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On Moments and Entropy of Kumaraswamy Power Function Distribution based on Generalized Order Statistics with its Application

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Abstract: In this paper, we have obtained exact and explicit expressions for ratio, inverse and conditional moments of generalized order statistics (gos) from Kumaraswamy Power function distribution (KPFD) and reduced the aforesaid results for Kumaraswamy distribution. An exact and analytic expression of Shannon entropy for KPFD based on gos have been obtained. Also, we have derived the expressions for maximum likelihood estimator (MLE) of all shape parameters, uniformly minimum variance unbiased estimator (UMVUE) of one of the shape parameter by considering others to be known and best linear unbiased estimator (BLUE) for location and scale parameters of KPFD based on gos. We have computed means, variance and covariance matrices based on order statistics, progressive type-II censored order statistics and gos. Further, we have considered a real example for illustration purpose of the findings.

Keywords: Kumaraswamy Power function distribution; Ratio, Inverse and Conditional moments; Generalized order statistics; Maximum Likelihood estimator; Uniformly minimum variance unbiased estimator; Best linear unbiased estimator.
***AMS Classification*:** 62G30; 62E10.

1 Introduction

Kamps [1] introduced the concept of generalized order statistics (gos) as a unified distribution theoretical setup which contains several important models of ordered random variables arranged in increasing order of magnitude such as ordinary order statistics, record values, progressive type II censored order statistics, sequential order statistics etc. and is defined as: Let $n \in \mathbb{N}$, $n \geq 2$, $k > 0$, $\tilde{m} = (m_1, m_2, ..., m_{n-1}) \in \mathbb{R}^{n-1}$, $M_r = \sum_{j=r}^{n-1} m_j$ be the parameters such that $\gamma_r = k + n - r + M_r > 0$ for all $r \in 1, 2, ..., n-1$. Then based on the absolute continuous cumulative distribution function (*cdf*) F with probability density function (*pdf*) f, $X(1,n,\tilde{m},k)$, $X(2,n,\tilde{m},k)$, ..., $X(n,n,\tilde{m},k)$ are said to be gos, if their joint *pdf* is given by

$$f_{X(1,n,\tilde{m},k),...,X(n,n,\tilde{m},k)}(x_1,\ldots,x_n) = k\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{i=1}^{n-1}[\bar{F}(x_i)]^{m_i} f(x_i)\right)[\bar{F}(x_n)]^{k-1} f(x_n) \tag{1}$$

on the cone $F^{-1}(0) < x_1 \leq x_2 \leq ... \leq x_n < F^{-1}(1)$ of $\mathbb{R}^n$, where $\bar{F}(x) = 1 - F(x)$ denotes the survival function. Appropriately choosing the specific values of parameters, (1) can be reduced to ordinary order statistics $(\gamma_i = n - i + 1;\ i = 1,2,...,n$ *i.e.* $m_1 = m_2 = ... = m_{n-1} = 0, k = 1)$, k^{th} record values $(\gamma_i = k,\ i = 1,2,...,n$ *i.e.* $m_1 = m_2 = ... = m_{n-1} = -1,\ k \in \mathbb{N})$, Pfeifer's record values $(\gamma_i = \beta_i;\ \beta_1, ..., \beta_n > 0)$, sequential order statistics $(\gamma_i = (n-i+1)\delta_i;\ \ \delta_1, \delta_2, \ldots, \delta_n > 0)$, order statistics with non-integral sample size $(\gamma_i = \alpha - i + 1;\ \alpha > 0)$ and progressive type II censored order statistics $(m_i = R_i, n = m_0 + \sum_{j=1}^{m_0} R_j, R_j \in \mathbb{N}_0$ and $\gamma_i = n - \sum_{t=1}^{i-1} R_t - i + 1, 1 \leq i \leq m_0)$ (where m_0 is the fixed number of units of failure to be observed and $\mathbb{N}_0 = 0 \cup \mathbb{N}$), *see* [2]. The following two cases appear in the theory of gos, which are considered separately.
Case I: When $\gamma_i \neq \gamma_j$, $i \neq j$ for all $i, j \in (1, 2, \ldots, n)$, *i.e.* γ_i's are pairwise different.

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In this case, the *pdf* of $X(r,n,\tilde{m},k)$ is given by (Kamps and Cramer, 2001):

$$f_{X(r,n,\tilde{m},k)}(x) = c_{r-1}\left(\sum_{i=1}^{r} a_i(r)[\bar{F}(x)]^{\gamma_i-1}\right) f(x), \quad -\infty < x < \infty \tag{2}$$

and the joint *pdf* of $X(r,n,\tilde{m},k)$ and $X(s,n,\tilde{m},k)$ where $1 \le r < s \le n$, is given by (Kamps and Cramer [3])

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y) = c_{s-1}\left(\sum_{i=r+1}^{s} a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\right)\left(\sum_{j=1}^{r} a_j(r)[\bar{F}(x)]^{\gamma_j}\right)\frac{f(x)}{\bar{F}(x)}\frac{f(y)}{\bar{F}(y)}, \quad -\infty < x < y < \infty \tag{3}$$

therefore, the conditional *pdf* of $X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k) = x$, $1 \le r < s \le n$, is given by

$$f_{X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)}(y|x) = \frac{c_{s-1}}{c_{r-1}}\sum_{i=r+1}^{s} a_i^{(r)}(s)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}\frac{f(y)}{\bar{F}(y)}, \quad -\infty < x < y < \infty \tag{4}$$

similarly, the conditional *pdf* of $X(r,n,\tilde{m},k)|X(s,n,\tilde{m},k) = y$, $1 \le r < s \le n$, is given by

$$f_{X(r,n,\tilde{m},k)|X(s,n,\tilde{m},k)}(x|y) = \frac{\sum_{i=r+1}^{s} a_i^{(r)}(s)\sum_{j=1}^{r} a_j(r)\left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i}[\bar{F}(x)]^{\gamma_j}\frac{f(x)}{\bar{F}(x)}}{\sum_{t=1}^{s} a_t(s)[\bar{F}(y)]^{\gamma_t}}, \quad -\infty < x < y < \infty \tag{5}$$

where

$$c_{r-1} = \prod_{i=1}^{r}\gamma_i, \quad a_i(r) = \prod_{\substack{j=1\\ j\neq i}}^{r}\frac{1}{\gamma_j - \gamma_i}, \quad \gamma_i \neq \gamma_j, \quad 1 \le i \le r \le n$$

and

$$a_j^r(s) = \prod_{\substack{j=r+1\\ j\neq i}}^{s}\frac{1}{\gamma_j - \gamma_i}, \quad \gamma_i \neq \gamma_j, \quad r+1 \le j \le s \le n.$$

Case II: When $m_1 = m_2 = \ldots = m_{n-1} = m$ (say).
In this case, the *pdf* of $X(r,n,m,k)$ is given by (Kamps [1])

$$f_{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}[\bar{F}(x)]^{\gamma_r-1} g_m^{r-1}(F(x)) f(x), \quad -\infty < x < \infty \tag{6}$$

and the joint *pdf* of $X(r,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < s \le n$, is given by (Kamps [1])

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m g_m^{r-1}(F(x)) \times [h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1} f(x) f(y), \quad -\infty < x < y < \infty \tag{7}$$

therefore, the conditional *pdf* of $X(s,n,m,k)$ given $X(r,n,m,k) = x$, $1 \le r < s \le n$, is given by (Kamps [1])

$$f_{X(s,n,m,k)|X(r,n,m,k)}(y|x) = \frac{c_{s-1}}{c_{r-1}(s-r-1)!}\frac{[h_m(F(y)) - h_m(F(x))]^{s-r-1}[\bar{F}(y)]^{\gamma_s-1} f(y)}{[\bar{F}(x)]^{\gamma_{r+1}}}, \quad -\infty < x < y < \infty \tag{8}$$

similarly, the conditional *pdf* of $X(r,n,m,k)$ given $X(s,n,m,k) = y$, $1 \le r < s \le n$, is given by (Kamps [1])

$$f_{X(r,n,m,k)|X(s,n,m,k)}(x|y) = \frac{(s-1)!}{(r-1)!(s-r-1)!}[\bar{F}(x)]^m \times \frac{g_m^{r-1}(F(x))[h_m(F(y)) - h_m(F(x))]^{s-r-1} f(x)}{g_m^{s-1}(F(y))}, \quad -\infty < x < y < \infty \tag{9}$$

where

$$h_m(x) = \begin{cases} -\frac{(1-x)^{m+1}}{m+1}, & m \neq -1 \\ -ln(1-x), & m = -1 \end{cases}$$

and

$$g_m(x) = h_m(x) - h_m(0), \quad x \in (0,1).$$

Here, we first consider the case $\gamma_i \neq \gamma_j$ and then reduced the results for another case $m_1 = \ldots = m_{n-1} = m \neq -1$. Khan and Khan [4] defined the relationship between both the cases, I & II and it can be seen for $m_1 = m_2 = \ldots = m_{n-1} = -1$,

$$a_i(r) = \frac{(-1)^{r-i}}{(m+1)^{r-1}(r-1)!}\binom{r-1}{r-i} \tag{10}$$

and

$$a_j^{(r)}(s) = \frac{(-1)^{s-j}}{(m+1)^{s-r-1}(s-r-1)!}\binom{s-r-1}{s-j} \tag{11}$$

