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Characterizations of Regular Semigroups

Bingxue Yao

School of Mathematics Science, Liaocheng University, Liaocheng, Shandong 252059, China

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Abstract: The concepts of $(\lambda,\mu)-$fuzzy subsemigroup and various $(\lambda,\mu)-$fuzzy ideals of a semigroup were introduced by generalizing $(\in,\in\vee q)-$fuzzy subsemigroup and $(\in,\in\vee q)-$fuzzy ideal. The regular semigroup was characterized by the properties of the middle parts of various $(\lambda,\mu)-$fuzzy ideals, and several equivalence conditions of regular semigroups were obtained.

Keywords: $(\lambda,\mu)-$fuzzy ideal; $(\lambda,\mu)-$fuzzy quasi-ideal; $(\lambda,\mu)-$fuzzy bi-ideal; regular semigroup

1 Introduction

The concept of fuzzy set, introduced by Zadeh [11], was applied to the theory of groups by Rosenfeld [7]. Since then, many scholars have been engaged in the fuzzification of some algebraic structures. Kuruki [3,4] initiated the theory of fuzzy semigroups, and introduced the concepts of fuzzy ideal and fuzzy bi-ideal. A systemtic exposition of fuzzy semigroup by Mordeson et al appeared in [5] , where one can find the theoretical results of fuzzy semigroup and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages.

It is worth to be pointed out that Bhakat and Das [1,2] introduced the concepts of $(\alpha,\beta)-$fuzzy subgroup by using the "belongs to" relation and "quasi-coincident with" relation between a fuzzy point and a fuzzy subset, and gave the concepts of $(\in,\in\vee q)-$fuzzy subgroup and $(\in,\in\vee q)-$fuzzy subring. Muhammad et al [6] studied the characterizations of regular semigroups using $(\in,\in\vee q)-$fuzzy ideals.

It is well known that a fuzzy subset A of a group G is a Rosenfeld's fuzzy subgroup if and only if $A_t = \{x \in G \mid A(x) \geqslant t\}$ is a subgroup of G for all $t \in (0,1]$ (for our convenience, here $\emptyset$ is regarded as a subgroup of G). Similarly, A is an $(\in,\in\vee q)-$fuzzy subgroup if and only if A_t is a subgroup of G for all $t \in (0,0.5]$. A corresponding result should be considered naturally when A_t is a subgroup of G for all $t \in (a,b]$,where $(a,b]$ is an arbitrary subinterval of [0,1].

Motivated by above problem, Yuan et al [10] introduced fuzzy subgroup with thresholds of a group. In order to generalize the concepts of $(\in,\in\vee q)-$fuzzy subsemigroup and $(\in,\in\vee q)-$fuzzy ideal of a semigroup, Yao [9] introduced the notions of $(\lambda,\mu)-$fuzzy subsemigroup and various $(\lambda,\mu)-$fuzzy ideals, and discussed their fundamental properties. In this paper, we will characterize regular semigroups by the properties of $(\lambda,\mu)-$fuzzy ideal, $(\lambda,\mu)-$ fuzzy quasi-ideal and $(\lambda,\mu)-$fuzzy bi-ideal of a semigroup.

2 $(\lambda,\mu)-$fuzzy ideal, fuzzy quasi-ideal and fuzzy bi-ideal

A semigroup is an algebraic systems $(S,.)$ consisting of a nonempty set S together with an associative binary operation "$\cdot$". A subsemigroup of S is a nonempty subset T of S such that $TT \subseteq T$. A nonempty subset T of S is called a left (right) ideal of S if $ST \subseteq T(TS \subseteq T)$. A nonempty subset T of S is called an ideal of S if it is both a left ideal and a right ideal of S. A nonempty subset Q of S is called a quasi-ideal of S if $QS\bigcap SQ \subseteq Q$. A nonempty subset T of S is called a generalized bi-ideal of S if $TST \subseteq T$. A subsemigroup T of S is called a bi-ideal of S if $TST \subseteq T$. A subsemigroup T of S is called an interior ideal of S if $STS \subseteq T$. Obviously, each left (right) ideal of S is a quasi-ideal of S and each quasi-ideal of S is a bi-ideal of S.

