Y_m^{(k)})^j]\right\}$$

$$+\frac{m\beta}{\alpha^{\beta}(i+\beta)}\left\{E[(Y_{m+1}^{(k)})^{i+j+\beta}] - E[(Y_m^{(k)})^{i+\beta}(Y_{m+1}^{(k)})^j]\right\} \quad (8)$$

and for $1 \leq m \leq n-2$, $i, j = 0, 1, ...$

$$E[(Y_m^{(k)})^i(Y_n^{(k)})^j] = \frac{\lambda\beta k}{\alpha^{\beta-1}(i+\beta)}\left\{E[(Y_m^{(k)})^{i+\beta}(Y_{n-1}^{(k)})^j] - E[(Y_{m-1}^{(k)})^{i+\beta}(Y_{n-1}^{(k)})^j]\right\}$$

$$+\frac{m\beta}{\alpha^{\beta}(i+\beta)}\left\{E[(Y_{m+1}^{(k)})^{i+\beta}(Y_n^{(k)})^j] - E[(Y_m^{(k)})^{i+\beta}(Y_n^{(k)})^j]\right\} \quad (9)$$

Proof. *From* (2) *and* (5) *we have*

$$E[(Y_m^{(k)})^i(Y_n^{(k)})^j] = \frac{\lambda\beta k^n}{(m-1)!(n-m-1)!\alpha^{\beta-1}}\int_0^\infty\int_0^y x^i y^{j+\beta-1}[-ln\bar{F}(x)]^{m-1}\frac{f(x)}{\bar{F}(x)}$$

$$\times[ln\bar{F}(x)-ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^k dydx + \frac{\beta k^n}{(m-1)!(n-m-1)!\alpha^{\beta}}\int_0^\infty\int_0^y x^i y^{j+\beta-1}$$

$$\times[-ln\bar{F}(x)]^m\frac{f(x)}{\bar{F}(x)}[ln\bar{F}(x)-ln\bar{F}(y)]^{n-m-1}[\bar{F}(y)]^k dydx. \quad (10)$$

After simplification we get the result as given in (9).
Proceeding in a similar manner for the case $n = m+1$ *, the recurrence relation given in* (8) *can easily be established.*
One can also note that Theorem 3.1 can be deduced from Theorem 3.2 by taking $j = 0$.

Remarks
i)Setting $\alpha = 1$ *in* (9)*, we deduce the recurrence relation for product moments of k-th upper record values from the Chen distribution established by Kumari and Pathak (2013).*
ii)Setting $\beta = 1$ *and* $\alpha = 1/\alpha$ *in* (9)*, we get the recurrence relation for product moments of k-th upper record values from the Gompertz distribution, obtained by Minimol and Thomas (2014).*

Corollary 3.2. *The recurrence relation for product moments of upper record values from the NMWD has the form*

$$E(X_{U_m}^i X_{U_n}^j) = \frac{\lambda\beta}{(i+\beta)\alpha^{\beta-1}}\{E(X_{U_m}^{i+\beta}X_{U_{n-1}}^j) - E(X_{U_{m-1}}^{i+\beta}X_{U_{n-1}}^j)\}$$

$$+\frac{m\beta}{(i+\beta)\alpha^{\beta}}\{E(X_{U_{m+1}}^{i+\beta}X_{U_n}^j) - E(X_{U_m}^{i+\beta}X_{U_n}^j)\} \quad (11)$$

Remarks
i) If $\beta = 1$ *and* $\alpha = \frac{1}{\alpha}$ *in* (11)*, we obtain the recurrence relation for product moments of upper record values from the Gompertz distribution as established by Khan and Zia (2009).*

3.3 Characterization

Theorem 3.3. *Let* X *be a non-negative random variable having an absolutely continuous* df $F(x)$ *with* $F(0) = 0$ *and* $0 \leq F(x) \leq 1$ *for all* $x > 0$ *, then*

$$E[\xi(Y_n^{(k)}) \mid (Y_l^{(k)}) = x] = e^{\lambda\alpha(1-e^{(x|\alpha)^\beta})}\left(\frac{k}{k+1}\right)^{n-l}, \quad l = m, \quad m+1 \quad (12)$$

if and only if

$$F(x) = 1 - e^{\lambda\alpha(1-e^{(x|\alpha)^\beta})},$$

where

$$\xi(y) = e^{\lambda\alpha(1-e^{(y|\alpha)^\beta})},$$

Proof. *From* (1) *and* (2) *, we have*

$$E[\xi(Y_n^{(k)}) \mid (Y_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!}\int_x^\infty e^{\lambda\alpha(1-e^{(y|\alpha)^\beta})}$$

$$\times[ln\bar{F}(x)-ln\bar{F}(y)]^{n-m-1}\left(\frac{\bar{F}(y)}{\bar{F}(x)}\right)^{k-1}\frac{f(y)}{\bar{F}(x)}dy. \quad (13)$$

