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On the Joint Distribution of Two Continuous Independent Random Variables

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Abstract: In this paper, we propose a novel special distribution. The proposed distribution is developed by analyzing the Joint distribution of two continuous independent random variables under some mathematical operations. One of these variables belongs to Chi-square distribution. The second variable belongs to the exponential distribution. The proposed method is based on the Change of variables and distribution function methods. One of main result of this analysis shows that the distribution of sum of these two variables is an exponential distribution. The graphs of the joint distribution function and various cases studies are discussed in detail.
Keywords: Random Variable, Chi-Square, and Exponential Distributions, Change of variables, Distribution Function.

1 Introduction

The probability density function for the sum of two discrete independent random variables together with an enforcement of the algorithm in the algebraic computer system has been discussed to propose some algorithms [1]. Moreover, they explained some case studies of their algorithm using brute force method [1]. S. Nadarajah S. Kotz [2] explained the distribution that yields from the division of two random variables which is, known as the stress-strength model. Moreover, they showed the graphical presentation of this distribution [3]. Recently, J. Osiewalski and J. Marzec [4] proposed a joint statistical model for two variables: The first is zero or a countable variable, and the second is a regular countable variable by applying the ZIP–CP bivariate model and the standard univariate Poisson regression model. The main results of this model are that: The inference on individual parameters is not affected by the sample selection error. Nowadays, quantum physics [5-8], and different areas of sciences and engineering are developed based on the distribution of random variables [9,10].
Ware and Lad [11] explained extensively which factors have more impact on the appearance of normality for the multiplication of two normally independent variables. They deduced that for small values of the coefficient of variation inverse (< 1), the normal distribution will not be a perfect approach for the multiplication. Furthermore, the impact of the joint ratio value is smaller than that of the coefficient of variation inverse value. P. E. Oguntunde et al. [12] proposed the Exponential distribution by deriving a model of two-parameter based on the sum of two exponentially distributed independent random variables. K. Teerapabolarn [13] used beta binomial w-functions and Stein's method to determine the limit of the distance of total variation between the distribution of the sum of n independent beta binomial random variables (with parameters $n_i, \alpha_i, and \beta_i$), and a binomial distribution:
(with parameters $m = \sum_{i=1}^{n} n_i \; and \; p = \frac{1}{m}\sum_{i=1}^{n} \frac{n_i \alpha_i}{\alpha_i + \beta_i}$). The method gives a perfect approach when all β_i are big comparing with all $n_i, and \alpha_i$. T. Kadri and K. Smaili [14] studied the division of two Hypo exponential independent distributions. They got the accurate terms of the probability density function, moment generating function, the cumulative distribution function, the reliability function, and hazard function. Moreover, they proved that all of these mentioned functions are a linear combination of the Generalized-F distribution. In this paper, we propose a novel special distribution very close to F-Distribution.
The proposed distribution is developed by evaluating the probability density function of the following two independent random variables: The sum, ratio, and the product of the two random variables X and Y, to find the distribution of two random variables. One of these random variables belongs to exponential distribution, and the second belongs to chi-square distribution.

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The rest of this paper is organized ,as follows: Section 2 presents some basic concepts which will be used in this paper. Section 3 explains the methods which are used to propose our distribution. Section 4 illustrates the proposed distribution in detail. Section 5 shows the experimental results. Section 6 is devoted to the main findings of this paper.

2 Backgrounds

Random Variable: it is a variable for a random operation and its values are numeric outcomes, commonly written X.
Exponential Distribution: it is continuous distribution with the following probability density function:

$$f(x) = \begin{cases} \lambda\ e^{-\lambda x} & , \quad x \geq 0 \\ 0 & , \quad x < 0 \end{cases}$$

with mean $\mu = \frac{1}{\lambda}$ of and variance of $\sigma^2 = \frac{1}{\lambda^2}$.
Chi-Squared Distribution: It is continuous distribution with the following probability density function:

$$f(x) = \frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)}\ x^{\left(\frac{r}{2}\right)-1}\ e^{-\frac{x}{2}} \quad ,x \geq 0$$

where r = degrees of freedom, the mean $\mu = r$ and the variance of $\sigma^2 = 2\text{r}$. The Probability density function (p.d.f) for the Continuous random variable is written, as follows:

$$P(a \leq X \leq b) = \int_a^b f(x)dx.$$

The Cumulative distribution function for the Continuous random variable is written ,as follows:
$\text{F(x)} = \text{P(X} \leq \text{x)} = \int_{-\infty}^{\text{x}} \text{f(t)dt}$. The Mathematical Expectation for the Continuous random variable is written, as follows:

