Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/100323

Some Results on Parametric Weighted Generalized Inaccuracy Measure

M. Sultan Shah and M. A. K. Baig*

Department of Statistics, University of Kashmir, Srinagar-190006, J and K, India

Received: 1 Feb. 2020, Revised: 2 May 2020, Accepted: 25 May 2020

Published online: 1 Nov. 2021

Abstract: In this paper, a new weighted generalized measure of inaccuracy of order β and its dynamic (residual) version is introduced. It is shown that the weighted generalized residual inaccuracy uniquely determines the survival function. Based on proportional hazard rate model (PHRM), some characterization results of the proposed dynamic measure of inaccuracy are focused. Further, some important properties and their relationships with the other reliability measures of the dynamic measure have also been studied under proportional hazard rate model (PHRM).

Keywords: Length-biased entropy, Inaccuracy, generalized inaccuracy, Length-biased dynamic inaccuracy measure, Proportional hazard rate model, Characterization results.

1 Introduction

Entropy, the quantitative measure of uncertainty, generally known as Shannon's entropy was originally introduced by Claude Shannon in [1]. The measure has promoted a wide interest and attention among the researchers towards the development of its generalizations and one of the remarkable generalizations of this measure known as inaccuracy measure , has been proposed by Kerridge in [2]. The proposed measure has been widely used as a fundamental tool for the measurement of error in experimental results.
Let X and Y be two non-negative random variables describing time to failure of two systems with density functions $f(x)$ and $g(x)$ respectively and let $\overline{F}(x)$ and $\overline{G}(x)$ be the survival functions of X and Y respectively. Then Shannon's measure of entropy and Kerridge's measure of inaccuracy are respectively defined as

$$H(X) = -\int_0^\infty f(x)\log f(x)dx, \tag{1}$$

and

$$\xi(X,Y) = -\int_0^\infty f(x)\log g(x)dx. \tag{2}$$

In case, $g(x) = f(x)$, then (2) reduces to (1).

The information measures defined above are shift-independent, as they consider only probability density function of the observed random variable. But sometimes in several real life situations, it is necessary to also take into account the value of random variable known as weight or preference, which depends on the goal set by the experimenter. See Rao [3]. So, we need to study shift-dependent (weighted) version of the above measures. In this situation, Belis and Guiasu [4], introduced a new measure of entropy called as "length biased" entropy, or weighted entropy, defined as

*Corresponding author e-mail: msultanshahh01@gmail.com

$$H^w(X) = -\int_0^\infty x f(x) \log f(x) dx . \tag{3}$$

Based on (3), the weighted (length biased) measure of inaccuracy is defined as

$$\xi^w(X,Y) = -\int_0^\infty x f(x) \log g(x) dx , \tag{4}$$

where, the coefficient x in the integral represents the weight function $w(x)$ of the elementary events. when $w(x) = x$, i.e. the weight function depends on the length of unit of interest, X^w is said to be length (size) biased random variable.

The measures (1) and (2) cannot be applied to a system which has survived for some units of time say t. Ebrahimi [5], introduced the measure of uncertainty for the residual lifetime random variable $X_t = [X - t / X > t]$, known as residual measure of entropy given as

$$H(X;t) = -\int_t^\infty \frac{f(x)}{\overline{F}(t)} \log \frac{f(x)}{\overline{F}(t)} dx . \tag{5}$$

Taneja et al. [6] defined a residual measure of inaccuracy as

$$\xi(X,Y;t) = -\int_t^\infty \frac{f(x)}{\overline{F}(t)} \log \frac{g(x)}{\overline{G}(t)} dx . \tag{6}$$

Dicrescenzo and Longobardi [7] developed the weighted form of (5) as follows

$$H^w(X;t) = -\int_t^\infty x \frac{f(x)}{\overline{F}(t)} \log \frac{f(x)}{\overline{F}(t)} dx , \tag{7}$$

and

Kumar et al. [8] developed the weighted form of (6) as follows

$$\xi^w(X,Y;t) = -\int_t^\infty x \frac{f(x)}{\overline{F}(t)} \log \frac{g(x)}{\overline{G}(t)} dx . \tag{8}$$

Recently, there has been an ample interest among scholars in developing the new and important length biased inaccuracy measures. For more in this direction we refer to, Kumar and Taneja [9], Kundu [10], M. Khorshadizadeh [11], Daneshi et al. [12].

Various researchers have developed several generalizations of Kerridge's measure of inaccuracy in different ways and in this direction, Nath [13], introduced a new generalized inaccuracy measure of order β defined as follows

$$\xi_\beta(X,Y) = \frac{1}{1-\beta} \log \left[\int_0^\infty f(x) (g(x))^{\beta-1} dx \right], \beta \neq 1, \beta > 0 \ , \tag{9}$$

where,

$\lim_{\beta \to 1} \xi_\beta(X,Y) = -\int_0^\infty f(x) \log g(x) dx$, which is the Kerridge's inaccuracy given in (2). Further, it should be noted that, when, $g(x) = f(x)$, then (9) reduces to Renyi's entropy of order β and even when $g(x) = f(x)$, $\beta \to 1$, then (9) reduces to (1), the Shannon's entropy.