Recently, Kumaraswamy distribution, [5] has received considerable attention among the researcher due to its closed form of cumulative distribution. Kumaraswamy distribution is very similar to Beta distribution but the major difference is its invertible form of distribution function over double bounded support. For a comprehensive prospect of Kumaraswamy distribution, one may refer to [6,7]. The *cdf* and *pdf* of Kumaraswamy distribution are given by

$$F(x|a,b) = 1-(1-x^a)^b, \qquad 0 \leq x \leq 1,\ a,b>0 \tag{12}$$

and

$$f(x|a,b) = abx^{a-1}(1-x^a)^{b-1}, \quad 0 \leq x \leq 1,\ a,b>0 \tag{13}$$

Cordeiro and De-Castro [8] defined a new family of generalized distribution named Kumaraswamy-G (Kum-G) distribution which is defined as for an arbitrary baseline *cdf* $G(x)$, the *cdf* of Kum-G distribution is given by

$$F(x) = 1-[1-G^a(x)]^b, \quad a,b>0$$

and the *pdf* of Kum-G distribution is

$$f(x) = abG^{a-1}(x)[1-G^a(x)]^{b-1}, \quad a,b>0$$

where, $g(x) = \frac{dG(x)}{dx}$. Considering the power function distribution as a baseline distribution, *i.e.* $G(x) = x^\theta,\ \theta>0,\ 0 \leq x \leq 1$, we get the Kumaraswamy power function distribution (KPFD) with *pdf* and *cdf*

$$\begin{aligned} f(x|\alpha,\beta,\theta) &= \alpha\beta\theta x^{\alpha\theta-1}(1-x^{\alpha\theta})^{\beta-1}, \quad 0 \leq x \leq 1,\ \alpha,\beta,\theta>0 \\ \text{and} \qquad F(x|\alpha,\beta,\theta) &= 1-(1-x^{\alpha\theta})^{\beta}, \quad 0 \leq x \leq 1,\ \alpha,\beta,\theta>0. \end{aligned} \tag{14}$$

respectively. Clearly, at θ=1, KPFD$(a,b,1)$ reduces to Kumaraswamy distribution.

In last few years, various forms of Kumaraswamy-G family of distributions have appeared in the literature. Mitnik [9] described some important properties of Kumaraswamy distribution such as closeness under linear transformation, exponentiation and some limiting form of the distribution under regularity conditions. Cordeiro and De-Castro [8] have derived the explicit expression for moments of some Kumaraswamy generalized family of distribution using special functions. Further, these results are extended by Cordeiro and Bager [10] for Kumaraswamy-normal, Kumaraswamy-gamma, Kumaraswamy-beta and Kumaraswamy-t and Kumaraswamy-F distribution. Hassan and Elgarhy [11] introduced the Kumaraswamy-Weibull generated family of distribution and derived general explicit expressions for moments, quantile functions, order statistics. Based on *k*-records values, the maximum likelihood estimates and alternative point estimates for the parameters of the Kumaraswamy distribution have been considered by Wang [12]. Abdul-Moniem [13] discussed a Kumaraswamy-Power function distribution with its properties and applications. Wang [14] discussed various estimators of progressively censored competing risks data from Kumaraswamy distributions. For more details see [15,16,17,18,19,20,21,22].

The joint distribution, distribution of product and ratio of two generalized order statistics from Kumaraswamy distribution have been discussed in [23]. Athar *et al.* [24] have obtained the explicit expressions for ratio and inverse moments of *gos* from Weibull distribution. Khan and Khan [4] obtained the ratio and inverse moments of *gos* from Burr distribution using hypergeometric functions. Safi and Ahmed [25] have discussed the MLE for the parameter of Kumaraswamy distribution based on *gos*. MLE and Bayesian estimator for the parameter of Kumaraswamy distribution based on *gos* have been obtained in [26]. An exact and explicit expression of moments from Topp leone distribution based on dual *gos* have been given by Khan and Iqrar [27] and also deduced the expressions for MLE and UMVUE for the parameter of Topp - Leone distribution. Recurrence relation and expression for single and product moment of

Kumaraswamy distribution based on *gos* was obtained by Kumar [28]. However, his results regarding the expressions for single and product moments were not in closed form. We, in this paper have given a closed form expressions for single and product moments of KPFD and Kumaraswamy distribution and obtained the exact and explicit expressions for ratio and inverse moments, conditional moments and Shannon entropy of KPFD based on *gos*. Further, by simple adjustment in the power of the main result of moments, it can be reduced for the moments of quotients and moments of the ratio of two *gos* of different powers. Also we have obtained MLE and UMVUE for the parameter of KPFD based on *gos*.

This paper is categorized in several sections. In Section 2, we deduced the exact and explicit expression for single moments of *gos* from KPFD. Expression for product moments of *gos* from KPFD is derived in Section 3. Exact expression for conditional moments and Shannon entropy of KPFD based on *gos* are obtained in Section 4 and 5 respectively. In Section 6, we obtained the expression for MLE of the parameters of KPFD and UMVUE of the single shape parameter. In this section, we also computed BLUE of location and scale parameter of KPFD. Section 7 included a real example which is used to interpret the findings obtained in preceding Sections.

Auxiliary results:

Here, we have summarised the results which have been used in the subsequent Sections.
1. Let X be a continuous random variable over the support (0,1). For any $a,b \in \mathbb{R}^+$, and $0 < t < 1$ we have

$$\mathscr{B}_t(a,b) = \int_0^t x^{a-1}(1-x)^{b-1}dx = \frac{t^a}{a}\, {}_2F_1(a,1-b;a+1;t) \tag{15}$$

(Mathai and Saxena [29, p–43]).
where $\mathscr{B}_t(a,b)$ is the upper incomplete beta function with parameters a,b and ${}_2F_1(\alpha,\beta;\gamma;\nu)$ is the Gauss hypergeometric function defined as (Prudnikov *et. al.* [30, p–430])

$${}_2F_1(\alpha,\beta;\gamma;\nu) = \sum_{p=0}^{\infty} \frac{(\alpha)_p(\beta)_p}{(\gamma)_p}\frac{\nu^p}{p!}$$

2. Let (X,Y) be continuous random variables and (a,b,c,d) are non negative real constants, then the value of integral is given by

$$\int_0^1 t^{a-1}(1-t)^{b-1}\, {}_2F_1(c,d,e;t)dt = \mathscr{B}(a,b)\, {}_3F_2(c,d,a;e,a+b;1) \tag{16}$$

where

$${}_\mu F_\nu(a_1,a_2,\dots,a_\mu;b_1,b_2,\dots,b_\nu;1) = \sum_{t=0}^{\infty}\left[\prod_{k=1}^{\mu}\frac{\Gamma(a_k+t)}{\Gamma(a_k)}\right]\left[\prod_{k=1}^{\nu}\frac{\Gamma(b_k)}{\Gamma(b_k+t)}\right]\frac{1}{t!}$$

for $\mu = \nu+1$ and $\sum_{k=1}^{\nu} b_k - \sum_{k=1}^{\mu} a_k > 0$, (Mathai and Saxena [29]). **3.** For any positive real number a, we have

$$\int_0^1 z^{a-1}\ln(1-z)dz = -\frac{1}{a}[\Psi(a+1)-\Psi(1)], \tag{17}$$

where $\Psi(z) = \frac{d}{dz}\ln\Gamma z$ is digamma function. (Gradshteyn and Ryzhik [31, p–558]).

2 Single moments of *gos* from KPFD

In this Section, we have derived the explicit expressions for single moments of *gos* from KPFD in terms of beta function. Further, by adjusting the parameter m and k, we have deduce the expression for single moment of order statistics, sequential order statistics and progressive type-II censored order statistics. Means ans variances of KPFD based on order statistics, progressive type-II censored order statistics and *gos* for values of the parameters $\alpha = 1$, $\theta = 2$ and $\beta = 1,2,\dots,8$ have been computed which are given in table 1 to 6 respectively.