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx,$$

based on the above formula, it was defined that the Population Mean:

$$\mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)dx,$$

and the Population Variance:

$$\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)dx = E(X^2) - [E(X)]^2.$$

The Moment generating function: Suppose that X is a continuous random variable, and its (p.d.f) is f(x). The moment generating function of X is written, as follows: $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$. Distribution Function: The Distribution Function for the continuous variable is given by:

$$F(x) = p(X \leq x) = \int_{-\infty}^{x} f(w)\,dw, where\colon -\infty \leq w \leq x$$

3 Methodologies

The proposed method depends on the change of variables, and the joint characteristic function.

3.1 Change of Variables: Let (Y_1, Y_2) be a function of (X_1, X_2) defined by $Y_1=u_1(X_1, X_2)$ and $Y_2=u_2(X_1, X_2)$ with the inverse of each single-valued given by: $X_1=v_1$ (Y_1, Y_2) and $X_2=v_2$ (Y_1, Y_2).Then, the joint probability density function of Y_1 andY_2 is given by:

$$g(y_1, y_2) = f[v_1(y_1, y_2), v_2(y_1, y_2)]\ |J|$$

where:

$$|J| = \begin{vmatrix} \frac{\partial v_1}{dy_1} & \frac{\partial v_1}{dy_2} \\ \frac{\partial v_2}{dy_1} & \frac{\partial v_2}{dy_2} \end{vmatrix}$$

3.2 The Joint Characteristic Function

Since we are talking about two random variables X and Y, we must define the joint characteristic function of two random variables by the following formula: $\phi_{X,Y}(\omega_1, \omega_2) = Ee^{j\omega_1 x + j\omega_2 y}$
If X and Yare jointly continuous random variables, then

$$\phi_{X,Y}(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)e^{j\omega_1 x + j\omega_2 y}\,dydx$$

Theorem1: If X_1, X_2 are independent, then $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$.

Theorem2: If X_1, X_2 are independent, then $M_{X_1+X_2}(t) = M_{X_1}(t)M_{X_2}(t)$.

Proof :

Let X_1, X_2 be two independent variables if and only if $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)f_{X_2}(x_2)$, So

$$M_{X_1+X_2}(t) = E(e^{t(X_1+X_2)}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t(x_1+x_2)} f_{X_1,X_2}(x_1,x_2)dx_1dx_2$$

$$= \int_{-\infty}^{\infty} e^{tx_1} f_{X_1}(x_1)dx_1 \int_{-\infty}^{\infty} e^{tx_2} f_{X_2}(x_2)dx_2$$

$$= M_{X_1}(t)M_{X_2}(t).$$

4 The Proposed Method

4.1 Distribution of the Sum

Assume that X and Y are two independent random variables such that:

$X \sim \chi^2(r)$ and $Y \sim exp(\lambda)$, then

$$f(x) = \frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)} x^{\frac{r}{2}-1} e^{-\frac{x}{2}}, \quad x > 0 \qquad (1)$$

$$and \quad f(y) = \lambda\, e^{-\lambda y} \quad , \; y > 0 \qquad (2)$$

$$f(x,y) = \frac{\lambda}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)} e^{-\lambda y}\; x^{\frac{r}{2}-1} e^{-\frac{x}{2}} \;, \; x > 0, y > 0 \qquad (3)$$

Applying the Distribution Function *technique, we get*:

$let \;\; Z = X + Y \;, F(Z) = p(Z \le z) = \; p(X + Y \le z) = \; p(Y \le z - x)$

$$= \int_0^{\infty}\int_0^{z-x} \frac{\lambda}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)} e^{-\frac{x}{2}} e^{-\lambda y} x^{\frac{r}{2}-1} dydx$$

$$= \frac{\lambda}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)}\left(\int_0^{\infty} e^{-\frac{x}{2}} x^{\frac{r}{2}-1}\left(\int_0^{z-x} e^{-\lambda y}\, dy\right) dx\right)$$

$$= \frac{\lambda}{2^{r/2}\Gamma(r/2)}\int_0^{\infty} e^{-x/2} x^{r/2-1}\left(\frac{-1}{\lambda}\left(e^{-\lambda(z-x)} - 1\right)dx\right)$$

$$= 1 - \frac{2e^{-\lambda z}}{2^{r/2}\Gamma(r/2)(1\text{-}2\lambda)} \qquad (4)$$