Analogous to (6) and on the basis of (9), the generalized dynamic (residual) inaccuracy of order β is defined as

$$\xi_\beta(X,Y;t) = \frac{1}{1-\beta} \log \left[\int_t^\infty \left(\frac{f(x)}{\overline{F}(t)} \right) \left(\frac{g(x)}{\overline{G}(t)} \right)^{\beta-1} dx \right], \beta \neq 1, \beta > 0 \ . \tag{10}$$

The objective of this research work is to develop a new one parametric weighted generalized inaccuracy measure (WGI) and its dynamic (residual) version (WGRI). The rest of the paper consists of the following sections. In section 2, we define weighted generalized inaccuracy (WGI) along with an example to compare the generalized inaccuracy with its weighted version. In section 3, we define the weighted generalized residual inaccuracy (WGRI) and based on proportional hazard rate model (PHRM), some significant characterization results of the proposed dynamic measure of inaccuracy are studied. In section 4, under the proportional hazard rate model (PHRM).we discuss some important properties and their relationships with the other reliability measures of WGRI. Finally, in section 5, we illustrate some concluding remarks.

2 Weighted Generalized Inaccuracy (WGI)

In this section, we discuss the weighted form of generalized inaccuracy (GI) (9), which is known as weighted generalized inaccuracy (WGI).

Definition 2.1. Analogous to (2) and on the basis of (9), the WGI is given by

$$\xi_\beta^w(X,Y)=\frac{1}{1-\beta}\log\left(\int_0^\infty x^\beta f(x)(g(x))^{\beta-1}dx\right);\ \beta\neq 1,\beta>0. \quad (11)$$

It can be further expressed as

$$\xi_\beta^w(X,Y)=\frac{1}{1-\beta}\log\int_0^\infty\underbrace{\int_0^x\ldots\int_0^x}_{\beta} f(x)(g(x))^{\beta-1}\,dx,$$

$$=\frac{1}{1-\beta}\log\left(E\left(x^\beta (g(x))^{\beta-1}\right)\right).$$

Remark 2.1. When $g(x)=f(x)$, then (11) reduces to weighted Renyi's entropy of order β.

The following pairs of examples exhibits the difference between GI (9) and its weighted version (11)

Example 2.1. Let the two random variables X and Y be distributed as

1. $f_1(x)=1;0<x<1,\ g_1(x)=2(1-x);0<x<1.$
2. $f_2(x)=1;0<x<1,\ g_2(x)=2x;0<x<1.$

then, by simple calculation, we obtain

$$\xi_{1(\beta)}(X,Y)=\xi_{2(\beta)}(X,Y)=\frac{1}{1-\beta}\log\left(\frac{2^{\beta-1}}{\beta}\right).$$

and

$$\xi_{1(\beta)}^w(X,Y)=\frac{1}{1-\beta}\log\left(\frac{2^{\beta-1}\Gamma\beta\Gamma(\beta+1)}{\Gamma(2\beta+1)}\right),$$

$$\xi_{2(\beta)}^w(X,Y)=\frac{1}{1-\beta}\log\left(\frac{2^{\beta-2}}{\beta}\right).$$

where,

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx=\frac{\Gamma m\Gamma n}{\Gamma m+n},\text{ is a beta function.}$$

Thus, from the above calculations, we observe that the generalized inaccuracy (GI) of both the numerical examples is same, but their weighted versions are different. i.e., $\xi_{1(\beta)}(X,Y)=\xi_{2(\beta)}(X,Y)$ but $\xi_{1(\beta)}^w(X,Y)\neq\xi_{2(\beta)}^w(X,Y)$.

3 Weighted Generalized Residual Inaccuracy (WGRI)

In this section, we discuss the weighted version of GRI (10) which is called as weighted generalized residual inaccuracy (WGRI). Some characterization results by using the proportional hazard rate model (PHRM) of this measure are also discussed.

Definition 3.1. Analogous to (4), the weighted form of (10), known as weighted generalized residual inaccuracy (WGRI) is defined as

$$\xi_\beta^w(X,Y;t)=\frac{1}{1-\beta}\log\left(\int_t^\infty x^\beta\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx\right),\ \beta\neq 1,\beta>0. \tag{12}$$

Remark 3.1. When $t=0$, then (12) reduces to (11), the weighted generalized inaccuracy measure.

In order to provide characterization results we define the proportional hazard rate model (PHRM). The notion of this model was introduced by Cox [14]. The model has been widely used in the variety of fields such as survival analysis, reliability, economics etc. For the application of this model one may refer to Cox and Oakes [15], Ebrahimi and Kirmani [16] and Nair and Gupta [17].

Definition 3.2. Two random variables X and Y are said to satisfy proportional hazard rate model (PHRM), if there exists (proportionality constant) $\theta>0$ such that

$$\lambda_G(x)=\theta\lambda_F(x).\ \text{Or, equivalently, } \overline{G}(x)=\left[\overline{F}(x)\right]^\theta,\ \text{for some } \theta. \tag{13}$$

where, $\lambda_F(x)$ and $\lambda_G(x)$ represent the hazard rate functions of X and Y respectively.

Table 1: Expressions of WGI $\xi_\beta^w(X,Y)$ for some lifetime distributions.