Theorem 2.1: Let $X_1,X_2,\dots,X_n$ be n continuous random variables from KPFD defined in (14) and $X(r,n,\tilde{m},k),X(2,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k)$ be the corresponding *gos*, then the single moment of r^{th} *gos*, $1 \le r \le n$, is given by

$$\mu^{j-p}_{r,n,\tilde{m},k} = E(X^{j-p}(r,n,\tilde{m},k)) = \beta c_{r-1}\sum_{i=1}^{r} a_i(r)\mathscr{B}\left(\frac{j-p}{\alpha\theta}+1,\beta\gamma_i\right) \tag{18}$$

where $\mathscr{B}(a,b)$ is complete beta function and j,p are real numbers.

Proof: In view of (2), the single moment of r^{th} *gos* is given by

$$E(X^{j-p}_{r,n,\tilde{m},k}(x)) = c_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^1 x^{j-p}[\bar{F}]^{\gamma_i-1}f(x)dx$$

on putting the value of $f(x)$ and $\bar{F}(x)$ from (14) and after algebric simplification, we get

$$= \beta c_{r-1}\sum_{i=1}^{r} a_i(r)\int_0^1 t^{(j-p)/\alpha\theta}(1-t)^{\beta\gamma_i-1}dt$$

Now using the result given in (10), we get the theorem.

Corollary 2.1: Considering case II *i.e.* $m_1 = m_2,\ldots,m_{n-1} = m \neq -1$, from relation (10), the single moment of *gos* from KPFD is given by

$$\mu^{j-p}_{r,n,m,k} = \frac{\beta c_{r-1}}{(m+1)^{r-1}(r-1)!}\sum_{i=0}^{r-1}(-1)^{r-i}\binom{r-1}{i}\mathscr{B}\left(\frac{j-p}{\alpha\theta}+1,\beta\gamma_i\right) \tag{19}$$

Remark 2.1: If we put $\theta = 1$ in (18), we get an exact expression of single moments based on *gos* from Kumaraswamy distribution.

Remark 2.2: Putting $m = 0, k = 1$ in (19), the single moments from KPFD based on order statistics is given by

$$\mu^{j-p}_{r,n,0,1} = \frac{\beta n!}{(n-r)!(r-1)!}\sum_{i=0}^{r-1}\binom{r-1}{i}\mathscr{B}\left(\frac{j-p}{\alpha\theta}+1,\beta(n-i+1)\right) \tag{20}$$

Remark 2.3: Putting $m_i = R_i, n = m_0 + \sum_{j=1}^{m_0} R_j,\ R_j \in \mathbb{N}_0$ and $\gamma_i = n - \sum_{t=1}^{i-1} R_t - i + 1, 1 \leq j \leq m_0$ (where m_0 is fixed number of failure of units to be observed) in (18), we get the single moment based on progressive type II censored order statistics from KPFD.

Remark 2.4: Putting $\gamma_i = (n-i+1)\delta_i;\ \delta_1,\delta_2,\ldots,\delta_n > 0$ in (18), the single moment of KPFD from sequential order statistics can be obtained.

Remark 2.5: The result given in (18) is more general in the sense that for $(j < p)$, inverse moments and for $(j > p)$, the simple moments of KPFD based on *gos* can be obtained.

Table 1: Mean of KPFD based on order statistics
$n = 6, m = 0, k = 1, \alpha = 1, \theta = 2$

	β=1	β=2	β=3	β=4	β=5	β=6	β=7	β=8
$r=1$	0.3410	0.2482	0.2047	0.1781	0.1598	0.1462	0.1355	0.1269
$r=2$	0.5115	0.3807	0.3165	0.2766	0.2488	0.2279	0.2116	0.1983
$r=3$	0.6394	0.4885	0.4100	0.3602	0.3249	0.2983	0.2774	0.2603
$r=4$	0.7459	0.5883	0.4999	0.4419	0.4003	0.3685	0.3433	0.3226
$r=5$	0.8392	0.6894	0.5958	0.5316	0.4844	0.4477	0.4183	0.3939
$r=6$	0.9231	0.8049	0.7160	0.6496	0.5983	0.5572	0.5235	0.4952

From table 1 to 6, it can be clearly seen that means and variances are decrease as β increases for order statistics and *gos* and progressive type-II censored order statistics and *gos*. Moreover, in table 3 and 4, by putting different values of R_is, one can obtain mean and variance for various censoring schemes in the similar way.

3 Product moments of *gos* from KPFD

In this Section, we have deduced the exact and explicit expressions for product moments of KPFD defined in (14) based on *gos* in terms of Gauss hypergeometric function. We have also obtained the expression for order statistics, progressive

Table 2: Variance of KPFD based on order statistics
$n=6, m=0, k=1, \alpha=1, \theta=2$

	β=1	β=2	β=3	β=4	β=5	β=6	β=7	β=8
$r=1$	0.02662	0.01530	0.01070	0.00828	0.00676	0.00563	0.00494	0.00430
$r=2$	0.02407	0.01587	0.01163	0.00919	0.00760	0.00646	0.00563	0.00498
$r=3$	0.01977	0.01547	0.01210	0.00976	0.00824	0.00712	0.00615	0.00554
$r=4$	0.01503	0.01450	0.01220	0.01042	0.00896	0.00791	0.00695	0.00623
$r=5$	0.01004	0.01323	0.01262	0.01130	0.01006	0.00906	0.00823	0.00754
$r=6$	0.00499	0.01114	0.01304	0.01322	0.01264	0.01193	0.01125	0.01048

Table 3: Mean of KPFD based on progressive type-II censored order statistics
$n=20, m_0=6, \alpha=1, \theta=2$, censoring scheme $(R_1,R_2,R_3,R_4,R_5,R_6)=(2,0,4,0,0,8)$

	β=1	β=2	β=3	β=4	β=5	β=6	β=7	β=8
$r=1$	0.00430	0.00120	0.00053	0.00030	0.00019	0.00014	0.00010	0.00008
$r=2$	0.01450	0.00400	0.00180	0.00110	0.00068	0.00048	0.00035	0.00027
$r=3$	0.03270	0.00940	0.00440	0.00250	0.00160	0.00120	0.00085	0.00066
$r=4$	0.06310	0.01900	0.00900	0.00520	0.00340	0.00240	0.00180	0.00140
$r=5$	0.11370	0.03650	0.01770	0.01040	0.00690	0.00490	0.00360	0.00280
$r=6$	0.20410	0.07220	0.03640	0.02190	0.01460	0.01040	0.00780	0.00610

Table 4: Variance($\times 10^{-3}$) of KPFD based on progressive type-II censored order statistics
$n=20, m_0=6, \alpha=1, \theta=2$, censoring scheme $(R_1,R_2,R_3,R_4,R_5,R_6)=(2,0,4,0,0,8)$

	β=1	β=2	β=3	β=4	β=5	β=6	β=7	β=8
$r=1$	0.75619	0.05926	0.01294	0.00428	0.00180	0.00088	0.00048	0.00029
$r=2$	3.70360	0.32573	0.07389	0.02333	0.01033	0.00510	0.00280	0.00167
$r=3$	11.3071	1.12840	0.25981	0.09098	0.03990	0.01807	0.01063	0.00634
$r=4$	29.1839	3.32050	0.81270	0.29000	0.12653	0.06344	0.03465	0.02048
$r=5$	67.7231	9.67750	2.50000	0.92080	0.40475	0.20567	0.11952	0.07151
$r=6$	159.4319	30.87160	8.75040	3.59210	1.67720	0.88860	0.51040	0.30944

Table 5: Mean of KPFD based on *gos*
$n=6, m=1, k=2, \alpha=1, \theta=2$

	β=1	β=2	β=3	β=4	β=5	β=6	β=7	β=8
$r=1$	0.2482	0.1781	0.1462	0.1269	0.1137	0.1039	0.0963	0.0901
$r=2$	0.3807	0.2766	0.2279	0.1983	0.1779	0.1627	0.1508	0.1412
$r=3$	0.4885	0.3602	0.2983	0.2603	0.2338	0.2141	0.1986	0.1861
$r=4$	0.5883	0.4419	0.3685	0.3226	0.2904	0.2663	0.2473	0.2319
$r=5$	0.6894	0.5316	0.4477	0.3939	0.3558	0.3269	0.3040	0.2854
$r=6$	0.8049	0.6496	0.5572	0.4952	0.4500	0.4152	0.3874	0.3645