By setting $u = \frac{\bar{F}(y)}{\bar{F}(x)}$ *from* (4) *in above equation, we have*

$$E[\xi(Y_n^{(k)}) \mid (Y_m^{(k)}) = x] = \frac{k^{n-m}}{(n-m-1)!}e^{\lambda\alpha(1-e^{(x|\alpha)^\beta})}\int_0^1 u^k(-lnu)^{n-m-1}du. \quad (14)$$

We have Gradshteyn and Ryzhik (2007)

$$\int_0^1 (-lnx)^{\mu-1}x^{\nu-1}dx = \frac{\Gamma\mu}{\nu^{\mu}}, \quad \mu > 0, \;\; \nu > 0. \tag{15}$$

On using (15)*, we have the result given in* (12)*.*
To prove sufficient part, we have

$$\frac{k^{n-m}}{(n-m-1)!}\int_x^{\infty} e^{\lambda\alpha(1-e^{(y|\alpha)^{\beta}})}[ln\bar{F}(x)-ln\bar{F}(y)]^{n-m-1}$$

$$\times[\bar{F}(y)]^{k-1}f(y)dy = [\bar{F}(x)]^k g_{n|m}(x), \tag{16}$$

where
$g_{n|m}(x) = e^{\lambda\alpha(1-e^{(x|\alpha)^{\beta}})}\left(\frac{k}{k+1}\right)^{n-m}$.
Differentiating (16) *both the sides with respect to x , we get*

$$-\frac{k^{n-m}f(x)}{\bar{F}(x)(n-m-2)!}\int_x^{\infty} e^{\lambda\alpha(1-e^{(y|\alpha)^{\beta}})}[ln\bar{F}(x)-ln\bar{F}(y)]^{n-m-2}$$

$$\times[\bar{F}(y)]^{k-1}f(y)dy = g'_{n|m}(x)[\bar{F}(x)]^k - k\,g_{n|m}(x)[\bar{F}(x)]^{k-1}f(x)$$

or

$$-k\,g_{n|m+1}(x)[\bar{F}(x)]^{k-1}f(x) = g'_{n|m}(x)[\bar{F}(x)]^k - k\,g_{n|m}(x)[\bar{F}(x)]^{k-1}f(x).$$

which proves that

$$F(x) = 1-e^{\lambda\alpha(1-e^{(x|\alpha)^{\beta}})}, \;\; x \geq 0, \;\; \alpha, \; \lambda, \; \beta > 0.$$

4 Bayes Estimation of Model Parameters

Xie et al. (2002) addressed the method of maximum likelihood estimation for estimating the model parameters where equations cannot be solved analytically. We propose Bayesian method of estimation for the same. Bayesian method of reasoning is deductive in nature and also it has advantages over the classical method (see Martz and Waller (1982)). Bayesian method is more preferable when sample size is small or complete information is not available because it makes use of additional information on modal parameters in the form of prior distribution. Bayesian paradigm combines the prior distribution with the likelihood function via 'Bayes theorem' to update our knowledge regarding the uncertainty and defines the posterior distribution. This posterior distribution forms the basis of Bayesian inferences.

If the model parameter is in a vector form, the posterior-based inferences involve high dimensional integration to evaluate various joint and marginal distributions of parameters. The problem becomes quite difficult or sometimes even impossible if the study involves censored data or constrained parameter surfaces. Among the various developments in Bayesian computing, one can employ numerical integration and/or analytic approximation techniques (see, for example, Smith (1991)). As an alternative, sample based approaches have become popular in the last two decades. In sample based approaches, MCMC methods such as Gibbs sampler and Metropolis-Hastings algorithms have received maximum attention in the literature. We shall discusses a brief on these techniques.