Distribution	$f(x)$	$g(x)$	x	$\xi_\beta^w(X,Y)$
Uniform	$\frac{1}{n}$	$\frac{\theta(n-x)^{\theta-1}}{n^\theta}$	$0<x<n;\theta>0$	$S\log\left[\frac{n\theta^p\Gamma(p+2)\Gamma(w-p)}{\Gamma(w+2)}\right]$
Pareto	$\frac{\mu\lambda^\mu}{x^{\mu+1}}$	$\frac{\mu\theta\lambda^{\mu\theta}}{x^{\mu\theta+1}}$	$x\geq\lambda;\lambda,\mu,\theta>0$	$S\log\left[\frac{\lambda\theta^p\mu^{p+1}}{\mu w-1}\right]$
Weibull	$\frac{1}{r}e^{-\left(\frac{x-\lambda}{r}\right)}$	$\frac{\theta}{r}e^{-\theta\left(\frac{x-\lambda}{r}\right)}$	$x>\lambda;\lambda,r,\theta>0$	$S\log\left[\frac{r\theta^p e^{-\frac{w\lambda}{r}}\Gamma\left((p+2),\frac{w\lambda}{r}\right)}{w^{p+2}}\right]$
Exponential	me^{-mx}	$m\theta e^{-m\theta x}$	$x\geq 0;m,\theta>0$	$S\log\left[\frac{\theta^p\Gamma(p+2)}{mw^{p+2}}\right]$
Finite range	$a(1-x)^{a-1}$	$a\theta(1-x)^{a\theta-1}$	$0\leq x\leq 1,a>1,\theta>0$	$S\log\left[\frac{a^{p+1}\theta^p\Gamma(p+2)\Gamma(aw-p)}{\Gamma(aw+2)}\right]$
Rayleigh	$\frac{x}{\sigma^2}e^{-\frac{x^2}{\sigma^2}}$	$\frac{\theta x}{\sigma^2}e^{-\frac{\theta x^2}{\sigma^2}}$	$0<x<\infty,\theta>0$	$S\log\left[\frac{2^{p+\frac{1}{2}}\sigma\theta^p\Gamma\left(p+\frac{3}{2}\right)}{w^{p+\frac{3}{2}}}\right]$

where, $S=\frac{1}{1-\beta}, p=\beta-1, w=p\theta+1$ and $B(a,b)=\int_0^1 u^{a-1}(1-u)^{b-1}du=\int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}}du=\frac{\Gamma a\Gamma b}{\Gamma(a+b)}$ is the complete beta function.

In the following theorem, an alternative way of expressing the WGRI (12) is obtained.

Theorem 3.1. If random variables X and Y satisfy the proportional hazard rate model (13) with proportionality constant $\theta>0$, then for all $t>0$, the following equality holds

$$\xi_\beta^w(X,Y;t)=\frac{1}{1-\beta}\log\left[t^\beta\exp((1-\beta)\xi_\beta(X,Y;t))\right.$$

$$+\beta\int_{z=t}^{\infty} z^{\beta-1}\left(\frac{\overline{F}(z)}{\overline{F}(t)}\right)^{\theta(\beta-1)+1}\exp\big((1-\beta)\xi_\beta(X,Y;z)\big)dz\Bigg]. \quad (14)$$

Proof. $$\int_t^\infty x^\beta\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx=\int_t^\infty\left(\int_0^x \beta z^{\beta-1}dz\right)\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx$$

$$=\beta\int_t^\infty\left[\int_0^t z^{\beta-1}dz+\int_t^x z^{\beta-1}dz\right]\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx$$

$$=t^\beta\int_t^\infty\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx+\beta\int_{z=t}^\infty z^{\beta-1}\left(\int_{x=z}^\infty\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx\right)dz. \quad (15)$$

Since, from (10), we have

$$\int_t^\infty\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx=\exp\big((1-\beta)\xi_\beta(X,Y;t)\big), \quad (16)$$

and

$$\int_t^\infty f(x)(g(x))^{\beta-1}dx=\overline{F}(t)\big(\overline{G}(t)\big)^{\beta-1}\exp\big((1-\beta)\xi_\beta(X,Y;t)\big).$$

Using proportional hazard rate model (13), we obtain

$$\int_t^\infty f(x)(g(x))^{\beta-1}dx=\big(\overline{F}(t)\big)^{\theta(\beta-1)+1}\exp\big((1-\beta)\xi_\beta(X,Y;t)\big). \quad (17)$$

Using (15), (16) and (17) in (12), the desired result is satisfied.

Theorem 3.2. Let the two non-negative random variables X and Y with survival functions $\overline{F}(x)$ and $\overline{G}(x)$ respectively, satisfying the proportional hazard rate model (13) and Let $\xi_\beta^w(X,Y;t)<\infty, \forall t\ge 0$ be an increasing function of t, then $\xi_\beta^w(X,Y;t)$, determines $\overline{F}(t)$ uniquely.

Proof. Rewriting (12) as

$$\exp\big((1-\beta)\xi_\beta^w(X,Y;t)\big)=\int_t^\infty x^\beta\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx \quad (18)$$

Differentiating (18) w.r.t.t, we have

$$\frac{\partial}{\partial t}\exp\big((1-\beta)\xi_\beta^w(X,Y;t)\big)=-t^\beta\lambda_F(t)(\lambda_G(t))^{\beta-1}$$
$$+\big(\lambda_F(t)+(\beta-1)\lambda_G(t)\big)\exp\big((1-\beta)\xi_\beta^w(X,Y;t)\big). \quad (19)$$

Where, $\lambda_F(t)=\frac{f(t)}{\overline{F}(t)}$ and $\lambda_G(t)=\frac{g(t)}{\overline{G}(t)}$ denoting the hazard rates of X and Y respectively.