Table 6: Variance of KPFD based on *gos*
$n=6, m=1, k=2, \alpha=1, \theta=2$

	β=1	β=2	β=3	β=4	β=5	β=6	β=7	β=8
$r=1$	0.01530	0.00828	0.00563	0.00430	0.00347	0.00290	0.00253	0.00218
$r=2$	0.01587	0.00919	0.00646	0.00498	0.00405	0.00343	0.00296	0.00256
$r=3$	0.01547	0.00976	0.00712	0.00554	0.00454	0.00386	0.00336	0.00297
$r=4$	0.01450	0.01042	0.00791	0.00623	0.00517	0.00448	0.00384	0.00342
$r=5$	0.01323	0.01130	0.00906	0.00754	0.00631	0.00544	0.00488	0.00435
$r=6$	0.01114	0.01322	0.01193	0.01048	0.00920	0.00821	0.00742	0.00674

type-II censored order statistics and sequential order statistics by adjusting the parameter of *gos i.e. m, k*. Covariances based on order statistics, progressive type-II censored order statistics and *gos* from KPFD for the parameters values $\alpha = 1$, $\theta = 2$ and $\beta = 1, 2, \ldots, 8$ have been computed which are given in table 7, 8 and 9 respectively. The result for the product moment is summarized in form of theorem which is given as:

Theorem 3.1: For any $p, q, l \in \mathbb{R}$ and $\gamma_{1:r} = min(\gamma_1, \gamma_2, \ldots, \gamma_r) > \gamma_t, \forall t = r+1, \ldots, s$, the product moment of r^{th} and s^{th} *gos*, $1 \le r \le s \le n$ from KPFD is given by

$$\begin{aligned} \mu^{q,l-p}_{r,s,n,\tilde{m},k} &= E[X^q(r,n,\tilde{m},k)X^{l-p}(s,n,\tilde{m},k)] \\ &= \frac{\beta^2}{\left(\frac{q}{\alpha\theta}+1\right)} c_{s-1} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^{(r)}(s) \mathscr{B}\left(\frac{l-p+q}{\alpha\theta}+2, \beta\gamma_i\right) \\ &\quad \times {}_3F_2\left(\frac{q}{\alpha\theta}+1, 1-\beta(\gamma_j-\gamma_i), \frac{l-p+q}{\alpha\theta}+2; \frac{q}{\alpha\theta}+2, \frac{l-p+q}{\alpha\theta}+\beta\gamma_i+2; 1\right) \end{aligned} \tag{21}$$

where ${}_3F_2(a,b,c;d,e;f)$ defined in (16).

Proof: The product moment of r^{th} and s^{th} *gos* is given by

$$\mu^{q,l-p}_{r,s,n,\tilde{m},k} = c_{s-1} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^{(r)}(s) \int_0^1 \int_0^y x^q y^{l-p} \left[\frac{\bar{F}(y)}{\bar{F}(x)}\right]^{\gamma_i} [\bar{F}(x)]^{\gamma_j} \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)} dxdy, \qquad 0 \le x < y \le 1$$

from (14), we can re-write the above equation as

$$\begin{aligned} &= (\alpha\beta\theta)^2 c_{s-1} \sum_{j=1}^{r} \sum_{i=r+1}^{s} a_j(r) a_i^{(r)}(s) \\ &\quad \times \int_0^1 \int_0^y y^{l-p+\alpha\theta-1}(1-y^{\alpha\theta})^{\beta\gamma_i-1} x^{q+\alpha\theta-1}(1-x^{\alpha\theta})^{\beta(\gamma_j-\gamma_i)-1} dxdy, \qquad 0 \le x < y \le 1 \end{aligned}$$

solving the aforesaid integral by using the results given in (15) and (16), we obtained the theorem.

Corollary 3.1: For $m_1 = m_2, \ldots, m_{n-1} = m \neq -1$, the product moment from KPFD of the *gos* by using (10) and (11) is given as

$$\begin{aligned} \mu^{q,l-p}_{r,s,n,m,k} &= \frac{c_{s-1}}{(m+1)^{s-2}(r-1)!(s-r-1)!} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \mathscr{B}\left(\frac{l-p+q}{\alpha\theta}+2, \beta\gamma_i\right) \\ &\quad \times {}_3F_2\left(\frac{q}{\alpha\theta}+1, 1-\beta(\gamma_j-\gamma_i), \frac{l-p+q}{\alpha\theta}+2; \frac{q}{\alpha\theta}+2, \frac{l-p+q}{\alpha\theta}+\beta\gamma_i+2; 1\right) \end{aligned} \tag{22}$$

Remark 3.1: For $\theta = 1$, we get exact expression for product moments of Kumaraswamy distribution based on *gos*.

Remark 3.2: Putting $m = 0, k = 1$ in (22), an exact expression of product moment of order statistics from KPFD is given by

$$\begin{aligned} \mu^{q,l-p}_{r,s,n,0,1} &= \frac{n!}{(n-s)!(r-1)!(s-r-1)!} \sum_{i=0}^{r-1} \sum_{j=0}^{s-r-1} (-1)^{i+j} \binom{r-1}{i} \binom{s-r-1}{j} \mathscr{B}\left(\frac{l-p+q}{\alpha\theta}+2, \beta(n-i+1)\right) \\ &\quad \times {}_3F_2\left(\frac{q}{\alpha\theta}+1, 1+\beta(j-i), \frac{l-p+q}{\alpha\theta}+2; \frac{q}{\alpha\theta}+2, \frac{l-p+q}{\alpha\theta}+\beta(n-i+1)+2; 1\right) \end{aligned} \tag{23}$$

Remark 3.3: At $m_i = R_i, n = m_0 + \sum_{j=1}^{m_0} R_j,\ R_j \in \mathbb{N}_0$ and $\gamma_i = n - \sum_{t=1}^{i-1} R_t - i + 1, 1 \le j \le m_0$ (where m_0 is fixed number of failure of units to be observed) in (21), we get the product moment of KPFD from progressive type II censored order statistics.

Remark 3.4: If we put $\gamma_i = (n-i+1)\delta_i;\ \delta_1, \delta_2, \ldots, \delta_n > 0$ in (21), the product moment of KPFD from sequential order statistics can be obtained.

Remark 3.5: By simply adjusting $l-p$ in (21), we can see that it contains some interesting results. For example if $l-p=-q$, then

$$\mu_{r,s,n,m,k}^{q,l-p} = \left[\frac{X(r,n,\tilde{m},k)}{X(s,n,\tilde{m},k)}\right]^q$$

gives the moments of quotients. For $l-p<0$, $\mu_{r,s,n,m,k}^{q,l-p}$ represents the moments of the ratio of two *gos* of different powers while for $l-p>0$, it is a product moments of two *gos*.

Table 7: Covariance of KPFD based on order statistics
$m=0, k=1, \alpha=1, \theta=2,$

r	s	β=1	β=2	β=3	β=4	β=5	β=6	β=7	β=8
1	2	0.11128	0.06211	0.04311	0.03304	0.02674	0.02248	0.01933	0.01704
1	3	0.12486	0.07125	0.04987	0.03835	0.03108	0.02619	0.02261	0.01987
1	4	0.13745	0.08068	0.05707	0.04420	0.03603	0.03033	0.02628	0.02316
1	5	0.14923	0.09089	0.06524	0.05092	0.04179	0.03535	0.03072	0.02711
1	6	0.16022	0.10292	0.07603	0.06031	0.04989	0.04264	0.03717	0.03296
2	3	0.18725	0.11013	0.07784	0.06017	0.04906	0.04142	0.03580	0.03148
2	4	0.20627	0.12433	0.08878	0.06907	0.05651	0.04782	0.04136	0.03653
2	5	0.22385	0.13985	0.10143	0.07956	0.06538	0.05547	0.04819	0.04259
2	6	0.24023	0.15827	0.11789	0.09382	0.07794	0.06671	0.05823	0.05170
3	4	0.25777	0.16052	0.11614	0.09093	0.07464	0.06338	0.05497	0.04853
3	5	0.27972	0.18003	0.13212	0.10422	0.08602	0.07325	0.06376	0.05647
3	6	0.30027	0.20341	0.15324	0.12271	0.10231	0.08769	0.07668	0.06820
4	5	0.32644	0.21793	0.16246	0.12929	0.10719	0.09162	0.08000	0.07093
4	6	0.35046	0.24568	0.18757	0.15154	0.12690	0.10917	0.09578	0.08525
5	6	0.39413	0.28910	0.22541	0.18417	0.15548	0.13444	0.11842	0.10574