4.1 Model formulation and MCMC implementation

Let $\mathbf{x}$: $x_1, x_2, \cdots, x_n$ be the observed failure times of n items which are put into an experiment and the failure time distribution is given by (3). The corresponding likelihood function can be given as

$$L(\mathbf{x},\lambda,\alpha,\beta) = \lambda^n\beta^n\alpha^{n-n\beta}\prod_{i=1}^{n}x_i^{\beta-1}\exp\left(\sum_{i=1}^{n}(x_i/\alpha)^{\beta}\right)X\exp\left(\lambda\alpha\sum_{i=1}^{n}(1-\exp(x_i/\alpha)^{\beta})\right) \tag{17}$$

We can consider independent priors for parameters as

$$\Psi_1(\lambda) = \frac{1}{U_1}, 0 < \lambda < U_1 \tag{18}$$

$$\Psi_2(\alpha) \propto \alpha^{\gamma-1} \exp(-\frac{\alpha}{\omega}), \gamma, \omega > 0 \tag{19}$$

$$\Psi_3(\beta) = \frac{1}{U_2}, 0 < \beta < U_2 \tag{20}$$

It is worth to mention that if very concrete prior information is not available one can choose the hyper parameters of priors in such a way that priors become diffused and inference are driven by observed data only. Combining likelihood with prior distributions, the expression of posterior can be obtained as follows

$$p(\lambda, \alpha, \beta|\mathbf{x}) \propto \lambda^n \beta^n \alpha^{n-n\beta+\gamma-1} \prod_{i=1}^{n} x_i^{\beta-1} \exp(\sum_{i=1}^{n} (x_i/\alpha)^{\beta}) X$$
$$\times \exp(\lambda\alpha \sum_{i=1}^{n} (1 - \exp(x_i/\alpha)^{\beta})) X \exp(-\frac{\alpha}{\omega}) \tag{21}$$

The posterior is too complicated to entertain for a close form inferences. We, therefore, recommend sampling-based approaches namely Gibbs sampler and the Metropolis algorithm discussed in next section to simulate realizations from the posterior. Simulated samples can be further used to estimate model parameter.

4.2 Gibbs Sampler

A Markov chain Monte Carlo method for simulating from a distribution f can be defined as a method producing an ergodic Markov chain whose stationary distribution is f (see Robert and Casella (2004)). Gibbs sampler is a particular form of MCMC technique that can be described as a Markovian-updating scheme designed to extract samples from the joint posterior distribution by simulating through its various full conditionals. The joint posterior and all its full conditionals need to be specified up to proportionality only.

To clarify, let us suppose parameter $\boldsymbol{\theta}$ which is a vector-valued parameter having k (≥ 2) components $(\theta_1, \theta_2, \cdots, \theta_k)$. From the posterior distribution $p(\boldsymbol{\theta}|\underline{y})$, one can easily identify k full conditionals up to proportionality. These full conditionals can be defined as $p(\theta_i|\underline{y}, \theta_j, j \neq i)$, $i, j = 1, 2, \cdots, k$, by regarding $p(\theta_i|\underline{y})$ as the function of θ_i alone and considering all other $\theta_j, j \neq i$, as fixed constants. Suppose $\theta_1^0, \theta_2^0, \cdots, \theta_k^0$ are the initial values to start the chain, the algorithm then proceeds as follows.

$$\theta_1^1 \sim p(\theta_1|\theta_2^0, \theta_3^0, \cdots, \theta_k^0, \underline{y})$$
$$\theta_2^1 \sim p(\theta_2|\theta_1^1, \theta_3^0, \cdots, \theta_k^0, \underline{y})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\theta_k^1 \sim p(\theta_k|\theta_1^1, \theta_2^1, \cdots, \theta_{k-1}^1, \underline{y})$$