By using the relationship $\lambda_G(t)=\theta\lambda_F(t)$ in (19), we obtain

$$t^\beta\theta^{\beta-1}(\lambda_F(t))^\beta-(\theta(\beta-1)+1)\exp\big((1-\beta)\xi_\beta^w(X,Y;t)\big)\lambda_F(t)$$
$$+\frac{\partial}{\partial t}\exp\big((1-\beta)\xi_\beta^w(X,Y;t)\big)=0 \quad (20)$$

Hence for fixed $t>0$, $\lambda_F(t)$, is a solution of the equation $\tau(x_t)=0$, where

$$\tau(x_t)=t^\beta\theta^{\beta-1}x_t^\beta-(\theta(\beta-1)+1)\exp\big((1-\beta)\xi_\beta^w(X,Y;t)\big)x_t$$
$$+\frac{\partial}{\partial t}\exp\big((1-\beta)\xi_\beta^w(X,Y;t)\big). \quad (21)$$

Here, $\tau(0)=\frac{\partial}{\partial t}\exp\big((1-\beta)\xi_\beta^w(X,Y;t)\big)\ge 0$, since we have assumed that $\xi_\beta^w(X,Y;t)$ is increasing in t, and also as

$x_t \to \infty$, $\tau(\infty) = \infty$.

Differentiating (21) both sides w.r.t. x_t, we have

$$\frac{\partial}{\partial x_t}\tau(x_t) = \beta t^{\beta}\theta^{\beta-1}\ x_t^{\beta-1} - (\theta(\beta-1)+1)\exp((1-\beta)\xi_{\beta}^{w}(X,Y;t)).$$

and

$$\frac{\partial^2}{\partial x_t^2}\tau(x_t) = \beta(\beta-1)t^{\beta}\theta^{\beta-1}\ x_t^{\beta-2}.$$

Now, $\frac{\partial}{\partial x_t}\tau(x_t) = 0$, gives

$$x_t = \left[\frac{(\theta(\beta-1)+1)\exp((1-\beta)\xi_{\beta}^{w}(X,Y;t))}{\beta t^{\beta}\theta^{\beta-1}}\right]^{\frac{1}{\beta-1}} = x_0 \text{ (say).}$$

So, in view of the above, the unique solution to $\tau(x_t) = 0$ is given by $x_t = \lambda_F(t)$. Thus, $\xi_{\beta}^{w}(X,Y;t)$, the Weighted generalized inaccuracy uniquely determines $\lambda_F(t)$, which in turn determines $\overline{F}(t)$.

In the below given table 2, we derive the expressions of weighted generalized residual inaccuracy corresponding to some well-known lifetime distributions.

Table 2: Weighted generalized residual inaccuracy $\xi_{\beta}^{w}(X,Y;t)$ of some lifetime distributions.

Distribution	$f(x)$	$g(x)$	x	$\xi_{\beta}^{w}(X,Y;t)$
Uniform	$\frac{1}{n}$	$\frac{\theta(n-x)^{\theta-1}}{n^{\theta}}$	$0 < x < n; \theta > 0$	$S\log\left[\frac{n^{w+1}\theta^{p}K(p,w)}{(n-t)^{w}}\right]$
Pareto	$\frac{\mu\lambda^{\mu}}{x^{\mu+1}}$	$\frac{\mu\theta\lambda^{\mu\theta}}{x^{\mu\theta+1}}$	$x \ge \lambda; \lambda,\mu,\theta > 0$	$S\log\left[\frac{t\theta^{p}\mu^{p+1}}{\mu w-1}\right]$
Weibull	$\frac{1}{r}e^{-\left(\frac{x-\lambda}{r}\right)}$	$\frac{\theta}{r}e^{-\theta\left(\frac{x-\lambda}{r}\right)}$	$x > \lambda; \lambda, r, \theta > 0$	$S\log\left[\frac{r\theta^{p}e^{\frac{wt}{r}}\Gamma\left(p+2,\frac{wt}{r}\right)}{w^{p+2}}\right]$
Exponential	me^{-mx}	$m\theta e^{-m\theta x}$	$x \ge 0; m,\theta > 0$	$S\log\left[\frac{\theta^{p}e^{mwt}\Gamma(p+2,mwt)}{mw^{p+2}}\right]$
Finite range	$a(1-x)^{a-1}$	$a\theta(1-x)^{a\theta-1}$	$0 \le x \le 1, a > 1, \theta > 0$	$S\log\left[\frac{a^{p+1}\theta^{p}B^{*}(t;p+2,aw-p)}{(1-t)^{aw}}\right]$
Rayleigh	$\frac{x}{\sigma^2}e^{-\frac{x^2}{\sigma^2}}$	$\frac{\theta x}{\sigma^2}e^{-\frac{\theta x^2}{\sigma^2}}$	$0 < x < \infty, \theta > 0$	$S\log\left[\frac{2^{p+\frac{1}{2}}\sigma\theta^{p}\Gamma\left(p+\frac{3}{2},\frac{wt}{2}\right)}{e^{-\frac{wt^2}{2\sigma^2}}w^{p+\frac{3}{2}}}\right]$

where, $K(p,w) = B(p+1, v-p) - \overline{B}\left(\frac{t}{b}; p+2, w-p\right)$,

$\overline{B}(x;a,b) = \int_0^x u^{a-1}(1-u)^{b-1}du, a > 0, b > 0$ and $B^{*}(x;a,b) = \int_x^1 u^{a-1}(1-u)^{b-1}du, a > 0, b > 0$ are the lower and upper incomplete beta functions, respectively and $\Gamma(b,ax)=a^b\int_x^\infty u^{b-1}e^{-au}du; a>0, b>0$, is the upper incomplete gamma function.

In the following example, we study the monotonic behavior of weighted generalized residual inaccuracy $\xi_\beta^w(X,Y;t)$ with respect to exponential distribution.

