[Information Sciences Letters](https://digitalcommons.aaru.edu.jo/isl)

Volume 11
Issue 5 *Sep. 2022*

Article 7

2022

Remediation of pollution in a river by releasing clean water using the solution of advection-diffusion equation in two dimensions

Fayez Ibrahim
Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt, ahmed.saleh9167@gmail.com

Ahmed Saleh
Department of Basic Science, Egyptian Academy for Engineering and Advanced Technology, Cairo, Egypt, ahmed.saleh9167@gmail.com

Ali Wadi
Fasher University, Darfur, Sudan, ahmed.saleh9167@gmail.com

Mohamed Hadhouda
Elgazeera High Institute for Engineering and Technology, Cairo, Egypt, ahmed.saleh9167@gmail.com

Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/isl/110507

Remediation of pollution in a river by releasing clean water using the solution of advection-diffusion equation in two dimensions

Fayez Ibrahim [1], *Ahmed Saleh*[2,*], *Ali Wadi* [3] *and Mohamed Hadhouda* [4]

[1]Department of Mathematics, Faculty of Science, Ain Shams University, Cairo, Egypt
[2]Department of Basic Science, Egyptian Academy for Engineering and Advanced Technology, Cairo, Egypt
[3]Fasher University, Darfur, Sudan
[4]Elgazeera High Institute for Engineering and Technology, Cairo, Egypt

Received: 26 Feb. 2022, Revised: 22 Apr. 2022, Accepted: 6 May 2022
Published online: 1 Sep. 2022

Abstract: Analytical and numerical solutions are obtained for two-dimensional advection-diffusion equation, using Laplace transformation technique and explicit finite difference method for the pollutant concentration in a river or in shallow aquifer with time-dependent dispersion coefficients. We take two cases, first case: concentration of increasing nature (mixed type or third type) is considered at the origin and initially the domain is solute free. Second case: the river's water is polluted initially (at time $t = 0$) while at the origin, at time $t > 0$, the source of pollution is removed by releasing fresh water. We have proved mathematically the fact that the high concentration of pollutant can be reduced by releasing adequate discharges from barrage in a river. Both the dispersion coefficients, the velocity components and first order decay term are considered exponentially decreasing function of time. The different effects of the parameters controlling the pollutant dispersion along the river at any time are studied separately with the help of figures. The parameters that have a role in removing or reducing concentration of pollutant along the river have been studied in detail. When comparing the analytical solution with the numerical solution, we found a very good agreement between them. For a real situation, our simple model can provide decision support for planning restrictions to be imposed on farming and urban practices.

Keywords: Concentration of pollutant, Advection-diffusion equation, Explicit finite difference method, Laplace transformation, Solutions of partial differential equations.

1 Introduction

Many rivers such as the river Nile in Egypt, the Tha Chin River in Thailand [1] and, the Mississippi River in the U.S.A. pass through agricultural and industrial areas and through settled communities. These rivers are polluted from the combined discharges of industrial, domestic, and rural inflows before reaching the sea. Every year, approximately 25 million persons die as a result of water pollution[1]. Hence, water pollution is a major problem in many countries. The advection-diffusion equation describes the pollutant concentration distribution due to the combined effect of diffusion and convection in a porous medium.

The advection-diffusion equation is applicable in many disciplines like groundwater hydrology, chemical engineering, biosciences, environmental sciences and, petroleum engineering [2]. Its solutions along with an initial condition and two boundary conditions help to understand the pollutant concentration distribution behavior through an open medium like air, rivers, lakes and, porous medium like an aquifer, on the basis of which remedial processes to reduce or eliminate the damages may be enforced, Kumar et al. [3]. Yadav et al. [4] obtained an analytical solution for two-dimensional dispersion through a semi-infinite homogeneous porous medium when a point source concentration of pulse-type is considered at the origin. Ibrahim et al. [5] investigated pollution remediation in a river using unsteady aeration with arbitrary initial and boundary conditions. Yadav and Kumar [6] investigated analytical solutions for the two-dimensional advection-dispersion equation in a semi-infinite heterogeneous porous medium with a