Table 8: Covariance($\times 10^{-3}$) of KPFD based on progressive type-II censored order statistics
$n=20, m_0=6, \alpha=1, \theta=2$, censoring scheme $(R_1,R_2,R_3,R_4,R_5,R_6)=(2,0,4,0,0,8)$

r	s	β=1	β=2	β=3	β=4	β=5	β=6	β=7	β=8
1	2	0.86110	0.08234	0.02381	0.00621	0.00263	0.00109	0.00080	0.00066
1	3	0.97125	0.13185	0.02880	0.00863	0.00452	0.00167	0.00122	0.00089
1	4	1.27068	0.15720	0.03635	0.01354	0.00687	0.00288	0.00144	0.00111
1	5	1.97291	0.19579	0.04366	0.01717	0.00815	0.00368	0.00167	0.00128
1	6	2.10292	0.26302	0.06234	0.02003	0.00947	0.00413	0.00199	0.00143
2	3	4.32113	0.46167	0.05012	0.03486	0.01846	0.00712	0.00314	0.00312
2	4	5.52033	0.68971	0.06872	0.04390	0.02138	0.00889	0.00415	0.00386
2	5	7.38454	0.83655	0.09957	0.05589	0.02643	0.01272	0.00656	0.00445
2	6	9.51827	0.94329	0.15674	0.07170	0.03325	0.01542	0.00944	0.00522
3	4	14.66555	1.49928	0.33368	0.09986	0.04791	0.02794	0.01786	0.00987
3	5	17.10680	2.01068	0.43726	0.12647	0.07009	0.03616	0.02039	0.01214
3	6	22.03614	2.62775	0.68789	0.18690	0.09588	0.04269	0.02646	0.01677
4	5	36.19298	5.22588	0.96726	0.37109	0.19881	0.09983	0.05312	0.03772
4	6	49.25673	7.71122	1.79768	0.65485	0.32324	0.13542	0.09911	0.04832
5	6	79.89013	18.08176	5.31742	1.94588	0.89872	0.57747	0.34524	0.14367

Table 9: Covariance of KPFD based on *gos*
$m=1, k=2, \alpha=1, \theta=2,$

r	s	β=1	β=2	β=3	β=4	β=5	β=6	β=7	β=8
1	2	0.06211	0.03304	0.02248	0.01704	0.01367	0.01150	0.00988	0.00868
1	3	0.07125	0.03835	0.02619	0.01987	0.01602	0.01346	0.01157	0.01013
1	4	0.08068	0.04420	0.03033	0.02316	0.01868	0.01563	0.01349	0.01181
1	5	0.09089	0.05092	0.03535	0.02711	0.02195	0.01844	0.01592	0.01399
1	6	0.10292	0.06031	0.04264	0.03296	0.02684	0.02266	0.01959	0.01726
2	3	0.11013	0.06017	0.04142	0.03148	0.02551	0.02137	0.01835	0.01612
2	4	0.12433	0.06907	0.04782	0.03653	0.02954	0.02477	0.02141	0.01876
2	5	0.13985	0.07956	0.05547	0.04259	0.03460	0.02911	0.02516	0.02210
2	6	0.15827	0.09382	0.06671	0.05170	0.04215	0.03565	0.03088	0.02723
3	4	0.16052	0.09093	0.06338	0.04853	0.03940	0.03309	0.02859	0.02514
3	5	0.18003	0.10422	0.07325	0.05647	0.04591	0.03861	0.03343	0.02939
3	6	0.20341	0.12271	0.08769	0.06820	0.05579	0.04721	0.04086	0.03607
4	5	0.21793	0.12929	0.09162	0.07093	0.05788	0.04885	0.04232	0.03722
4	6	0.24568	0.15154	0.10917	0.08525	0.06992	0.05923	0.05140	0.04537
5	6	0.28910	0.18417	0.13444	0.10574	0.08709	0.07407	0.06443	0.05697

4 Conditional moments of *gos* from KPFD

In this Section, we have obtained the conditional moments of s^{th} *gos* given r^{th} *gos* of the KPFD in terms of Gauss hypergeometric function. The result can be reduced in form of order statistics, progressive type II censored order statistics and sequential order statistics by adjusting the parameters of *gos*.

Theorem 4.1: The conditional moment of s^{th} *gos* given r^{th} *gos* $(X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)=x)$, $1 \le r \le s \le n$, from KPFD is given by

$$\mu^{q}_{s|r,n,\tilde{m},k}(y|x) = \int_x^1 y^q f_{s|r}(x,y|x=r)dy = \frac{c_{s-1}}{c_{r-1}} \sum_{j=r+1}^{s} \frac{a_j^{(r)}(s)}{\gamma_j} \, {}_2F_1\left(\beta\gamma_j, -\frac{q}{\alpha\theta}; \beta\gamma_j+1; 1-x^{\alpha\theta}\right) \tag{24}$$

provided that $q>0$ and ${}_2F_1(a,b;c;d)$ defined in (15).

Proof: In view of (4) and (14), the q^{th} conditional moment of $X(s,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k)=x$, from the KPFD is written as

$$\begin{aligned}\mu^{q}_{X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)}(y|x) &= \int_x^1 y^q f_{X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)}(y|x)dy \\ &= \alpha\beta\theta\frac{c_{s-1}}{c_{r-1}} \sum_{j=r+1}^{s} a_j^r(s)(1-x^{\alpha\theta})^{-\beta\gamma_j} \int_x^1 y^{q+\alpha\theta-1}(1-y^{\alpha\theta})^{\beta\gamma_j-1}dy\end{aligned}$$

Setting $z=y^{\alpha\theta}$ and after some algebraic simplification, we get

$$\mu^{q}_{X(s,n,\tilde{m},k)|X(r,n,\tilde{m},k)}(y|x) = \beta\frac{c_{s-1}}{c_{r-1}} \sum_{j=r+1}^{s} a_j^r(s)(1-x^{\alpha\theta})^{-\beta\gamma_j} \int_{x^{\alpha\theta}}^1 z^{q/\alpha\theta}(1-z)^{\beta\gamma_j-1}dz$$

Now, by using (15), the result can be established.

Theorem 4.2: For any $q>0$, the conditional moment of r^{th} given s^{th} *gos* $(X(r,n,\tilde{m},k)|X(s,n,\tilde{m},k)=y)$, $1 \le r \le s \le n$, from KPFD is given by

$$\mu^{q}_{r|s,n,m,k}(x|y) = \frac{\sum_{j=1}^{r}\sum_{i=r+1}^{s} a_j(r)a_i^{(r)}(s)}{\sum_{t=1}^{s} a_t(s)} \frac{(1-y^{\alpha\theta})^{\beta(\gamma_j-\gamma_i)}}{(\gamma_j-\gamma_i)} \, {}_2F_1[\beta(\gamma_j-\gamma_i), -\frac{q}{\alpha\theta}; \beta(\gamma_j-\gamma_i)+1; 1-y^{\alpha\theta}] \tag{25}$$

Proof: In view of (5) and (14), we can proof the theorem in the similar way of theorem 4.1.

5 Shannon entropy of KPFD based on *gos*

Shannon [32] introduced the concept of entropy and it is a mathematical measure of information which measures the average reduction of uncertainty of a random variable X. For a continuous X with *pdf* $f(x)$, the Shannon entropy is defined as

$$H(X) = -\int_{-\infty}^{+\infty} f_X(x) \ln f_X(x) dx \tag{26}$$

Wong and Chen [33] have deduced the entropy of order sequence and order statistics. Shannon entropy in record data was discussed in [34,?]. An exact form of Shannon entropy based on *gos* from Pareto-type distributions given in [36]. Khan and Sharma [37] have obtained an analytic and exact expression for shannon entropy based on *gos* from Nadarajah and Haghighi distribution. Here, we have deduced an exact and analytical expression of Shanon entropy based on *gos* from KPFD.

Let $\mathbf{X}(n,\tilde{m},k) = (X(1,n,\tilde{m},k),\dots,X(n,n,\tilde{m},k))$ be the vector of first n *gos* then the joint *pdf* of these n *gos* from KPFD is given by

$$f_{\mathbf{X}(n,\tilde{m},k)}(\mathbf{x}) = k(\alpha\beta\theta)^n \left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{j=1}^{n-1} x_j^{\alpha\theta-1}(1-x_j^{\alpha\theta})^{\beta(m_j+1)-1}\right) x_n^{\alpha\theta-1}(1-x_n^{\alpha\theta})^{\beta k-1} \tag{27}$$

Theorem 5.1: The Shannon entropy of the vector $\mathbf{X}(n,\tilde{m},k)$ is given by

$$H(\mathbf{X}(n,\tilde{m},k)) = -\ln k - n\ln\alpha - n\ln\beta - n\ln\theta - \sum_{j=1}^{n-1}\ln\gamma_j + \left(1-\frac{1}{\alpha\theta}\right) c_{j-1}\sum_{i=1}^{j}\sum_{j=1}^{n}\frac{a_i(j)}{\gamma_i}[\Psi(\beta\gamma_i+1) - \Psi(1)] + \frac{1}{\beta}\sum_{i=1}^{j}\sum_{j=1}^{n-1}\frac{\beta(m_j+1)-1}{\gamma_i} + \frac{(\beta k-1)}{\beta}\sum_{i=1}^{n}\frac{1}{\gamma_i} \tag{28}$$

where $\Psi(z) = \frac{d}{dz}\ln\Gamma z$.