Example 3.1. If a random variable X is exponentially distributed with parameter $m>0$ and X and Y satisfy the PHRM with proportionality constant $\theta>0$, then from the above table 2, we have

$$\xi_\beta^w(X,Y;t)=\frac{1}{1-\beta}\left[(\beta-1)\log\theta+m(1+\theta(\beta-1))t+\log\left(\frac{\Gamma(\beta+1,m(1+\theta(\beta-1)t))}{m(1+\theta(\beta-1))^{\beta+1}}\right)\right].$$

In the below given table 3, assuming $\beta=0.4, \theta=0.5$ *and* $m=1.2$ in the expression of $\xi_\beta^w(X,Y;t)$ corresponding to exponential distribution and then calculate the values of the expression for different values of t as shown in the following table

Table 3. Different values of $\xi_\beta^w(X,Y;t)$ for different values of t

t	6	7	8	9	10	11	12	13	14	15
$\xi_\beta^w(X,Y;t)$	3.485	3.611	3.723	3.823	3.914	3.997	4.073	4.143	4.209	4.271

The graph of this table is shown in Fig. 1 and it clearly shows that $\xi_\beta^w(X,Y;t)$ is monotonic increasing in $t\in[6,15]$.

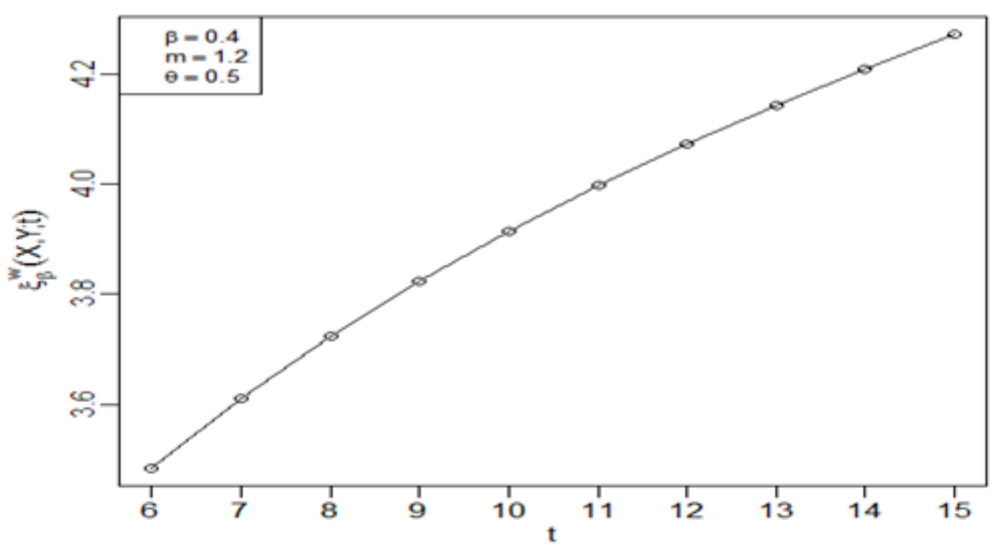

Fig.1: Weighted Generalized Inaccuracy for Exponential Distribution.

4 Some Properties and Inequalities of $\xi_\beta^w(X,Y;t)$

In this section, we present some significant properties and inequalities of WGRI.

Definition 4.1. The survival function $\overline{F}$ is said to have increasing (decreasing) WGRI of order β represented by IWGRI or DWGRI, if $\xi_\beta^w(X,Y;t)$ is increasing (decreasing) in t, $t>0$.

It means $\overline{F}$ has IWGRI or DWGRI if $\frac{\partial}{\partial t}\xi_\beta^w(X,Y;t)\geq(\leq)0$.

Theorem 4.1. Let the random variables X and Y have IWGRI, then under the PHRM, $\xi_\beta^w(X,Y;t)$ obtains a lower bound as follows

$$\xi_\beta^w(X,Y;t)\ge\frac{1}{1-\beta}\log\left[\frac{\theta^{\beta-1}t^\beta}{(\theta(\beta-1)+1)}\left(\frac{1+\frac{\partial}{\partial t}m_F(t)}{m_F(t)}\right)^{\beta-1}\right].$$

Proof. From equation (12), we have

$$(1-\beta)\xi_\beta^w(X,Y;t)=\log\int_t^\infty x^\beta\frac{f(x)}{\overline{F}(t)}\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx.$$

Differentiating both sides w.r.t. t, we have

$$(1-\beta)\frac{\partial}{\partial t}\xi_\beta^w(X,Y;t)=\lambda_F(t)[\theta(\beta-1)+1]$$
$$-t^\beta\theta^{\beta-1}(\lambda_F(t))^\beta\exp\left(-(1-\beta)\xi_\beta^w(X,Y;t)\right). \quad (22)$$

Using $\lambda_F(t)=\frac{1+\frac{\partial}{\partial t}m_F(t)}{m_F(t)}$, where $m_F(t)$, represents the mean residual life function of X, we obtain

$$(1-\beta)\frac{\partial}{\partial t}\xi_\beta^w(X,Y;t)=[\theta(\beta-1)+1]\left(\frac{1+\frac{\partial}{\partial t}m_F(t)}{m_F(t)}\right)$$
$$-t^\beta\theta^{\beta-1}\exp\left(-(1-\beta)\xi_\beta^w(X,Y;t)\right)\left(\frac{1+\frac{\partial}{\partial t}m_F(t)}{m_F(t)}\right)^\beta.$$

Since, $\xi_\beta^w(X,Y;t)$ is increasing w.r.t. t. Therefore

$$\xi_\beta^w(X,Y;t)\ge\frac{1}{1-\beta}\log\left[\frac{\theta^{\beta-1}t^\beta}{(\theta(\beta-1)+1)}\left(\frac{1+\frac{\partial}{\partial t}m_F(t)}{m_F(t)}\right)^{\beta-1}\right].$$