* Corresponding author e-mail: ahmed.saleh9167@gmail.com

uniform nature pulse-type input point source for conservative solute transport. Kumar et al. [7] investigated solute dispersion in a semi-infinite porous medium with a source/trough effect. Suryani et al. [8] solved the diffusion-convection equation with variable coefficients and for anisotropic media by using the boundary element method. Manitcharoen and Pimpunchat [9] used a mathematical model in a one-dimensional advection-dispersion equation that included terms of decay and enlargement process to study the motion of flowing pollution. Saleh et al. [10] obtained analytical and numerical solutions for a one-dimensional advection-diffusion equation with constant coefficients, the initial condition and the boundary condition at the source of pollution were applied to describe the exponential variation in pollutant concentration. They proved mathematically the fact that the high concentration of pollutants can be reduced with great efficiency by releasing clean water discharges from barrage in a river. The propagation of pollution in water bodies can be studied in several ways [11], [12] and [13].

The objective of this study is to obtain an analytical solution of the advection-dispersion equation by using Laplace transformation and numerical solution by using the explicit finite difference method. The flow is assumed to be unsteady through the semi-infinite porous medium $x \geq 0$. Both components of dispersion coefficient and flow velocity components in x and y directions are considered as decreasing functions of time t. We take two cases, first case: the concentration of increasing nature (mixed type or third type) is considered at the origin and initially the domain is solute-free. Second case: the river's water is polluted initially (at time $t=0$) while at the origin, at time $t>0$, the source of pollution is removed by releasing fresh water. Also, there is no pollutant concentration exchange at end of both boundaries x and y. Four special cases are obtained from the analytical solution.

2 Formulation of the problem

The general partial differential equation describing hydrodynamic dispersion in a homogeneous, isotropic porous media in two dimensions can be written as [4] and [14].

$$R\frac{\partial C}{\partial t} = D_x(t)\frac{\partial^2 C}{\partial x^2} + D_y(t)\frac{\partial^2 C}{\partial y^2} - u(t)\frac{\partial C}{\partial x} - v(t)\frac{\partial C}{\partial y} - \gamma(t)C \quad (1)$$

where R is the retardation coefficient accounting for equilibrium linear sorption processes, $C(x,y,t)$ $(\mathrm{kg\,m^{-3}})$ is the pollutant concentration which depends on the longitudinal direction along the river x (m), the transversal direction $y(m)$ and time $t(\text{day})$, $D_x\left(m^2 day^{-1}\right)$ and $D_y\left(m^2 day^{-1}\right)$ are the hydrodynamic dispersion coefficients in x and y-directions respectively, $u(t)\left(m\,day^{-1}\right)$ and $v(t)\left(m\,day^{-1}\right)$ are the average fluid velocities in the x and y-directions respectively and $\gamma(t)$ $\left(day^{-1}\right)$ is the first order decay term. Yadav et al. [4] and Ebach and White [15] have established that the dispersion coefficients vary approximately directly with flow velocity for different types of the porous medium, hence:

$$\left.\begin{array}{l} D_x = D_{x_0}\exp(-nt), \quad D_y = D_{y_0}\exp(-nt) \\ u = u_0\exp(-nt), \quad v = v_0\exp(-nt), \\ \gamma = \gamma_0\exp(-nt), \end{array}\right\} \quad (2)$$

where $u_0\left(m\,day^{-1}\right)$ and $v_0\left(m\,day^{-1}\right)$ are initial velocity components of the fluid along the x and y-directions respectively, $D_{x_0} = au_0\left(m^2 day^{-1}\right), D_{y_0} = av_0\left(m^2 day^{-1}\right)$ are initial dispersion coefficient components along two respective directions x and y, a (m) is a constant depending upon pore geometry of the medium, $n\left(day^{-1}\right)$ is flow resistance constant-coefficient and $\gamma_0\left(day^{-1}\right)$ is the initial first-order decay term. By using equation (2), equation (1) is solved by Yadav et al. [4] for the case of a uniform pulse-type input point source condition. Introducing a distance variable $\chi(m)$ and a time variable T (day) defined by Crank [16], Kumar et al. [2], Jaiswal et al. [17] and, Yadav et al. [4]

$$\left.\begin{array}{l} \chi = x + y\sqrt{\frac{D_{y_0}}{D_{x_0}}}, \\ T = \int_0^t \frac{\exp(-nt)}{R}dt = \frac{1}{nR}[1-\exp(-nt)]. \end{array}\right\} \quad (3)$$