$\lim_{z\to\infty}(1 - \frac{e^{-z}}{(1-2\lambda)}) = 1$

assume that r =2 and λ =1, then F(Z)=$(1 - \frac{e^{-z}}{(1-2\lambda)})$, *So* $\lim_{z\to\infty}(1 - \frac{e^{-z}}{(1-2\lambda)}) = 1.$

Thus , $\frac{dF}{dz} = f(z) = \frac{2\lambda e^{-\lambda^z}}{2^{r/2}\Gamma(r/2)(1\text{-}2\lambda)}$ (z > 0), suppose that z*=$\lambda e^{-\lambda^z} \approx exp(\lambda)$

f(z) is p.d.f since $\int_0^{\infty} \frac{2\lambda e^{-\lambda^z}}{2^{r/2}\Gamma(r/2)(1\text{-}2\lambda)} dz = 1$.

Based on the above calculations, we conclude that the distribution of sum of two independent random variables , one form chi-square distribution and the second from exponential distribution, is an exponentially distribution.

4.2 Distribution of Ratio

Assume that X and Y are two independent random variables such that:

$X \sim \chi^2(r)$ and $Y \sim exp(\lambda)$, then

$$f(x)=\frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)}x^{\frac{r}{2}-1}e^{-\frac{x}{2}},\quad x>0 \tag{5}$$

and $$f(y)=\lambda\, e^{-\lambda y}\quad,\ y>0 \tag{6}$$

$$f(x,y)=\frac{\lambda}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)}e^{-\lambda y}\ x^{\frac{r}{2}-1}e^{-\frac{x}{2}}\ ,\ x>0,y>0 \tag{7}$$

Applying the Distribution Function technique, we get:

$$let\ Z=Y/X,\qquad F(Z)=p(Z\le z)=\ p\left(\frac{Y}{X}\le z\right)=\ p(Y\le zx)$$

$$=\int_0^\infty\int_0^{zx}\frac{\lambda}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)}e^{-\frac{x}{2}}e^{-\lambda y}x^{\frac{r}{2}-1}dydx$$

$$=\frac{\lambda}{2^{r/2}\Gamma(r/2)}\left(\int_0^\infty e^{-x/2}x^{r/2-1}\left(\int_0^{zx}e^{-\lambda y}\,dy\right)dx\right)$$

$$=\frac{\lambda}{2^{r/2}\Gamma(r/2)}\int_0^\infty e^{-x/2}x^{r/2-1}\left(\frac{-1}{\lambda}\left(e^{-\lambda zx}-1\right)dx\right)$$

$$=1-\frac{1}{(1-2\lambda z)^{\frac{r}{2}}},\qquad z>0 \tag{8}$$

$\lim_{z\to\infty}(1-\frac{1}{(1-2\lambda z)^{\frac{r}{2}}})=1$, if r =2 and λ =1 then F(Z)=$1-\frac{1}{(1-2Z)^{\frac{1}{2}}}$

$so\ that\ \lim_{z\to\infty}(1-\frac{1}{(1-2Z)^{\frac{1}{2}}})=1\ .\ now\ \frac{dF}{dz}=f(z)=\ \frac{r\lambda}{(1\text{-}2z\lambda)^{\frac{r}{2}+1}}(z\ >\ 0)$

$$\Rightarrow f(z)\ is\ p.d.f\ since\ \int_0^\infty\frac{r\lambda}{(1\text{-}2z\lambda)^{\frac{r}{2}+1}}dz=1\quad,\qquad (z\ >\ 0).$$

Therefore, the following special results were obtained:

$$E(Z)=\mu=\int_0^\infty\frac{zr\lambda}{(1-2\lambda z)^{-(r/2+1)}}dz=\frac{r\lambda}{2(1\text{-}r/2)} \tag{9}$$

$$E(Z^2)=\int_0^\infty\frac{z^2r\lambda}{(1-2\lambda z)^{-\left(\frac{r}{2}+1\right)}}dz=\frac{4\lambda}{(1\text{-}r/2)(2\text{-}r/2)} \tag{10}$$

$$V(Z)=\sigma^2=E(Z^2)-E(Z)^2=\ \frac{\left[16\text{-}8r\text{-}2r^2\lambda^2+\left(\frac{r^3}{2}\right)\lambda\right]}{4(1\text{-}r/2)^2(2\text{-}r/2)} \tag{11}$$

Based on the above calculations, it is obvious that the distribution of ratio of the two variables, form chi-square distribution and exponential distribution is a novel distribution close to the behavior of F-Distribution .