Theorem 4.2. Let the random variables X and Y be lifetimes of two components of a system with probability density functions $f(x)$ and $g(x)$and with survival functions $\overline{F}(t)$ and $\overline{G}(t)$ respectively, $t>0$, then for $0<\beta<1$ $\xi_\beta^w(X,Y;t)$ attains a lower bound as follows

$$\xi_\beta^w(X,Y;t)\ge\xi(X,Y;t)+\frac{\beta}{1-\beta}\int_t^\infty\frac{f(x)}{\overline{F}(t)}\log x\,dx. \quad (23)$$

Proof. From log-sum inequality, we have

$$\int_t^\infty\left(\frac{f(x)}{\overline{F}(t)}\right)\log\frac{\left(\frac{f(x)}{\overline{F}(t)}\right)}{x^\beta\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}}dx\ge\int_t^\infty\left(\frac{f(x)}{\overline{F}(t)}\right)dx\log\frac{\int_t^\infty\left(\frac{f(x)}{\overline{F}(t)}\right)dx}{\int_t^\infty x^\beta\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx}$$
$$=-\log\int_t^\infty x^\beta\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx$$

$$=-(1-\beta)\xi_{\beta}^{w}(X,Y;t). \tag{24}$$

Where (24) is obtained from (12).
The L.H.S of (24) leads to

$$(1-\beta)\int_{t}^{\infty}\left(\frac{f(x)}{\overline{F}(t)}\right)\log\left(\frac{g(x)}{\overline{G}(t)}\right)dx-\beta\int_{t}^{\infty}\left(\frac{f(x)}{\overline{F}(t)}\right)\log x\ dx. \tag{25}$$

Using definition of $\xi(X,Y;t)$ and (25) in (24), we obtain (23).

Theorem 4.3. Let $\overline{F}$ be an IWGRI (DWGRI) and $\beta<1$, then under PHRM

$$\lambda_F(t)\le(\ge)\left[\frac{(\theta(\beta-1)+1)\exp\left((1-\beta)\xi_{\beta}^{w}(X,Y;t)\right)}{t^{\beta}\theta^{\beta-1}}\right]^{\frac{1}{\beta-1}}.$$

Proof. From (22), we have

$$(1-\beta)\frac{\partial}{\partial t}\xi_{\beta}^{w}(X,Y;t)=\lambda_F(t)[\theta(\beta-1)+1]-t^{\beta}\theta^{\beta-1}(\lambda_F(t))^{\beta}\exp\left(-(1-\beta)\xi_{\beta}^{w}(X,Y;t)\right).$$

Since $\overline{F}$ is IWGRI (DWGRI) and $\beta<1$, therefore, we have

$$\lambda_F(t)\left[(\theta(\beta-1)+1)-t^{\beta}\theta^{\beta-1}(\lambda_F(t))^{\beta-1}\exp\left(-(1-\beta)\xi_{\beta}^{w}(X,Y;t)\right)\right]\ge(\le)0,$$

which leads to

$$\lambda_F(t)\le(\ge)\left[\frac{(\theta(\beta-1)+1)\exp\left((1-\beta)\xi_{\beta}^{w}(X,Y;t)\right)}{t^{\beta}\theta^{\beta-1}}\right]^{\frac{1}{\beta-1}}.$$

Theorem 4.4. For the random variables X and Y having support $(0,b]$, probability density functions $f(x)$ and $g(x)$ and survival functions $\overline{F}(t)$ and $\overline{G}(t)$ respectively, $t>0$, then for $0<\beta<1$, the following upper bound of $\xi_{\beta}^{w}(X,Y;t)$ holds

$$\xi_{\beta}^{w}(X,Y;t)\le\frac{1}{1-\beta}\left[\frac{\int_{t}^{b}x^{\beta}\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}\log x^{\beta}\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx}{\int_{t}^{b}x^{\beta}\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx}+\log(b-t)\right].$$

Proof. From log-sum inequality and (12)

$$\int_{t}^{b}x^{\beta}\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}\log x^{\beta}\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx$$

$$\ge\int_{t}^{b}x^{\beta}\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx\log\frac{\int_{t}^{b}x^{\beta}f(x)(g(x))^{\beta-1}dx}{\int_{t}^{b}\overline{F}(t)(\overline{G}(t))^{\beta-1}dx}$$

$$=\int_{t}^{b}x^{\beta}\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx\left[(1-\beta)\xi_{\beta}^{w}(X,Y;t)-\log(b-t)\right].$$

After the simplification, the desired result is obtained.

Proposition 4.1. For the random variables X and Y having WGRI $\xi_{\beta}^{w}(X,Y;t)$ and $\beta>1$, we have

$$\xi_{\beta}^{w}(X,Y;t)\ge\frac{1}{\beta-1}\left(1-\int_{t}^{\infty}x^{\beta}\left(\frac{f(x)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^{\beta-1}dx\right).$$

Proof. Since, $-\log x \geq 1-x$, we have

$$\xi_\beta^w(X,Y;t)=\frac{1}{1-\beta}\log\int_t^\infty x^\beta\left(\frac{f(x)}{\bar{F}(t)}\right)\left(\frac{g(x)}{\bar{G}(t)}\right)^{\beta-1}dx$$

$$=-\frac{1}{\beta-1}\log\int_t^\infty x^\beta\left(\frac{f(x)}{\bar{F}(t)}\right)\left(\frac{g(x)}{\bar{G}(t)}\right)^{\beta-1}dx$$

$$\geq\frac{1}{\beta-1}\left(1-\int_t^\infty x^\beta\left(\frac{f(x)}{\bar{F}(t)}\right)\left(\frac{g(x)}{\bar{G}(t)}\right)^{\beta-1}dx\right).$$