The expression $\exp[-nt]=1$, for $n=0$ or $t=0$, thus the new time variable T obtained from equation (3) satisfies the conditions $T=0$ for $t=0$ and $T=t/R$ for $n=0$. Equations (2) and (3) transform equation (1) into:

$$\frac{\partial C}{\partial T} = D\frac{\partial^2 C}{\partial \chi^2} - U\frac{\partial C}{\partial \chi} - \gamma_0 C, \quad (4)$$

where

$$D = D_{x_0}\left(1+\frac{D_{y_0}^2}{D_{x_0}^2}\right), \; U = \left(u_0 + v_0\sqrt{\frac{D_{y_0}}{D_{x_0}}}\right), \quad (5)$$

such that D is new variable represents the dispersion and U represents the velocity. We studied two different cases where:
first case: $C(x,y,0)=0, C(0,0,t)\neq 0$.
Second case: $C(x,y,0)\neq 0, C(0,0,t)=0$.

Case study one: in our study, we will assume that the river is initially free from the pollutant. The source of input pollutant concentration may increase with time due to a variety of reasons. This type of situation may be described by a mixed type (third type). Also at infinity from the source $(x=y=0)$, it is assumed that there is no pollutant concentration exchange with the system. Hence the initial and boundary conditions associated with equation (1) are:

$$C(x,y,t)=0 \quad , \quad x\geq 0, \quad y\geq 0, t=0, \quad (6)$$

$$
\begin{aligned}
&-\beta\left[D_x\frac{\partial C}{\partial x}-\sqrt{\frac{D_y^3}{D_x}}\frac{\partial C}{\partial y}\right]+\left[u+v\sqrt{\frac{D_y}{D_x}}\right]C \\
&=\left[u+v\sqrt{\frac{D_y}{D_x}}\right]C_0,\ x=0,\ y=0,\ t>0,
\end{aligned} \tag{7}
$$

$$
\frac{\partial C}{\partial x}=\frac{\partial C}{\partial y}=0,\ x\to\infty,\ y\to\infty,\ t\geq 0, \tag{8}
$$

where C_0 is constant at the inlet boundary and β is constant. Equations (2),(3) and (5) transform equations (6-8) into:

$$
C(\chi,T)=0\,,\ \chi\geq 0,\ T=0, \tag{9}
$$

$$
-\beta\, D\frac{\partial C}{\partial \chi}+UC=UC_0\,,\ \chi=0,\ T>0, \tag{10}
$$

$$
\frac{\partial C}{\partial \chi}=0,\ \chi\to\infty,\ T\geq 0, \tag{11}
$$

Case study two: Assume that the river's water is polluted at the initial time $(t=0)$ along the river. Assume also that at $x=y=0$ at any time $(t>0)$, the source of pollution is removed. Then the initial and boundary conditions associated with equation (1) are: (Saleh et al. [10], Hadhouda and Hassan [18])

$$
C(x,y,t)=C_1 e^{\frac{-\left(x+y\sqrt{\frac{D_{y_0}}{D_{x_0}}}\right)}{k}},\quad x\geq 0,\ y\geq 0,\ t=0, \tag{12}
$$

$$
C(x,y,t)=0,\qquad x=y=0,\ t>0, \tag{13}
$$

$$
\frac{\partial C}{\partial x}=\frac{\partial C}{\partial y}=0\,,\qquad x\to\infty,\ y\to\infty,\ t\geq 0, \tag{14}
$$

where C_1 $\left(\mathrm{kg\ m^{-3}}\right)$ is the value of pollutant concentration at $x=y=0$ and $t=0$, $k(\mathrm{\ m})$ is the initial pollutant-decay length. Equations (2) and (3) transform equations (12-14) into:

$$
C(\chi,T)=C_1\, e^{\frac{-\chi}{k}}\,,\qquad \chi\geq 0,\ T=0, \tag{15}
$$

$$
C(\chi,T)=0\,,\qquad \chi=0,\ T>0, \tag{16}
$$

$$
\frac{\partial C}{\partial \chi}=0\,,\qquad \chi\to\infty, T\geq 0 \tag{17}
$$