An element a of a semigroup S is called a regular element if there exists an element b of S such that $a = aba$. A semigroup S is called a regular semigroup if every element of S is a regular element.

* Corresponding author e-mail: yaobingxue@163.com

In the following discussions, S always stands for a semigroup, and λ and μ are two constant numbers such that $0 \leqslant \lambda < \mu \leqslant 1$.

By a fuzzy subset of S we mean a mapping A from S to the closed interval $[0,1]$. If T is a subset of S, then the characteristic function of T is denoted by 1_T.

In order to generalize the concepts of fuzzy subsemigroup and various fuzzy ideals of S defined in [3], we introduced the followings.

Definition 2.1. A fuzzy subset A of S is called a $(\lambda,\mu)-$fuzzy subsemigroup of S if for all $x,y \in S$,

$$A(xy) \vee \lambda \geqslant A(x) \wedge A(y) \wedge \mu.$$

Definition 2.2.[9] A fuzzy subset A of S is called a $(\lambda,\mu)-$fuzzy left (right) ideal of S if for all $x,y \in S$,

$$A(xy) \vee \lambda \geqslant A(y) \wedge \mu \, (A(xy) \vee \lambda \geqslant A(x) \wedge \mu).$$

A fuzzy subset A of S is called a $(\lambda,\mu)-$fuzzy ideal of S if it is both a $(\lambda,\mu)-$fuzzy left ideal and a $(\lambda,\mu)-$fuzzy right ideal of S.

Definition 2.3. A fuzzy subset A of S is called a $(\lambda,\mu)-$fuzzy generalized bi-ideal of S if for all $x,y,z \in S$,

$$A(xyz) \vee \lambda \geqslant A(x) \wedge A(z) \wedge \mu.$$

Let A,B be fuzzy subsets of S, then the fuzzy subset $A \cdot B$ of S is defined as the following. For all $x \in S$, if $\exists x_1,x_2 \in S$, such that $x = x_1x_2$, then $(A \cdot B)(x) = \sup\{A(x_1) \wedge B(x_2) \mid x = x_1x_2\}$, otherwise $(A \cdot B)(x) = 0$.

Definition 2.4. A fuzzy subset A of S is called a $(\lambda,\mu)-$fuzzy quasi-ideal of S if for all $x \in S$,

$$A(x) \vee \lambda \geqslant (A \cdot 1_S)(x) \wedge (1_S \cdot A)(x) \wedge \mu.$$

Definition 2.5. A $(\lambda,\mu)-$fuzzy subsemigroup A of S is called a $(\lambda,\mu)-$fuzzy bi-ideal(interior ideal) of S if for all $x,y,z \in S$,

$$A(xyz) \vee \lambda \geqslant A(x) \wedge A(z) \wedge \mu \; (A(xyz) \vee \lambda \geqslant A(y) \wedge \mu).$$

It is obvious that an $(\in, \in\vee q)-$fuzzy subsemigroup (left ideal, right ideal, ideal, bi-ideal, quasi-ideal) is a $(0,0.5)-$fuzzy subsemigroup (left ideal, right ideal, ideal, bi-ideal, quasi-ideal).

The following two theorems show the relations among $(\lambda,\mu)-$fuzzy left (right) ideal, $(\lambda,\mu)-$fuzzy quasi-ideal and $(\lambda,\mu)-$fuzzy bi-ideal.

Theorem 2.1. Let A be a $(\lambda,\mu)-$fuzzy left ideal (right ideal) of S. Then A is a $(\lambda,\mu)-$fuzzy quasi-ideal of S.