Theorem 4.5. If the hazard rate $\lambda_G(t)$ is decreasing in t, then

$$\xi_\beta^w(X,Y;t)\geq\frac{1}{1-\beta}\log\int_t^\infty x^\beta\left(\frac{f(x)}{\bar{F}(t)}\right)dx-\log\lambda_G(t).$$

Proof. Rewriting the equation (12) as

$$\xi_\beta^w(X,Y;t)=\frac{1}{1-\beta}\log\left(\int_t^\infty x^\beta\left(\frac{f(x)}{\bar{F}(t)}\right)(\lambda_G(x))^{\beta-1}\left(\frac{\bar{G}(x)}{\bar{G}(t)}\right)^{\beta-1}dx\right).$$

Since, $\bar{G}(x)\leq\bar{G}(t)$ for $x\geq t$, implies $\lambda_G(x)\leq\lambda_G(t)$, therefore

$$\xi_\beta^w(X,Y;t)\geq\frac{1}{1-\beta}\log\left((\lambda_G(t))^{\beta-1}\int_t^\infty x^\beta\left(\frac{f(x)}{\bar{F}(t)}\right)\left(\frac{\bar{G}(x)}{\bar{G}(t)}\right)^{\beta-1}dx\right)$$

$$=\frac{1}{1-\beta}\log\left((\lambda_G(t))^{\beta-1}\int_t^\infty x^\beta\left(\frac{f(x)}{\bar{F}(t)}\right)dx\right).$$

which gives the required result.

Definition 4.2. If X and Y are the two non-negative random variables with density functions $f(x)$ and $g(x)$respectively, then X is said to be less than or equal to Y in the likelihood ratio ordering, denoted by $X\overset{lhr}{\leq}Y$, if $\frac{f(x)}{g(x)}$ is decreasing in x that is if $\frac{f(x)}{g(x)}\leq\frac{f(t)}{g(t)}$for all $x>t$.

Theorem 4.6. If $X\overset{lhr}{\leq}Y$, then

(I) $\xi_\beta^w(X,Y;t)\leq\frac{1}{1-\beta}\log\frac{\lambda_F(t)}{\lambda_G(t)}+H_\beta^w(X;t)$; If $0<\beta<1$.

(II) $\xi_\beta^w(X,Y;t)\geq\frac{1}{1-\beta}\log\frac{\lambda_F(t)}{\lambda_G(t)}+H_\beta^w(X;t)$; $\beta>1$.

where,

$H_\beta^w(X;t)=\frac{1}{1-\beta}\log\int_t^\infty\left(x\frac{g(x)}{\bar{G}(t)}\right)^\beta dx$, is the weighted residual Renyi's entropy of order β. See Nourbakhsh and Yari [18].

Proof. (I) From equation (12), we have

$$\xi_\beta^w(X,Y;t)=\frac{1}{1-\beta}\log\left(\int_t^\infty x^\beta\left(\frac{f(x)}{\bar{F}(t)}\right)\left(\frac{g(x)}{\bar{G}(t)}\right)^{\beta-1}dx\right)$$

$$=\frac{1}{1-\beta}\log\left(\int_t^\infty x^\beta\left(\frac{f(x)}{g(x)}\frac{\bar{G}(t)}{\bar{F}(t)}\right)\left(\frac{g(x)}{\bar{G}(t)}\right)^{\beta}dx\right). \tag{26}$$

Since, $\frac{f(x)}{g(x)}\leq\frac{f(t)}{g(t)}$, for all $x>t$, Therefore equation (26) reduces to

$$\xi_\beta^w(X,Y;t) \le \frac{1}{1-\beta}\log\left(\int_t^\infty x^\beta\left(\frac{f(t)}{\overline{F}(t)}\frac{\overline{G}(t)}{\overline{F}(t)}\right)\left(\frac{g(x)}{\overline{G}(t)}\right)^\beta dx\right),$$

which leads to

$$\xi_\beta^w(X,Y;t) \le \frac{1}{1-\beta}\log\frac{\lambda_F(t)}{\lambda_G(t)} + H_\beta^w(X;t), \text{ if } 0<\beta<1.$$

Similarly, we can prove (II) for $\beta>1$.

In the following theorems, we consider more than two variables and obtain some bounds for WGRI.

Theorem 4.7. Let the random variables $X_1, X_2 \,\&\, X_3$ have the density functions $f_1, f_2 \,\&\, f_3$, survival functions $\overline{F}_1, \overline{F}_2 \,\&\, \overline{F}_3$ and hazard rate functions $\lambda_{F_1}, \lambda_{F_2} \,\&\, \lambda_{F_3}$ respectively. Further, let $X_1 \overset{lhr}{\le} X_2$, that is $\frac{f_2}{f_1}$ is increasing in x, then

(I) $$\xi_\beta^w(X_1,X_3;t) - \xi_\beta^w(X_2,X_3;t) \le \frac{1}{1-\beta}\log\frac{\lambda_{F_1}(t)}{\lambda_{F_2}(t)}, \text{ if } 0<\beta<1.$$

(II) $$\xi_\beta^w(X_1,X_3;t) - \xi_\beta^w(X_2,X_3;t) \ge \frac{1}{1-\beta}\log\frac{\lambda_{F_1}(t)}{\lambda_{F_2}(t)}, \text{ if } \beta>1.$$

Proof. (I) since, by the given condition, we have $\frac{f_1(x)}{f_2(x)} \le \frac{f_1(t)}{f_2(t)}$. Therefore, for $0<\beta<1$ from (12), we obtain

$$\xi_\beta^w(X_1,X_3;t) \le \frac{1}{1-\beta}\log\int_t^\infty x^\beta \frac{f_1(t)f_2(x)}{\overline{F}_1(t)\overline{F}_2(t)}\left(\frac{f_3(x)}{\overline{F}_3(t)}\right)^{\beta-1} dx$$

$$= \xi_\beta^w(X_2,X_3;t) + \frac{1}{1-\beta}\log\left(\frac{\lambda_{F_1}(t)}{\lambda_{F_2}(t)}\right).$$

This completes the proof of (I). The proof of (II) follows similarly.