The above cycle of generations completes a transition from $\boldsymbol{\theta}^0 = (\theta_1^0, \theta_2^0, \cdots, \theta_k^0)$ to $\boldsymbol{\theta}^1 = (\theta_1^1, \theta_2^1, \cdots, \theta_k^1)$. After t such iterations we arrive at $\boldsymbol{\theta}^t = (\theta_1^t, \theta_2^t, \cdots, \theta_k^t)$. It is to be noted that each generation of θ_i uses the most recent value of other unknowns $\theta_j, j \neq i$. Repetition of the cycle of generation from each of the full conditionals, in turn, produces a Markov chain $\boldsymbol{\theta}^0, \boldsymbol{\theta}^1, \cdots, \boldsymbol{\theta}^t$ and, under the regularity conditions, which are satisfied by all commonly used models, it can be shown that $p(\boldsymbol{\theta}|\underline{y})$ is the equilibrium distribution of the Markov chain produced by the Gibbs sampler with component of $\boldsymbol{\theta}^t$ converging in distribution to a random sample from the corresponding marginal posterior distribution (see, Roberts and Smith (1993)). The details about the algorithm, the choice of initial values and the specific convergence diagnostic issues discussed in Robert and Casella (2004) (see also Smith and Roberts (1993), Upadhyay et al. (2001)).

For implementing Gibbs algorithm the full conditionals of parameters can be derived from posterior (21) as

$$p_1(\lambda|\alpha, \beta, \mathbf{x}) \propto \lambda^n \exp(\lambda\alpha \sum_{i=1}^{n} (1 - \exp(x_i/\alpha)^{\beta})) \tag{22}$$

$$p_2(\alpha, |\lambda, \beta, \mathbf{x}) \propto \alpha^{n-n\beta+\gamma-1} \prod_{i=1}^{n} x_i^{\beta-1} \exp(\sum_{i=1}^{n} (x_i/\alpha)^{\beta}) X$$
$$\times \exp(\lambda\alpha \sum_{i=1}^{n} (1 - \exp(x_i/\alpha)^{\beta})) X \exp(-\frac{\alpha}{\omega}) \tag{23}$$

$$p(\beta|\lambda,\alpha,\mathbf{x}) \propto \beta^{n}\alpha^{-n\beta}\prod_{i=1}^{n}x_i^{\beta-1}\exp\left(\sum_{i=1}^{n}(x_i/\alpha)^{\beta}\right)X\exp\left(\lambda\alpha\sum_{i=1}^{n}(1-\exp(x_i/\alpha)^{\beta})\right) \tag{24}$$

4.3 Metropolis algorithm

The Metropolis algorithm gives a different way of simulation from the complicated posterior distribution. It generates parameter vector θ completely in each iteration i.e. it does not require full conditional upto proportionality. By this algorithm one can easily simulate a Markovian chain with the equilibrium distribution as posterior $P(\theta|x)$. Suppose for the current realization of parameter θ, $q(\theta,\theta^{'})$ denotes an arbitrary Markov kernal where $\theta^{'}$ is the next realization, we accept the value $\theta^{'}$ with probability as

$$a(\theta,\theta^{'}) = min\left(\frac{P(\theta^{'}|x).q(\theta,\theta^{'})}{P(\theta|x).q(\theta^{'},\theta)},1\right) \tag{25}$$

otherwise retain θ as the next proposal.

For the new modified Weibull model, sample generation from the full conditionals from posterior (21) is still a daunting task. Therefore, we propose to mix above two sample generation algorithms that considers Metropolis algorithm for full conditionals. In the hybrid strategy Metropolis sampler is used to generate samples from some of the full conditions for the completion of each cycle of Gibbs sampler.

It can be seen that full conditional (22) belong to gamma family. Therefore, generation from (22) can be done by using gamma-generating scheme. It can also be observed that full conditionals (23) and (24) are complicated and not easily tractable. Metropolis algorithm, however, can be used to generate from (23) and (24). So Metropolis within the Gibbs sampler can complete the cycles of Gibbs chain. On may refer to Smith and Robert (1993), Upadhyay and Smith (1994), Ghosh et al. (2006) for the detail discussion on Metropolis algorithm.

5 Conclusion

In this paper, we have proposed a recurrence relation for single and product moments of generalized record values from NMWD. These relations are used to find the mean, variance and covariance for all sample sizes in a simple recursive manner. Further, we characterize the NMWD through conditional expectation of function.The paper successfully describes the Bayesian way of analysis of new modified Weibull distribution. The complexity involved in the parameter estimation can be tackled by the Markov chain Monte Carlo technique in Weibull extension model. Hybrid strategy has also been explained for studying the characteristic of such complicated model extensions. The procedure can be used for other complicated distributions also.

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