3 The analytical solution

The solution of equation (4) can be suggested in the form (Kumar et al. [2] and Yadav et al. [4])

$$
C(\chi,T)=K(\chi,T)\exp\left(\frac{U}{2D}\chi-\left(\frac{U^2}{4D}+\gamma_0\right)T\right). \tag{18}
$$

Equation (18) transforms equation (4) and equations (9-11) into :

$$
\frac{\partial K}{\partial T}=D\frac{\partial^2 K}{\partial \chi^2}. \tag{19}
$$

$$
K(\chi,T)=0\,,\ \chi\geq 0\,,\ T=0, \tag{20}
$$

$$
\begin{aligned}
&-\beta\mathrm{D}\frac{\partial \mathrm{K}}{\partial \chi}+\mathrm{U}\left(1-\frac{\beta}{2}\right)\mathrm{K} \\
&=\mathrm{UC_0}\exp\left\{\left[\frac{\mathrm{U}^2}{4\mathrm{D}}+\gamma_0\right]\mathrm{T}\right\},\ \chi=0,\ \mathrm{T}>0.
\end{aligned} \tag{21}
$$

$$
\frac{\partial \mathrm{K}}{\partial \chi}+\frac{\mathrm{U}}{2\mathrm{D}}\mathrm{K}=0\,,\quad \chi\to\infty\,,\ \mathrm{T}\geq 0, \tag{22}
$$

Applying Laplace transformation on equations (19, 21 and 22) and using equation (20) gives:

$$
D\frac{d^2\bar{K}(\chi,P)}{d\chi^2}=P\bar{K}(\chi,P), \tag{23}
$$

$$
\begin{aligned}
&-\beta D\frac{d\bar{K}(\chi,P)}{d\chi}+U\left(1-\frac{\beta}{2}\right)\bar{K}(\chi,P) \\
&=\frac{UC_0}{P-\alpha^2},\quad \chi=0,
\end{aligned} \tag{24}
$$

$$
\frac{d\bar{K}(\chi,P)}{d\chi}+\frac{U}{2D}\bar{K}(\chi,P)=0,\quad \chi\to\infty, \tag{25}
$$

where, α is constant which is given by $\alpha^2=\left(\frac{U^2}{4D}+\gamma_0\right)$ and P is Laplace transform of the time t, which is a complex variable and $\bar{K}$ is Laplace transform of K. Thus, the general solution of the ordinary differential equation (23) subject to boundary conditions (24) and (25), may be written as:

$$
\bar{K}(\chi,P)=\frac{UC_0}{\beta D}\frac{\exp\left(-\chi\sqrt{\frac{P}{D}}\right)}{(P-\alpha^2)\left[\sqrt{\frac{P}{D}}+\frac{U}{\beta D}\left(1-\frac{\beta}{2}\right)\right]}. \tag{26}
$$

Now, applying the inverse of Laplace transformation on equation (26) and using equation (18), hence the analytical solution of advection-diffusion equation (1) associated with the initial and boundary conditions (6-8) may be written in terms of (χ,T) as:

$$
\begin{aligned}
C(\chi,T)&=\frac{1}{2}UC_0\exp\left[\frac{U}{2D}\chi\right] \\
&*\left\{\begin{array}{l}\dfrac{\exp\left[-\frac{\alpha}{\sqrt{D}}\chi\right]\mathrm{erfc}\left(\frac{\chi}{2\sqrt{DT}}-\alpha\sqrt{T}\right)}{\left(U\left(1-\frac{\beta}{2}\right)+\alpha\beta\sqrt{D}\right)} \\ +\dfrac{\exp\left[\frac{\alpha}{\sqrt{D}}\chi\right]\mathrm{erfc}\left(\frac{\chi}{2\sqrt{DT}}+\alpha\sqrt{T}\right)}{\left(U\left(1-\frac{\beta}{2}\right)-\alpha\beta\sqrt{D}\right)}\end{array}\right\} \\
&-\frac{U^2C_0\left(1-\frac{\beta}{2}\right)\mathrm{erfc}\left(\frac{\chi}{2\sqrt{DT}}+\frac{U}{\beta\sqrt{D}}\left(1-\frac{\beta}{2}\right)\sqrt{T}\right)}{\left(U^2\left(1-\frac{\beta}{2}\right)^2-\alpha^2 D\right)} \\
&*\exp\left\{\begin{array}{l}\left(\frac{U}{2D}+\frac{U}{\beta D}\left(1-\frac{\beta}{2}\right)\right)\chi \\ -\left(\alpha^2-\frac{U^2}{\beta^2 D}\left(1-\frac{\beta}{2}\right)^2\right)T\end{array}\right\},
\end{aligned} \tag{27}
$$

where erfc is the complementary error function. The solution given by equation (27) is the same as that given by Cleary and Adrian [19] quoted as problem C_7 for the case of constant coefficients.