Proof. We only prove the case of $(\lambda,\mu)-$fuzzy left ideal. For all $x \in S$, if $\exists x_1,x_2 \in S$, such that $x = x_1x_2$, then

$$\begin{aligned}(1_S \cdot A)(x) \wedge \mu &= \sup\{1_S(x_1) \wedge A(x_2) \wedge \mu \mid x = x_1x_2\}\\ &= \sup\{A(x_2) \wedge \mu \mid x = x_1x_2\}\\ &\leqslant \sup\{A(x_1x_2) \vee \lambda \mid x = x_1x_2\} = A(x) \vee \lambda.\end{aligned}$$

Otherwise, $(1_S \cdot A)(x) \wedge \mu = 0$. So $A(x) \vee \lambda \geqslant (1_S \cdot A)(x) \wedge \mu \geqslant (1_S \cdot A)(x) \wedge (A \cdot 1_S)(x) \wedge \mu$. It means that A is a $(\lambda,\mu)-$fuzzy quasi-ideal of S.

Theorem 2.2. Let A be a $(\lambda,\mu)-$fuzzy quasi-ideal of S. Then A is a $(\lambda,\mu)-$fuzzy bi-ideal of S.

Proof. For all $x,y,z \in S$, we have

$$\begin{aligned}A(xy) \vee \lambda &\geqslant (A \cdot 1_S)(xy) \wedge (1_S \cdot A)(xy) \wedge \mu\\ &\geqslant A(x) \wedge 1_S(y) \wedge 1_S(x) \wedge A(y) \wedge \mu\\ &= A(x) \wedge A(y) \wedge \mu,\\ A(xyz) \vee \lambda &\geqslant (A \cdot 1_S)(xyz) \wedge (1_S \cdot A)(xyz) \wedge \mu\\ &\geqslant A(x) \wedge 1_S(yz) \wedge 1_S(xy) \wedge A(z) \wedge \mu\\ &= A(x) \wedge A(z) \wedge \mu.\end{aligned}$$

It follows that A is a $(\lambda,\mu)-$fuzzy bi-ideal of S.

$(\lambda,\mu)-$fuzzy subsemigroups and various $(\lambda,\mu)-$fuzzy ideals of S can be characterized by their cut sets.

Theorem 2.3.[9] A fuzzy subset A of S is a $(\lambda,\mu)-$fuzzy subsemigroup (left ideal, right ideal, ideal) of S if and only if for all $t \in (\lambda,\mu]$,A_t is a subsemigroup (left ideal, right ideal, ideal) of S whenever $A_t \neq \emptyset$.

Similarly, we have the following theorem.

Theorem 2.4. A fuzzy subset A of S is a $(\lambda,\mu)-$fuzzy quasi-ideal(bi-ideal, interior ideal) of S if and only if for all $t \in (\lambda,\mu]$,A_t is a quasi-ideal(bi-ideal, interior ideal) of S whenever $A_t \neq \emptyset$.

Proof. We only prove the case of $(\lambda,\mu)-$fuzzy quasi-ideal. Let A be a $(\lambda,\mu)-$fuzzy quasi-ideal of S and $t \in (\lambda,\mu]$. If $x \in A_tS \cap SA_t$, then there exist $a,b \in A_t, r,s \in S$, such that $x = ar = sb$. So we have

$$\begin{aligned}A(x) \vee \lambda &\geqslant (A \cdot 1_S)(x) \wedge (1_S \cdot A)(x) \wedge \mu\\ &\geqslant A(a) \wedge 1_S(r) \wedge 1_S(s) \wedge A(b) \wedge \mu\\ &= A(a) \wedge A(b) \wedge \mu\\ &\geqslant t \wedge \mu = t > \lambda.\end{aligned}$$

It implies that $A(x) \geqslant t$. So $x \in A_t$ and $A_tS \cap SA_t \subseteq A_t$. Hence A_t is a quasi-ideal of S.