Theorem 4.8. Consider the three random variables as mentioned in theorem 4.7. Further, assume $X_2 \overset{lhr}{\le} X_3$, that is $\frac{f_3}{f_2}$ increasing in $x>0$, then for $\beta \ne 1$

$$\xi_\beta^w(X_1,X_2;t) - \xi_\beta^w(X_1,X_3;t) \ge -\log\frac{\lambda_{F_2}(t)}{\lambda_{F_3}(t)}.$$

Proof. Since, we have $\frac{f_2(x)}{f_3(x)} \le \frac{f_2(t)}{f_3(t)}$, therefore for $\beta \ne 1$ from (12), we obtain

$$\xi_\beta^w(X_1,X_2;t) \ge \frac{1}{1-\beta}\log\int_t^\infty x^\beta\left(\frac{f_1(x)}{\overline{F}_1(t)}\right)\left(\frac{f_2(t)f_3(x)}{f_3(t)\overline{F}_2(t)}\right)^{\beta-1} dx$$

$$= \xi_\beta^w(X_1,X_3;t) - \log\left(\frac{\lambda_{F_2}(t)}{\lambda_{F_3}(t)}\right).$$

Hence we get the desired result.

5 Conclusions

In this paper, we have developed and studied a weighted inaccuracy measure of order β with its dynamic (residual) version. It is shown that the weighted generalized residual inaccuracy measure uniquely characterizes the survival function. Further, we study the monotonic behavior of the proposed dynamic measure on the basis of exponential distribution. Finally, we have presented some important properties and inequalities of the proposed inaccuracy measure.

Acknowledgement: The authors are thankful to referees for their valuable suggestions, which have immensely helped and improved the presentation of the paper.

Conflict of Interest: The authors declare that they have no conflict of interest.

References

[1] C. E. Shannon, A mathematical theory of communications, Bell System Technical Journal, **27**, 379-423, (1948).

[2] D.F. Kerridge, Inaccuracy and inference, Journal of the Royal Statistical Society Ser. **B23**, 184-194 (1961).

[3] C.R. Rao, On discrete distributions arising out of methods of ascertainment, In: Patil, G.P. (eds) Classical and Contagious Discrete Distributions. Statistical Publishing Society, Calcutta, **4**, 320-332 (1965).

[4] M. Belis and S. Guiasu, A quantitative-qualitative measure of information in cybernetic systems, IEEE Transactions on Information Theory, **14(4)**, 593–594, (1968).

[5] N. Ebrahimi, How to measure uncertainty about residual lifetime distribution, Sankhya Ser. **A58**, 48-57 (1996).

[6] H. C. Taneja, V. Kumar and R. Srivastava, A dynamic measure of inaccuracy between two residual lifetime distributions, International Mathematical forum, **4(25)**, 1213-1220, (2009).

[7] Di Crescenzo and M. Longobardi, On weighted residual and past entropies, Scientiae Mathematicae Japonicae, **64**, 255–266 (2006).

[8] V. Kumar, H.C. Taneja and R. Srivastava, Length biased weighted residual inaccuracy measure. Metron, LXVIII, **68**,153-160 (2010).

[9] V. Kumar, H.C. Taneja, On length biased dynamic measure of past inaccuracy, Metrika, **75**, 73-84, (2012).

[10] C. Kundu, Characterizations based on length-biased weighted measure of inaccuracy for truncated random variables, Applications of Mathematics, **59(6)**, 697-714 (2014).

[11] M. Khorashadizadeh, An extension of weighted cumulative inaccuracy measure based on doubly truncation, Pakistan Journal of Statistics and Operation Research, **XIII (1)**, 103-113 (2017).

[12] S. Daneshi, A. Nezakati and S. Tahmasebi, On weighted cumulative past (residual) inaccuracy for record values, Journal of Inequalities and Applications, **134,** 1-22 (2019).

[13] P. Nath, Inaccuracy and coding theory, Metrika, **13**, 123–135 (1968).

[14] D.R. Cox, The analysis of exponentially distributed life-times with two types of failure, Journal of the Royal Statistical Society, Ser. **B21**, 411–421 (1959).

[15] D. R. Cox and D. Oakes, Analysis of Survival Data, Chapman and Hall, London (1984).

[16] N. Ebrahimi and S.N.U.A. Kirmani, A characterization of the proportional hazards model through a measure of discrimination between two residual life distributions, Biometrika, **83(1)**, 233-235 (1996).

[17] N.U. Nair and R. P. Gupta, Characterization of proportional hazard models by properties of information measures, International Journal of Statistical Sciences, Special Issue, **6**, 223–231 (2007).

[18] M. Nourbakhsh and G. Yari, Weighted Renyi's entropy for lifetime distributions, Communications in Statistics-Theory and Methods, **46(14)**, 7085-7098 (2016).