4 Special cases

(I)Two- dimensional dispersion with constant coefficients and input conditions of increasing nature: the special case for which D_x, D_y, u, v and γ are constants can be derived by substituting $n = 0$ in equation (2), thus $T = \frac{t}{R}$ and equation (27) gives:

$$C(\chi,T) = \frac{1}{2} U C_0 \exp\left[\frac{U}{2D}\chi\right] * \left\{ \begin{array}{l} \frac{\exp\left[-\frac{\alpha}{\sqrt{D}}\chi\right] \operatorname{erfc}\left(\frac{\chi}{2\sqrt{DT}} - \alpha\sqrt{T}\right)}{\left(U\left(1-\frac{\beta}{2}\right)+\alpha\beta\sqrt{D}\right)} \\ + \frac{\exp\left[\frac{\alpha}{\sqrt{D}}\chi\right] \operatorname{erfc}\left(\frac{\chi}{2\sqrt{DT}} + \alpha\sqrt{T}\right)}{\left(U\left(1-\frac{\beta}{2}\right)-\alpha\beta\sqrt{D}\right)} \end{array} \right\} - \frac{U^2 C_0 \left(1-\frac{\beta}{2}\right) \operatorname{erfc}\left(\frac{\chi}{2\sqrt{DT}} + \frac{U}{\beta\sqrt{D}}\left(1-\frac{\beta}{2}\right)\sqrt{T}\right)}{\left(U^2\left(1-\frac{\beta}{2}\right)^2 - \alpha^2 D\right)} * \exp\left\{ \begin{array}{l} \left(\frac{U}{2D} + \frac{U}{\beta D}\left(1-\frac{\beta}{2}\right)\right)\chi \\ -\left(\alpha^2 - \frac{U^2}{\beta^2 D}\left(1-\frac{\beta}{2}\right)^2\right) T \end{array} \right\}. \tag{28}$$

(II) Two-dimensional dispersion solution for uniform input condition (case $\beta = 0$): the special case for which $\beta = 0$ is derived from equation (27) as:

$$C(\chi,T) = \frac{1}{2} C_0 \exp\left[\frac{U\chi}{2D}\right] \left\{ \begin{array}{l} \exp\left[-\frac{\alpha\chi}{\sqrt{D}}\right] * \\ \operatorname{erfc}\left(\frac{\chi - 2\alpha T\sqrt{D}}{2\sqrt{DT}}\right) \\ + \exp\left[\frac{\alpha\chi}{\sqrt{D}}\right] * \\ \operatorname{erfc}\left(\frac{\chi + 2\alpha T\sqrt{D}}{2\sqrt{DT}}\right) \end{array} \right\}. \tag{29}$$

(III) One-dimensional solution for input condition of increasing nature: the solution for one-dimensional dispersion can be derived by substituting $D_{y_0} = 0$ in equations (3) and (5), thus: $\chi = x, D = D_{x_0}, U = u_0$. In this case equation (27) by substituting $\beta = 1$ gives:

$$C(x,T) = \frac{1}{2} u_0 C_0 \exp\left[\frac{u_0}{2D_{x_0}} x\right] * \left\{ \begin{array}{l} \frac{\exp\left[-\frac{\eta}{\sqrt{D_{x_0}}} x\right] \operatorname{erfc}\left(\frac{x}{2\sqrt{D_{x_0} T}} - \eta\sqrt{T}\right)}{\left(\frac{u_0}{2} + \eta\sqrt{D_{x_0}}\right)} \\ + \frac{\exp\left[\frac{\eta}{\sqrt{D_{x_0}}} x\right] \operatorname{erfc}\left(\frac{x}{2\sqrt{D_{x_0} T}} + \eta\sqrt{T}\right)}{\left(\frac{u_0}{2} - \eta\sqrt{D_{x_0}}\right)} \end{array} \right\} - \frac{0.5\, u_0^2\, C_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{D_{x_0} T}} + \frac{u_0}{2\sqrt{D_{x_0}}}\sqrt{T}\right)}{0.25\, u_0^2 - \eta^2 D_{x_0}} * \exp\left\{ \frac{u_0\, x}{D_{x_0}} - \gamma_0\, T \right\}, \tag{30}$$