Conversely, for all $t \in (\lambda,\mu]$, let A_t be a quasi-ideal of S or $A_t = \emptyset$. We will show that $A(x) \vee \lambda \geqslant (A \cdot 1_S)(x) \wedge (1_S \cdot A)(x) \wedge \mu, \forall x \in S$. If possible, let $x_0 \in S$ such that $A(x_0) \vee \lambda < (A \cdot 1_S)(x_0) \wedge (1_S \cdot A)(x_0) \wedge \mu$. Choose t such that $A(x_0) \vee \lambda < t < (A \cdot 1_S)(x_0) \wedge (1_S \cdot A)(x_0) \wedge \mu$, then $t \in (\lambda,\mu]$, $x_0 \notin A_t$ and $(A \cdot 1_S)(x_0) \wedge (1_S \cdot A)(x_0) > t$. So there exist $a,b,r,s \in S$, such that $x_0 = ar = sb$ and $A(a) \wedge A(b) > t$. Thus $x_0 \in A_tS \cap SA_t$, a contradiction. So we have $A(x) \vee \lambda \geqslant (A \cdot 1_S)(x) \wedge (1_S \cdot A)(x) \wedge \mu, \forall x \in S$. Hence A is a $(\lambda,\mu)-$fuzzy quasi-ideal of S.

3 The characterization of regular semigroup

To characterize the regularity of the semigroup S, we define the fuzzy subset A^- of S, called the middle part of

A, as the following

$$A^-(x) = (A(x) \vee \lambda) \wedge \mu, \forall x \in S,$$

where A is a fuzzy subset of S.

Theorem 3.1. Let A, B be fuzzy subsets of S, then
(1)$(A \cap B)^- = A^- \cap B^-$,
(2)$(A \cup B)^- = A^- \cup B^-$,
(3)$(A \cdot B)^- \supseteq A^- \cdot B^-$.

Proof. For all $x \in S$, we have
$(A \cap B)^-(x) = ((A(x) \wedge B(x)) \vee \lambda) \wedge \mu$
$= A^-(x) \wedge B^-(x) = (A^- \cap B^-)(x)$.
$(A \cup B)^-(x) = ((A(x) \vee B(x)) \vee \lambda) \wedge \mu$
$= A^-(x) \vee B^-(x) = (A^- \cup B^-)(x)$.
This means (1) and (2) hold.
(3)For all $x \in S$, if x is not expressible as $x = x_1x_2$ for some $x_1, x_2 \in S$, then $(A \cdot B)^-(x) \geqslant 0 = (A^- \cdot B^-)(x)$. Otherwise,
$(A \cdot B)^-(x) = ((A \cdot B)(x) \vee \lambda) \wedge \mu$
$= (\sup\{A(x_1) \wedge B(x_2) \mid x = x_1x_2\} \vee \lambda) \wedge \mu$
$= \sup\{((A(x_1) \vee \lambda) \wedge \mu) \wedge ((B(x_2) \vee \lambda) \wedge \mu) \mid x = x_1x_2\}$
$= \sup\{A^-(x_1) \wedge B^-(x_2) \mid x = x_1x_2\}$
$= (A^- \cdot B^-)(x)$.
It shows that $(A \cdot B)^- \supseteq A^- \cdot B^-$.

Theorem 3.2. Let A be a $(\lambda, \mu)-$fuzzy subsemigroup (left ideal, right ideal, bi-ideal, quasi-ideal) of S. Then A^- is also a $(\lambda, \mu)-$fuzzy subsemigroup (left ideal, right ideal, bi-ideal, quasi-ideal) of S.

Proof. The proof is straightforward.

Now we characterize the regular semigroup by the properties of $(\lambda, \mu)-$fuzzy ideal, $(\lambda, \mu)-$fuzzy bi-ideal and $(\lambda, \mu)-$fuzzy quasi-ideal of S. First we need the following lemmas.

Lemma 3.1. Let X, Y be two nonempty subsets of S. Then
(1) $1_X \cap 1_Y = 1_{X \cap Y}, 1_X \cup 1_Y = 1_{X \cup Y}, 1_X \cdot 1_Y = 1_{X \cdot Y}$,
(2) $1_X^- = 1_Y^- \Longrightarrow 1_X = 1_Y \Longrightarrow X = Y$.

Proof. The proof is straightforward.

Lemma 3.2[8] For the semigroup S, the following conditions are equivalent.
(1) S is regular,
(2) $R \cap L = RL$ for every right ideal R and every left ideal L of S,
(3) $QSQ = Q$ for every quasi-ideal Q of S.