Proof: From (26),

$$H(\mathbf{X}(n,\tilde{m},k)) = -E(\ln f_{\mathbf{X}(n,\tilde{m},k)})$$

from (27), we have

$$H(\mathbf{X}(n,\tilde{m},k)) = -\ln k - n\ln\alpha - n\ln\beta - n\ln\theta - \sum_{j=1}^{n-1}\ln\gamma_j - (\alpha\theta-1)\sum_{j=1}^{n}E(\ln x_j) - \sum_{j=1}^{n-1}(\beta(m_j+1)-1)E(\ln(1-x_j^{\alpha\theta})) - (\beta k-1)E[\ln(1-x_n^{\alpha\theta})] \tag{29}$$

In view of (2),

$$E(\ln x_j) = \alpha\beta\theta c_{j-1}\sum_{i=1}^{j}a_i(j)\int_0^1 \ln x_j\, x_j^{\alpha\theta-1}(1-x_j^{\alpha\theta})^{\beta\gamma_i-1}dx_j$$

putting $(1-x_j^{\alpha\theta}) = u$ and doing some mathematical calculations, we get

$$E(\ln x_j) = \frac{\beta c_{j-1}}{\alpha\theta}\sum_{i=1}^{j}a_i(j)\int_0^1 u^{\beta\gamma_i-1}\ln(1-u)du$$

Using the result given in (17), we get

$$E(\ln x_j) = -\frac{c_{j-1}}{\alpha\theta}\sum_{i=1}^{j}\frac{a_i(j)}{\gamma_i}[\Psi(\beta\gamma_i+1) - \Psi(1)] \tag{30}$$

Similarly, we get

$$E[\ln(1-x_j^{\alpha\theta})] = -\frac{c_{j-1}}{\beta}\left(\sum_{i=1}^{j}\frac{a_i(j)}{\gamma_i^2}\right) = -\frac{1}{\beta}\left(\sum_{i=1}^{j}\frac{1}{\gamma_i}\right) \tag{31}$$

and

$$E[\ln(1-x_n^{\alpha\theta})] = -\frac{1}{\beta}\left(\sum_{i=1}^{n}\frac{1}{\gamma_i}\right) \tag{32}$$

(see [38], p-309).
Putting the values from (30), (31) and (32) in equation (29), we get the theorem.

6 Estimation for the parameters of KPFD based on *gos*

In this Section, we have obtained MLE and UMVUE for the parameters of KPFD defined in (14) based on *gos*. MLE is obtained for all the shape parameters of KPFD and UMVUE is derived for the parameter β. In order to obtain UMVUE, we have assumed that α and θ to be known. An appropriate literature is available on the theory of estimation for the continuous random variable based on *gos*. MLE and Bayesian estimators based on *gos* for the parameter of Burr type XII distribution have been obtained by Jaheen [39]. Malinowska *et. al.* [40] have derived MLE, minimum variance linear unbiased estimators (MVLUE) and Best linear invariant estimators (BLIE) for the location and scale parameters of Burr-XII distribution based on *gos*. El-Deen *et. al.* [26] obtained the MLE and Bayes estimators using different priors for Kumaraswamy distribution based on *gos*. Khan and Iqrar [27] obtained the expressions for MLE and UMVUE for the parameter of topp-Leone distribution based on dual *gos*. MLE, UMVUE and Bayes estimates of stress-strength reliability based on *gos* for exponential distribution have been obtained in [41].

6.1 MLE of KPFD based on gos

In this Section we have obtained MLE for the parameters of KPFD based on *gos*. From (1) and (14), the likelihood function based on *gos* from KPFD is written as

$$L(\alpha,\beta,\theta|\underline{X}) = k(\alpha\beta\theta)^n\left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{j=1}^{n}x_j^{\alpha\theta-1}\right)\left(\prod_{j=1}^{n-1}(1-x_j^{\alpha\theta})^{\beta(m_j+1)-1}\right)(1-x_n^{\alpha\theta})^{\beta k-1} \tag{33}$$

where $\underline{X} = (X(1,n,\tilde{m},k),X(2,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k))$.
Thus, the log-likelihood function is given by

$$\ln L(\alpha,\beta,\theta|\underline{X}) = \ln k + n\ln\alpha + n\ln\beta + n\ln\theta + \sum_{j=1}^{n-1}\ln\gamma_j + (\alpha\theta-1)\sum_{j=1}^{n}\ln x_j + \sum_{j=1}^{n-1}[\beta(m_j+1)-1]\ln(1-x_j^{\alpha\theta}) + (\beta k-1)\ln(1-x_n^{\alpha\theta}) \tag{34}$$

Differentiating (34) with respect to α, β and θ and equating to zero, we have

$$\frac{n}{\alpha} + \theta\sum_{j=1}^{n}\ln x_j - \sum_{j=1}^{n-1}[\beta(m_j+1)-1]\frac{\theta x_j^{\alpha\theta}lnx_j}{1-x_j^{\alpha\theta}} - (\beta k-1)\frac{\theta x_n^{\alpha\theta}lnx_n}{1-x_j^{\alpha\theta}} = 0 \tag{35}$$

$$\frac{n}{\beta} + \sum_{j=1}^{n-1}(m_j+1)\ln(1-x_j^{\alpha\theta}) + k\ln(1-x_n^{\alpha\theta}) = 0 \tag{36}$$

and

$$\frac{n}{\theta} + \alpha\sum_{j=1}^{n}\ln x_j - \sum_{j=1}^{n-1}[\beta(m_j+1)-1]\frac{\alpha x_j^{\alpha\theta}lnx_j}{1-x_j^{\alpha\theta}} - (\beta k-1)\frac{\alpha x_n^{\alpha\theta}lnx_n}{1-x_j^{\alpha\theta}} = 0 \tag{37}$$

Exact expression for MLE of α, β and θ can't be obtained directly so we use numerical computation technique to obtained the MLE of α, β and θ.

6.2 UMVUE of KPFD based on gos

Here, we have obtained UMVUE of parameter β of KPFD by assuming that α and θ are known. The joint *pdf* of n *gos* from KPFD can be re-written as

$$f(\underline{X}) = k(\alpha\beta\theta)^n \left(\prod_{j=1}^{n-1}\gamma_j\right)\left(\prod_{j=1}^{n}\frac{x_j^{\alpha\theta-1}}{1-x_j^{\alpha\theta}}\right)\left[\prod_{j=1}^{n-1}(1-x_j^{\alpha\theta})^{(m_j+1)}(1-x_n^{\alpha\theta})^k\right]^{\beta} \tag{38}$$

from (36), we have

$$\hat{\beta} = \frac{n}{\zeta} \tag{39}$$

where $\zeta = -\sum_{j=1}^{n-1}(m_j+1)\ln(1-x_j^{\alpha\theta}) - k\ln(1-x_n^{\alpha\theta})$.
Using normalized spacing of *gos* (Kamps[1], p–81), it can be clearly seen that ζ follows gamma distribution with shape parameter n and scale parameter β and the bias of $\hat{\beta}$ is $\frac{\beta}{n-1}$. Further, in view of (38), ζ is complete and sufficient statistics for β. From the property of gamma distribution, an unbiased estimator of ζ is $\frac{n-1}{\beta}$. Hence the UMVUE of β is

$$\hat{\beta}_{UMVUE} = \frac{n-1}{\zeta}$$

6.3 BLUE of KPFD based on gos

The results given in (18) and (21) allows us to evaluate means, variances and covariances of KPFD based on *gos*. By inserting location and scale parameter in KPFD, we have the *pdf* given by

$$f(x) = \frac{\alpha\beta\theta}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{\alpha\theta-1}\left[1-\left(\frac{x-\mu}{\sigma}\right)^{\alpha\theta}\right]^{\beta-1}, \ \mu < x \le \mu+\sigma, \ \alpha,\beta,\theta,\mu,\sigma > 0. \tag{40}$$

Let $X(1,n,\tilde{m},k), X(2,n,\tilde{m},k),\ldots,X(n,n,\tilde{m},k)$ be the n *gos* from the distribution given in (40), then

$$Y(i,n,\tilde{m},k) = \frac{X(i,n,\tilde{m},k)-\mu}{\sigma}, \quad i = 1,2,\ldots,n \tag{41}$$

be the vector of n *gos* from a population with *pdf* given in (40). Thus the best linear unbiased estimators (BLUE) of μ and σ can be written as

$$\hat{\mu} = a_1X(1,n,\tilde{m},k) + a_2X(2,n,\tilde{m},k) + \ldots + a_nX(n,n,\tilde{m},k) \tag{42}$$

$$\hat{\sigma} = b_1X(1,n,\tilde{m},k) + b_2X(2,n,\tilde{m},k) + \ldots + b_nX(n,n,\tilde{m},k) \tag{43}$$