4.3 Distribution of the Product

Assume that X and Y are two independent random variables such that:

$X\sim\chi^2(r)\quad and\ \ Y\sim exp(\lambda)\ \ ,then$

$$f(x)=\frac{1}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)}x^{\frac{r}{2}-1}e^{-\frac{x}{2}},\quad x>0 \tag{12}$$

$$and\ \ f(y)=\lambda\, e^{-\lambda y}\quad,\ y>0 \tag{13}$$

$$f(x,y)=\frac{\lambda}{2^{r/2}\Gamma(r/2)}e^{-\lambda y}\ x^{r/2-1}e^{-x/2}\ ,\ x>0,y>0 \tag{14}$$

Applying the change of variables technique, we get:

$$let\quad m_1=x.y,m_2=y\qquad\Rightarrow x=\frac{m_1}{y}=\frac{m_1}{m_2},y=m_2$$

$$\text{then }|J|=\begin{vmatrix}\frac{\partial m_1}{dx} & \frac{\partial m_1}{dy}\\ \frac{\partial m_2}{dx} & \frac{\partial m_2}{dy}\end{vmatrix}=\begin{vmatrix}y & x\\ 0 & 1\end{vmatrix}=y=m_2$$

The joint p.d.f. of m_1 and m_2 is given by

$$g(m_1,m_2) = \frac{\lambda}{2^{\frac{r}{2}}\Gamma\left(\frac{r}{2}\right)} e^{-\lambda(\mathrm{m1/m2})}\, m_2^{\frac{r}{2}-1} e^{-\frac{m_2}{2}} m_2 \qquad , m_1, m_2 > 0 \tag{15}$$

$$= \frac{\lambda}{2^{\frac{r}{2}}\,\Gamma\left(\frac{r}{2}\right)} e^{-\lambda(\mathrm{m1/m2})}\; m_2^{\frac{r}{2}}\, e^{-\frac{m_2}{2}} \tag{16}$$

Thus, the marginal probability density function of m_1 is given by:

$$g_1(m_1) = \int_0^{\infty} \frac{\lambda}{2^{\frac{r}{2}}\,\Gamma\left(\frac{r}{2}\right)} e^{-\lambda(\mathrm{m1/m2})}\; m_2^{\frac{r}{2}} e^{-\frac{m_2}{2}} dm_2 \tag{17}$$

$$where \;\; I = \int_0^{\infty} e^{-\lambda\,(m1/m2)}\; m_2{}^{r/2} e^{-m_2/2} dm_2$$

5 Experimental Results

In this section the graphs that explain the distribution of the proposed distribution are plotted and analyzed. Fig. 1 shows the distribution of the sum of the two random variables for the parameters r =2, and λ =1.

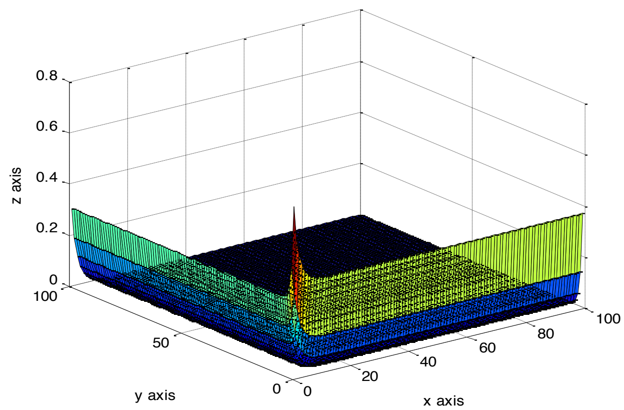

Fig. 1: The distribution of the sum of the two random variables for r =2 and λ =1.