Theorem 3.3. The semigroup S is regular if and only if for every $(\lambda, \mu)-$fuzzy right ideal A and every $(\lambda, \mu)-$fuzzy left ideal B of S, $(A \cap B)^- = (A \cdot B)^-$.

Proof. Assume S is regular. Let A and B be $(\lambda, \mu)-$fuzzy right ideal and $(\lambda, \mu)-$fuzzy left ideal of S, respectively. Then for all $x \in S$, on one hand, we have
$(A \cdot B)^-(x) = (\sup\{A(x_1) \wedge B(x_2) \mid x = x_1x_2\} \vee \lambda) \wedge \mu$
$= (\sup\{(A(x_1) \wedge \mu) \wedge (B(x_2) \wedge \mu) \mid x = x_1x_2\} \vee \lambda) \wedge \mu$
$\leqslant (\sup\{(A(x_1x_2) \vee \lambda) \wedge (B(x_1x_2) \vee \lambda) \mid x = x_1x_2\} \vee \lambda) \wedge \mu$
$= ((A(x) \wedge B(x)) \vee \lambda) \wedge \mu$
$= (A \cap B)^-(x)$.

On the other hand, considering the regularity of S, there exists $a \in S$, such that $x = xax$. Thus

$$\begin{aligned}(A \cdot B)^-(x) &= ((A \cdot B)(xax) \vee \lambda) \wedge \mu \\ &\geqslant ((A(xa) \wedge B(x)) \vee \lambda) \wedge \mu \\ &\geqslant ((A(x) \wedge B(x)) \vee \lambda) \wedge \mu \\ &= (A \cap B)^-(x).\end{aligned}$$

Hence $(A \cap B)^- = (A \cdot B)^-$.

Conversely, assume the condition holds. Let R and L be right ideal and left ideal of S, respectively. Then 1_R and 1_L are $(\lambda, \mu)-$fuzzy right ideal and $(\lambda, \mu)-$fuzzy left ideal of S, respectively. Thus from Lemma 3.1 we have $1_{RL}^- = (1_R \cdot 1_L)^- = (1_R \cap 1_L)^- = 1_{R \cap L}^-$ and hence $RL = R \cap L$. Therefore S is regular semigroup from Lemma 3.2.

Theorem 3.4. For the semigroup S, the following conditions are equivalent.
(1) S is regular.
(2) $A^- = (A \cdot 1_S \cdot A)^-$ for every $(\lambda, \mu)-$fuzzy generalized bi-ideal A of S.
(3) $A^- = (A \cdot 1_S \cdot A)^-$ for every $(\lambda, \mu)-$fuzzy bi-ideal A of S.
(4) $A^- = (A \cdot 1_S \cdot A)^-$ for every $(\lambda, \mu)-$fuzzy quasi-ideal A of S.

Proof. $(1) \Longrightarrow (2)$: For all $x \in S$, there exists $a \in S$, such that $x = xax$. Thus

$$\begin{aligned}(A \cdot 1_S \cdot A)^-(x) &= ((A \cdot 1_S \cdot A)(xax) \vee \lambda) \wedge \mu \\ &\geqslant ((A(x) \wedge 1_S(a) \wedge A(x)) \vee \lambda) \wedge \mu \\ &= (A(x) \vee \lambda) \wedge \mu \\ &= A^-(x).\end{aligned}$$

Conversely, if x is not expressible as $x = x_1x_2x_3$, then $A^-(x) \geqslant \lambda = (A \cdot 1_S \cdot A)^-(x)$. Otherwise

$$\begin{aligned}A^-(x) &= (A(x_1x_2x_3) \vee \lambda) \wedge \mu \\ &\geqslant ((A(x_1) \wedge A(x_3)) \vee \lambda) \wedge \mu \\ &= ((A(x_1) \wedge 1_S(x_2) \wedge A(x_3)) \vee \lambda) \wedge \mu.\end{aligned}$$