Here $a_i's$ and $b_i's$ are the entries of the matrix $\Delta = (A'\Sigma^{-1}A)^{-1}A'\Sigma^{-1}$ with $A = (\underset{\sim}{1} \ \underset{\sim}{\mu})$, $\underset{\sim}{1}' = (1,\ldots,1)_{1\times n}$, $\underset{\sim}{\mu}' = (\mu_1,\ldots,\mu_n)_{1\times n}$, where $\underset{\sim}{\mu}$ is the mean vector and Σ^{-1} is the inverse of covariance matrix $\Sigma = (\sigma_{r\times s})_{n\times n}$. Moreover, variances and covariances of these estimators are given by

$$V(\hat{\mu}) = d_{11}\sigma^2, \ V(\hat{\sigma}) = d_{22}\sigma^2 \text{ and } covar(\hat{\mu},\hat{\sigma}) = d_{12}\sigma^2,$$

where

$$D = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22}\end{pmatrix}\sigma^2 = (A'\Sigma^{-1}A)^{-1}.$$

In context of *gos*, BLUE for location and scale parameter of Weibull distribution was obtained in [42]. Similarly, in case of order statistics, record values and progressive type II censoring, the aforesaid written formula is used to obtain BLUE of location and scale parameter for a family of distribution (for order statistics see [44], in case of records see [45] and in case of progressive type II censoring, see [46]).
The results given in (18) and (21) can also be used to predict the future *gos*. Suppose we have observed only first r *gos*

i.e. $\underset{\sim}{Y} = (Y(1,n,\tilde{m},k), Y(2,n,\tilde{m},k), ..., Y(r,n,\tilde{m},k))$ and our target is to predict $Y(s,n,\tilde{m},k), 1 \leq r < s$, when F belongs to location scale family of distribution then the formula for best linear unbiased predictor (BLUP) is given by

$$\hat{Y}(s,n,\tilde{m},k) = (\hat{\mu} + \hat{\sigma}\underset{\sim}{\mu})\, W'\, \Sigma^{-1}\, (\underset{\sim}{X} - \hat{\mu}.\, 1 - \sigma\, \underset{\sim}{\mu})$$

where $\underset{\sim}{\mu}$ is the mean of first r gos and W' is the vector of the covariances between the s^{th} future gos and the first r recorded observations.

7 Real Example

In this Section, we consider a real data set as an application of estimation method described in this paper. The real data deals with the monthly water capacity of Shasta Reservoir in California in the month of Fabruary from 1991 to 2010, $(http://cdec.water.ca.gov/reservoir\_map.html)$ and used by several authors for inference purpose, like [22,**?**,**?**]. The maximum capacity of the reservoir is 4,552,000 AF. The data is given in the table-10 which contain the actual capacity and proportion of total capacity of water. We use proportion of total capacity of water for illustration purpose. Before

Table 10: Monthly capacity for August and proportion of total capacity for Shasta reservoir.

Year	Capacity	Proportion of Total Capacity	Year	Capacity	Proportion of Total Capacity
1991	1,542,838	0.338936	2001	3,495,969	0.768007
1992	1,966,077	0.431915	2002	3,839,544	0.843485
1993	3,459,209	0.759932	2003	3,584,283	0.787408
1994	3,298,496	0.724626	2004	3,868,600	0.849868
1995	3,448,519	0.757583	2005	3,168,056	0.695970
1996	3,694,201	0.811556	2006	3,834,224	0.842316
1997	3,574,861	0.785339	2007	3,772,193	0.828689
1998	3,567,220	0.783660	2008	2,641,041	0.580194
1999	3,712,733	0.815627	2009	1,960,458	0.430681
2000	3,857,423	0.847413	2010	3,380,147	0.742563

progressing further, first we check whether the given data follows KPFD or not. We use Kolmogorov-Smirnov (K-S) test which show that the K-S statistics and p-value for $KPFD(1,1.5,2,0,1)$ are 0.25 and 0.5713 respectively. Based on the value of K-S statistics and p-value, we conclude that that the given data is perfectly fitted for $KPFD(1,1.5,2,0,1)$.

Now, we find the BLUE based on progressive type II censoring for $n = 20, m_0 = 6, \alpha = 1, \beta = 1.5 and \theta = 2$, censoring scheme $(R_1,R_2,R_3,R_4,R_5,R_6) = (2,0,4,0,0,8)$. The Mean and variance-covariance matrix for the aforesaid values are given by

By following the procedure defined in Section (6.3) and using the values given in table 11-12, the coefficients of BLUEs are given by

where $i = 1,2,\ldots,6$, a and b are the coefficient of μ and σ respectively. Also, $\sum_{i=1}^{6} a_i = 1$ and $\sum_{i=1}^{6} b_i = 0$. The variance of μ, σ and variance-covariance of μ, σ are 0.06866991, 4.506055 and -2.740943 respectively. From table 10, we examine that there are $n = 20$ units and out of 20 we select $m_0 = 6$ units based on progressive type-II censoring procedure discussed in [46], for censoring scheme $(R_1,R_2,R_3,R_4,R_5,R_6) = (2,0,4,0,0,8)$ which are $0.580194, 0.695970, 0.724626, 0.78366, 0.815627, 0.843485$. The MLE of α, β and θ are 1.625015 1.013204 and 2.186006 respectively. If we fixed $\alpha = 1, \theta = 2$, the UMVUE for β is 1.303058 and mean squared error is 0.038786 for the censored samples. Now, by using the values of the coefficients of μ, σ given in table-13, the BLUEs of μ, σ are given by

Table 11: Mean of KPFD based on progressive type-II censored sample
$n = 20, m_0 = 6, \alpha = 1, \beta = 1.5, \theta = 2$, censoring scheme $(R_1,R_2,R_3,R_4,R_5,R_6) = (2,0,4,0,0,8)$

$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
0.00200	0.00690	0.01450	0.02990	0.05110	0.07860

Table 12: Variance-covariance of KPFD based on progressive type-II censored sample $n=20, m_0=6, \alpha=1, \beta=1.5, \theta=2$, censoring scheme $(R_1,R_2,R_3,R_4,R_5,R_6)=(2,0,4,0,0,8)$

	$s=1$	$s=2$	$s=3$	$s=4$	$s=5$	$s=6$
$r=1$	0.00002	0.11689	0.13477	0.15324	0.16870	0.18214
$r=2$	0.11689	0.00009	0.18390	0.20889	0.22975	0.24796
$r=3$	0.13477	0.18390	0.00025	0.24597	0.27036	0.29166
$r=4$	0.15324	0.20889	0.24597	0.00081	0.31107	0.33505
$r=5$	0.16870	0.22975	0.27036	0.31107	0.00169	0.37008
$r=6$	0.18214	0.24796	0.29166	0.33505	0.37008	0.00302

Table 13: Coefficients of BLUEs for μ and σ

a_i	0.363963	0.356948	0.277355	0.147521	0.005978	-0.151767
b_i	-14.527661	-3.530057	0.443028	3.576835	5.939415	8.098439

$$\begin{aligned}\hat{\mu} &= 0.580194\times 0.363963+0.695970\times 0.356948+0.724626\times 0.277355+0.78366\\ &\quad\times 0.147521+0.815627\times 0.005978+0.843485\times(-0.151767)\\ &= 0.655401\end{aligned}$$

$$\begin{aligned}\hat{\sigma} &= 0.580194\times(-14.527661)+0.695970\times(-3.530057)+0.724626\times 0.443028\\ &\quad+0.78366\times 3.576835+0.815627\times 5.939415+0.843485\times 8.098439\\ &= 3.765914\end{aligned}$$

8 Conclusion

In this paper, we consider the KPFD and Kumaraswamy distribution as the reduced form of KPFD. For these distributions, we deduce exact and explicit expressions for ratio, inverse and conditional moments based on *gos*. Moreover, based on *gos*, Shannon entropy and statistical estimation by using the methods of MLE, UMVUE and BLUE for the parameters of underlying distributions. Further, by adjusting the parameter values of *gos*, we obtain the expressions for order statistics, progressive type-II censored order statistics and sequential order statistics. Also, means and variance-covariance matrices based on order statistics, progressive type-II censored order statistics and *gos* are computed. We also computed MLE, UMVUE and BLUE for the parameters of KPFD of a real data set based on progressive type-II censored order statistics which shows that UMVUE perform better than the other described method of estimation.