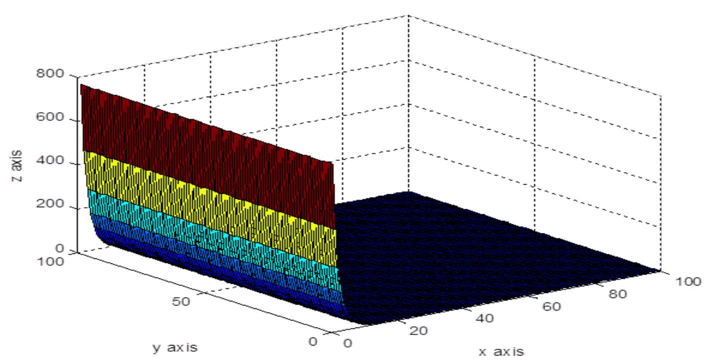

Fig.2: Shows the graph of the sum of the two random variables when r =4 and λ =5.

While Fig. 2 shows the distribution of the sum of the two random variables for the parameters r =4, and λ =5. It is evident from Fig.1 and Fig.2 that the distribution of the sum of the two random variables behaves like the exponential distributed. Also, it is obvious from Fig.1 and Fig.2 that the behavior of the proposed distribution is close to the behavior of F-Distribution as theoretically expected in Section 4.

6 Conclusion

In this paper, a novel special distribution is proposed. This distribution was proposed using the distribution function and the change of variables techniques has been applied to obtain the distribution of the sum, ratio, and the multiplication of two random variables. The main result of the proposed technique showed that the distribution of sum of two independent random variables form chi-square distribution and exponential distribution was an exponential distribution. Moreover, the distribution of ratio of two independent random variables, form chi-square distribution and exponential distribution was a special distribution.

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References

[1] D. L. Evans, L. M. Leemis, Algorithms for computing the distributions of sums of discrete random variables. Mathematical and Computer Modelling., **40(13)**, 1429-1452,(2004).

[2] S. Nadarajah, S. Kotz, On the product and ratio of t and Bessel random variables. BULLETIN-INSTITUTE OF MATHEMATICS ACADEMIA SINICA., **2(1)**, 55,(2007).

[3] J. Osiewalski, J. Marzec, Joint modelling of two count variables when one of them can be degenerate. Computational Statistics., **34(1)**, 153-171,(2019).

[4] M. Abdel-Aty, Quantum field entropy and entanglement of a three-level atom two-mode system with an arbitrary nonlinear medium, Journal of Modern Optics.,**50(2)**,161-177,(2003).

[5] M. Abdel-Aty, Quantum information entropy and multi-qubit entanglement, Progress in Quantum Electronics **31(1)**, 1-49 ,(2007).

[6] M. Abdel-Aty, An investigation of entanglement and quasiprobability distribution in a generalized Jaynes-Cummings model, Journal of Mathematical Physics., **44(4)**, 1457-1471,(2003).

[7] M. Abdel-Aty, T. Yu, Entanglement sudden birth of two trapped ions interacting with a time-dependent laser field, Journal of Physics B: Atomic, Molecular and Optical Physics., **41(23)**, 235503,(2008).

[8] J.-S. Zhang, , A.-X. Chen, M. Abdel-Aty, Two atoms in dissipative cavities in dispersive limit: Entanglement sudden death and long-lived entanglement, Journal of Physics B: Atomic, Molecular and Optical Physics., **43(2)**, 025501,(2010).

[9] M. Zidan, A. H. Abdel-Aty, A. El-Sadek, E. A. Zanaty, M. Abdel-Aty, Low-cost autonomous perceptron neural network inspired by quantum computation, AIP Conference Proceedings., **1905(1)**, 020005,(2017).

[10] A.-H. Abdel-Aty, H. Kadry, M. Zidan, Y. Al-Sbou, E. A. Zanaty, M. Abdel-Aty, A quantum classification algorithm for classification incomplete patterns based on entanglement measure, Journal of Intelligent & Fuzzy Systems., **38(3)**, 2809-2816,(2020).

[11] A. Seijas-Macías, A. Oliveira an approach to distribution of the product of two normal variables, Discussiones Mathematicae Probability and Statistics., **32**, 87-99,(2012).

[12] P. E. Oguntunde, , O. Odetunmibi, A. Adejumo, On the Sum of exponentially distributed random variables: A convolution approach. European Journal of Statistics and Probability., **2(1)**, 1-8, (2014).

[13] K. Teerapabolarn, Binomial approximation for a sum of independent binomial random variables, Appl. Math. Sci., **8**, 8929-8932 ,(2014).

[14] T. Kadri, K. Smaili, The exact distribution of the ratio of two independent hypoexponential random variables. Journal of Advances in Mathematics and Computer Science., 2665-2675, (2014).