It follows that
$A^-(x) \geqslant (\sup\{A(x_1) \wedge 1_S(x_2) \wedge A(x_3) \mid x = x_1x_2x_3\} \vee \lambda) \wedge \mu$
$= (A \cdot 1_S \cdot A)^-(x)$.
Hence $A^- = (A \cdot 1_S \cdot A)^-$.
$(4) \Longrightarrow (1)$: Let Q be any quasi-ideal of S. Then $QSQ \subseteq QS \cap SQ \subseteq Q$. In order to show $Q \subseteq QSQ$, we define fuzzy subset A of S by

$$A(s) = \begin{cases} \mu, & s \in Q \\ 0, & s \notin Q \end{cases}, \forall s \in S.$$

From Theorem 2.3, we have that A is a $(\lambda, \mu)-$fuzzy quasi-ideal of S and hence $A^- = (A \cdot 1_S \cdot A)^-$. For all $x \in Q$,
$(A \cdot 1_S \cdot A)^-(x) = A^-(x) = A(x) \vee \lambda \wedge \mu = \mu$.
This implies that there exist $x_1^0, x_3^0 \in Q, x_2^0 \in S$, such that $x_1^0x_2^0x_3^0 = x$, and
$\sup\{A(x_1) \wedge 1_S(x_2) \wedge A(x_3) \mid x = x_1x_2x_3\}$
$= A(x_1^0) \wedge 1_S(x_2^0) \wedge A(x_3^0) = \mu$.

So $x = x_1^0 x_2^0 x_3^0 \in QSQ$. It shows that $QSQ = Q$. From Lemma 3.2 S is a regular semigroup.
(2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) are obvious.

Theorem 3.5. For the semigroup S, the following conditions are equivalent.
(1) S is regular.
(2) $(A\cap B)^- = (A\cdot B\cdot A)^-$ for every $(\lambda,\mu)-$fuzzy quasi-ideal A and every $(\lambda,\mu)-$fuzzy ideal B of S.
(3)$(A\cap B)^- = (A\cdot B\cdot A)^-$ for every $(\lambda,\mu)-$fuzzy quasi-ideal A and every $(\lambda,\mu)-$fuzzy interior ideal B of S.
(4) $(A\cap B)^- = (A\cdot B\cdot A)^-$ for every $(\lambda,\mu)-$fuzzy bi-ideal A and every $(\lambda,\mu)-$fuzzy ideal B of S.
(5)$(A\cap B)^- = (A\cdot B\cdot A)^-$ for every $(\lambda,\mu)-$fuzzy bi-ideal A and every $(\lambda,\mu)-$fuzzy interior ideal B of S.
(6)$(A\cap B)^- = (A\cdot B\cdot A)^-$ for every $(\lambda,\mu)-$fuzzy generalized bi-ideal A and every $(\lambda,\mu)-$fuzzy ideal B of S.
(7)$(A\cap B)^- = (A\cdot B\cdot A)^-$ for every $(\lambda,\mu)-$fuzzy generalized bi-ideal A and every $(\lambda,\mu)-$fuzzy interior ideal B of S.

Proof. (1) $\Longrightarrow$ (7): Let A,B be $(\lambda,\mu)-$fuzzy generalized bi-ideal and $(\lambda,\mu)-$fuzzy interior ideal of S, respectively. For all $x\in S$, if $x = x_1x_2x_3$ for some $x_1,x_2,x_3\in S$, then

$$\begin{aligned}
A^-(x) &= A(x_1x_2x_3)\vee\lambda\wedge\mu\\
&\geqslant (A(x_1)\wedge A(x_3))\vee\lambda\wedge\mu\\
&\geqslant (A(x_1)\wedge B(x_2)\wedge A(x_3))\vee\lambda\wedge\mu,\\
B^-(x) &= B(x_1x_2x_3)\vee\lambda\wedge\mu\\
&\geqslant B(x_2)\vee\lambda\wedge\mu\\
&\geqslant (A(x_1)\wedge B(x_2)\wedge A(x_3))\vee\lambda\wedge\mu.
\end{aligned}$$

Hence we have
$(A\cap B)^-(x) = A^-(x)\wedge B^-(x)$
$\geqslant \sup\{A(x_1)\wedge B(x_2)\wedge A(x_3) \mid x = x_1x_2x_3\}\vee\lambda\wedge\mu$
$= (A\cdot B\cdot A)^-(x)$.