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Conflict of Interest

The authors declare that they have no conflict of interest.

References

[1] U. Kamps. *A concept of generalized order statistics*. B. G. Teubner Rtuttgart, Germany (1995).
[2] N. Balakrishnan, R. Aggarwala. *Progressive censoring: theory, methods, and applications*. Boston, Birkhuser (2000).
[3] U. Kamps and E. Cramer. On distributions of generalized order statistics. *Statistics*, **35(3)**, 269-280 (2001).
[4] A.H. Khan and M.J.S. Khan. On ratio and inverse moments of generalized order statistics from Burr distribution. *Pakistan J. Statist.*, **28(1)**, 59-68 (2012).

[5] P. Kumaraswamy. Generalized probability density function for double bounded random-processes. *J. Hydrol.*, **46(1-2)**, 79-88 (1980).
[6] S. Nadarajah. On the distribution of Kumaraswamy. *J. Hydrol.*, **348(3)**, 568-569 (2008).
[7] M.C. Jones. Kumaraswamy?s distribution: A beta-type distribution with some tractability advantages. *Stat. Methodol.*, **6(1)**, 70-81 (2009).
[8] G.M. Cordeiro and M. De-Castro. A new family of generalized distributions. *J. Statist. Comput. Simul.*, **81(7)**, 883-898 (2011).
[9] P.A. Mitnik. New Properties of the Kumaraswamy distribution. *Comm. Statist. Theory Method*, **42(5)**, 741-755 (2013).
[10] G.M. Cordeiro and R.D.S.B. Bager. Moments for Some Kumaraswamy Generalized Distributions. *comm. Statist. Theory Methods*, **44(13)**, 2720-2737 (2015).
[11] A.S. Hassan and M. Elgarhy. Kumaraswami Weibull-generated family of distributions with applications. *Advances and Applications in statistics*, **43(3)**, 205-239 (2016).
[12] L. Wang. Inference for the Kumaraswamy distribution under k-record values. *J. Comput. Appl. Math.*, **321**, 246-260 (2017).
[13] I.B. Abdul-Moniem. The Kumaraswamy Power Function Distribution. *J. Stat. Appl. Pro.*, **6(1)**, 81-90 (2017).
[14] L. Wang. Inference of progressively censored competing risks data from Kumaraswamy distributions. *J. Comput. Appl. Math.*, **343**, 719-736 (2018).
[15] G.M. Cordeiro, E.M.M. Ortegab and S. Nadarajah. The Kumaraswamy Weibull distribution with application to failure data. *J. Franklin Inst.*, **347(8)**, 1399-1429 (2010).
[16] M.Q. Shahbaz, S. Shahbaz and N.S. Butt. The Kumaraswamy inverse-Weibull distribution. *Pak. j. stat. oper. res.*, **8(3)**, 479-489 (2012).
[17] T.M. Shams. The Kumaraswamy generalized Lomax distribution. *Middle-East J Sci Res*, **17(5)**, 641-646 (2013).
[18] A.E. Gomes, C.Q. Da Silva, G.M. Cordeiro and E.M.M. Ortega. A new lifetime model: The Kumaraswamy generalized Rayleigh distribution. *J. Stat. Comput. Simul.*, **84(2)**, 290-309 (2014).
[19] D. Kumar, M. Kumar, J. Saran and N. Jain. The Kumaraswamy-Burr III Distribution Based on Upper Record Values. *Amer. J. Math. Management Sci.*, **36(3)**, 205-228 (2017).
[20] M.M. El-Din and M. Abu-Moussa. On estimation and prediction for the inverted Kumaraswamy distribution based on general progressive censored samples. *Pak. j. stat. oper. res.*, **14(3)**, 717-736 (2018).
[21] K.K. Jose and J. Varghese. Wrapped Log Kumaraswamy distribution and its applications. *International Journal of Mathematics and Computer Research*, **6(10)**, 1924-1930 (2018).
[22] R.M. El-Sagheer. Estimating the parameters of Kumaraswamy distribution using progressively censored data. *Journal of Testing and Evaluation*, **47(2)**, 905-926 (2019).
[23] M. Garg. On Generalized Order Statistics From Kumaraswamy Distribution. *Tamsui Oxford J. Infor. and Math. Sci.*, **25(2)**, 153-166 (2009).
[24] H. Athar, H.M. Islam and M. Yaqub. On ratio and inverse moments of generalized order statistics from weibull distribution. *J. Appl. Statist. Sci.*, **16(1)**, 37-46 (2007).
[25] S.K. Safi and R.H. Ahmed. Statistical estimation based on generalized order statistics from Kumaraswamy distribution. *Proceedings of the 15th Applied Stochastic Models and Data Analysis (ASMDA)*, Mataró (Barcelona), Spain, 25-28 (2013).
[26] M.M.S. El-Deen, G.R. AL-Dayian and A.A. El-Helbawy. Statistical inference for Kumaraswamy distribution based on generalized order statistics with applications. *British Journal of Mathematics & Computer Science*, **4(12)**, 1710-1743 (2014).
[27] M.J.S. Khan and S. Iqrar. On moments of dual generalized order statistics from Topp-Leon distribution. *comm. Statist. Theory Methods*, **48(3)**, 479-492 (2019).
[28] D. Kumar. Generalized order statistics from Kumaraswamy distribution and its characterization. *Tamsui Oxford J. Infor. and Math. Sci.*, **27(4)**, 463-476 (2011).
[29] A.M. Mathai and R.K. Saxena. Generalized hyper-geometric functions with applications in statistics and physical science. *Lecture Notes in Mathematics*. **348**, Springer-Verlag, Berlin (1973).
[30] A.P. Prudnikov, Y.A. Brychkov and O.I. Marichev. *Integral and series*. Vol–1, Gordon and Breach Science Publishers (1986).
[31] I.S. Gradshteyn and I.M. Ryzhik. *Table of Integrals, Series and Products*. Elsevier Academic Press Publications (2007).
[32] C. Shannon. A mathematical theory of communication. *Bell system technical journal*, **27**, 379-423 (1948).
[33] K.M. Wong and S. Chen. The entropy of order sequences and order statistics. *IEEE T. Inform. Theory*, **36(2)**, 276-284 (1990).
[34] M. Madadi and M. Tata. Shannon information in record data. *Metrika*, **74(1)**, 11-31 (2011).
[35] M. Madadi and M. Tata. Shannon information in k-records. *Comm. Statist. Theory Method*, **43(15)**, 3286-3301 (2014).
[36] B. Afhami and M. Madadi. Shannon entropy in generalized order statistics from Pareto-type distributions. *Int. J. Nonlinear Anal. Appl.*, **4(1)**, 79-91 (2013).
[37] M.J.S. Khan and A. Sharma. Shannon entropy and characterization of Nadarajah Haghighi distribution based on generalized order statistics. *Journal of Statistics: Advances in Theory and Applications*, **19(1)**, 43-69 (2018).
[38] N. Balakrishnan, E. Cramer and U. Kamps. Bounds for means and variances of progressive type II censored order statistics, *Statist. Probab. Lett.*, **54(3)**, 301-315 (2001).
[39] Z.F. Jaheen. Estimation based on generalized order statistics from the burr model. *Comm. statis. Theory Methods*, **34(4)**, 785-794 (2005).
[40] I. Malinowska, P. Pawlas and D. Szynal. Estimation of location and scale parameters for the Burr XII distribution using generalized order statistics. *Linear Algebra and Its Applications*, **417(1)**, 150-162 (2006).
[41] M.J.S. Khan and B. Khatoon. Statistical inferences of R=P(X<Y) for exponential distribution based on generalized order statistics. *Ann. Data Sci.*, **7**, 525-545 (2020).

[42] R. Jabeen, A. Ahmad, N. Feroze and G.M. Gilani. Estimation of location and scale parameters of Weibull distribution using generalized order statistics under type II singly and doubly censored data. *International Journal of Advanced Science and Technology*, **55**, 67-80 (2013).
[43] M. Nadar, A. Papadopoulos and F. Kizilaslan. Statistical analysis for Kumaraswamys distribution based on record data. *Stat. Papers*, **54(2)**, 355-369 (2013).
[44] H. A. David and H. N. Nagaraja. *Order Statistics*. John Wiley & Sons, INC., New York, (2003)
[45] B. C. Arnold, N. Balakrishnan and H. N. Nagaraja. *Records*. John Wiley & Sons, INC., New York, (1998).
[46] N. Balakrishnan and R. Aggarwala. *Progressive Censoring: Theory, Methods, and Applications*. Springer Science & Business Media, New York (2000).