Furthermore, from the condition (1), there exists $a\in S$, such that $x = xax = xaxax$. Thus

$$\begin{aligned}
(A\cdot B\cdot A)^-(x) &= (A\cdot B\cdot A)(xaxax)\vee\lambda\wedge\mu\\
&\geqslant (A(x)\wedge B(axa)\wedge A(x))\vee\lambda\wedge\mu\\
&\geqslant (A(x)\wedge B(x)\wedge A(x))\vee\lambda\wedge\mu\\
&= (A\cap B)^-(x).
\end{aligned}$$

It follows that $(A\cap B)^- = (A\cdot B\cdot A)^-$.
(2) $\Longrightarrow$ (1): Let A be a $(\lambda,\mu)-$fuzzy quasi-ideal of S. Considering that 1_S is a $(\lambda,\mu)-$fuzzy ideal of S and $1_S\supseteq A$, we have
$A^- = (A\cap 1_S)^- = (A\cdot 1_S\cdot A)^-$.
It implies that S is regular from Theorem 3.4.
(7) $\Longrightarrow$ (5) $\Longrightarrow$ (3) $\Longrightarrow$ (2) and (7) $\Longrightarrow$ (6) $\Longrightarrow$ (4) $\Longrightarrow$ (2) are obvious.

Lemma 3.3. Let A and B be $(\lambda,\mu)-$fuzzy right ideal and $(\lambda,\mu)-$fuzzy left ideal of S, respectively. Then
$(A\cdot 1_S\cdot B)^- \subseteq (A\cdot B)^- \subseteq (A\cap B)^-$.

Proof. The proof is straightforward.

Theorem 3.6. For the semigroup S, the following conditions are equivalent.
(1) S is regular.
(2) $(A\cap B)^- \subseteq (A\cdot B)^-$ for every $(\lambda,\mu)-$fuzzy right ideal A and every $(\lambda,\mu)-$fuzzy left ideal B of S.
(3) $(A\cap B)^- \subseteq (A\cdot B)^-$ for every $(\lambda,\mu)-$fuzzy quasi-ideal A and every $(\lambda,\mu)-$fuzzy left ideal B of S.
(4) $(A\cap B)^- \subseteq (A\cdot B)^-$ for every $(\lambda,\mu)-$ fuzzy bi-ideal A and every $(\lambda,\mu)-$fuzzy left ideal B of S.
(5) $(A\cap B)^- \subseteq (A\cdot B)^-$ for every $(\lambda,\mu)-$fuzzy generalized bi-ideal A and every $(\lambda,\mu)-$fuzzy left ideal B of S.
(6) $(A\cap B)^- \subseteq (A\cdot B)^-$ for every fuzzy subset A and every $(\lambda,\mu)-$fuzzy left ideal B of S.

Proof. (1) $\Longrightarrow$ (6): Let A and B be fuzzy subset and $(\lambda,\mu)-$fuzzy left ideal of S, respectively. Then for all $x\in S$, there exists an element $a\in S$, such that $x = xax$. Thus

$$\begin{aligned}
(A\cdot B)^-(x) &= (A\cdot B)(xax)\vee\lambda\wedge\mu\\
&\geqslant (A(x)\wedge B(ax))\vee\lambda\wedge\mu\\
&\geqslant (A(x)\wedge B(x))\vee\lambda\wedge\mu\\
&= (A\cap B)^-(x).
\end{aligned}$$

It shows that $(A\cap B)^- \subseteq (A\cdot B)^-$.
(2) $\Longrightarrow$ (1): Let A and B be $(\lambda,\mu)-$fuzzy right ideal and $(\lambda,\mu)-$fuzzy left ideal of S, respectively. Then from the condition (2) and Lemma 3, we have $(A\cap B)^- = (A\cdot B)^-$. From Theorem 9 S is regular.
(6) $\Longrightarrow$ (5) $\Longrightarrow$ (4) $\Longrightarrow$ (3) $\Longrightarrow$ (2) is obvious.

Symmetrically we have the following theorem.

Theorem 3.7. For the semigroup S, the following conditions are equivalent.
(1) S is regular.
(2) $(A\cap B)^- \subseteq (A\cdot B)^-$ for every $(\lambda,\mu)-$fuzzy right ideal A and every $(\lambda,\mu)-$fuzzy quasi-ideal B of S.
(3)$(A\cap B)^- \subseteq (A\cdot B)^-$ for every $(\lambda,\mu)-$fuzzy right ideal A and every $(\lambda,\mu)-$fuzzy bi-ideal B of S.
(4) $(A\cap B)^- \subseteq (A\cdot B)^-$ for every $(\lambda,\mu)-$fuzzy right ideal A and every $(\lambda,\mu)-$fuzzy generalized bi-ideal B of S.
(5) $(A\cap B)^- \subseteq (A\cdot B)^-$ for every $(\lambda,\mu)-$fuzzy right ideal A and every fuzzy subset B of S.

Theorem 3.8. For the semigroup S, the following assertions hold.
(1) If S is regular, then $(A\cap C\cap B)^- \subseteq (A\cdot C\cdot B)^-$ for every $(\lambda,\mu)-$fuzzy right ideal A, every $(\lambda,\mu)-$fuzzy left ideal B and every fuzzy subset C of S.
(2) If $(A\cap B)^- \subseteq (A\cdot 1_S\cdot B)^-$ for every $(\lambda,\mu)-$fuzzy right ideal A and every $(\lambda,\mu)-$fuzzy left ideal B, then S is regular.

Proof. (1) Let A and B be $(\lambda,\mu)-$fuzzy right ideal and $(\lambda,\mu)-$fuzzy left ideal of S, respectively. For any fuzzy subset C of S and $x\in S$, there exists an element a of S such that $x = xaxax$. Thus

$$\begin{aligned}(A\cdot C\cdot B)^{-}(x) &= (A\cdot C\cdot B)(xaxax)\vee\lambda\wedge\mu\\ &\geqslant (A(xa)\wedge C(x)\wedge B(ax))\vee\lambda\wedge\mu\\ &\geqslant (A(x)\wedge C(x)\wedge B(x))\vee\lambda\wedge\mu\\ &= (A\cap C\cap B)^{-}(x).\end{aligned}$$

Hence $(A\cap C\cap B)^{-}\subseteq(A\cdot C\cdot B)^{-}$.
(2) Let A and B be $(\lambda,\mu)-$fuzzy right ideal and $(\lambda,\mu)-$fuzzy left ideal of S, respectively. If $(A\cap B)^{-}\subseteq(A\cdot 1_S\cdot B)^{-}$ holds, then $(A\cap B)^{-}=(A\cdot B)^{-}$ holds from Lemma 3.3. Hence S is regular from Theorem 3.6.

The following corollary is an immediate consequence of Theorem 3.8.

Corollary For the semigroup S, the following conditions are equivalent.
(1) S is regular.
(2) $(A\cap C\cap B)^{-}\subseteq(A\cdot C\cdot B)^{-}$ for every $(\lambda,\mu)-$fuzzy right ideal A, every $(\lambda,\mu)-$fuzzy left ideal B and every $(\lambda,\mu)-$fuzzy bi-ideal C of S.
(3) $(A\cap C\cap B)^{-}\subseteq(A\cdot C\cdot B)^{-}$ for every $(\lambda,\mu)-$fuzzy right ideal A, every $(\lambda,\mu)-$fuzzy left ideal B and every $(\lambda,\mu)-$fuzzy quasi-ideal C of S.

4 Conclusion

In this paper, the properties of various $(\lambda,\mu)-$fuzzy ideals of a semigroup were discussed and the regular semigroup was characterized by the properties of the middle parts of various $(\lambda,\mu)-$fuzzy ideals. Then several equivalence conditions of regular semigroup were obtained.

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Bingxue Yao is a professor at Lioacheng University. He received his PhD degree in System Theorem at Shandong University in 2004. His research interests are in the areas of applied mathematics and fuzzy systems. He has published over 30 papers in international journals.