where $\eta^2 = \frac{u_0^2}{4D_{x_0}} + \gamma_0$, equation (30) is the same as that obtained by Kumar et al. [2].

(IV) One-dimensional solution for uniform input condition (case $\beta = 0$): the solution for one-dimensional dispersion can be derived by substituting $D_{y_0} = 0$ in equations (3) and (5), thus equation (27) gives:

$$C(x,T) = \frac{1}{2}\, C_0 \exp\left[\frac{u_0\, x}{2\, D_{x_0}}\right] \left\{ \begin{array}{l} \exp\left[-\frac{\eta\, x}{\sqrt{D_{x_0}}}\right] * \\ \operatorname{erfc}\left(\frac{x - 2\eta\, T\sqrt{D_{x_0}}}{2\sqrt{D_{x_0}\, T}}\right) \\ + \exp\left[\frac{\eta\, x}{\sqrt{D_{x_0}}}\right] * \\ \operatorname{erfc}\left(\frac{x + 2\eta\, T\sqrt{D_{x_0}}}{2\sqrt{D_{x_0} T}}\right) \end{array} \right\} \tag{31}$$

Equation (31) agrees with that obtained by Kumar et al. [2].

For Case study two: equation (1) associated with the initial and boundary conditions (12-14) is solved analytically, hence $C(x,y,t)$ may be written in terms of (χ, T) as:

$$C(\chi,T) = \frac{C_1}{2} \exp\left[\frac{D\, T - k\,(-T\, U + \chi + k\, T\, \gamma_0)}{2\, D}\right] * \left\{ \begin{array}{l} -2 + \operatorname{erfc}\left(\frac{1}{2}\left(-\frac{(2\, D + k\, U)\sqrt{T}}{k\,\sqrt{D}} + \frac{\chi}{\sqrt{D\, T}}\right)\right) \\ + \exp\left[\frac{2\,\chi}{k} + \frac{U\,\chi}{D}\right] \\ * \operatorname{erfc}\left[\frac{1}{2}\left(\frac{(2\, D + k\, U)\sqrt{T}}{k\,\sqrt{D}} + \frac{\chi}{\sqrt{D\, T}}\right)\right. \end{array} \right\}. \tag{32}$$

5 Numerical solution

In the one-dimensional case $\left(D_{y_0} = 0, \text{then } \chi = x, D = D_{x_0}\right.$ and $\left.U = u_0\right)$. Hence equation (4) can be written as:

$$\frac{\partial C}{\partial T} = D\, \frac{\partial^2 C}{\partial x^2} - U\, \frac{\partial C}{\partial x} - \gamma_0\, C. \tag{33}$$

The explicit finite difference method (EFDM) is applied to solve equation (33). The central difference scheme was used for $\frac{\partial^2 C}{\partial x^2}$ and $\frac{\partial C}{\partial x}$. The forward difference scheme was used for $\frac{\partial C}{\partial T}$. With these substitutions, equation (33) can be written as:

$$C_{i,j+1} = r_1\, C_{i+1,j} + r_2\, C_{i,j} + r_3\, C_{i-1,j}\ , \tag{34}$$

where i and j refer to the discrete step lengths Δx and ΔT for the coordinate x and time T, respectively, and:

$r_1 = \frac{D\,\Delta T}{(\Delta x)^2} - \frac{U\,\Delta T}{2\,(\Delta x)}\ ,\quad r_2 = 1 - \frac{2\,D\,\Delta T}{(\Delta x)^2} - \frac{\gamma_0\,\Delta T}{2\,(\Delta x)}\ ,$

$r_3 = \frac{D\,\Delta T}{(\Delta x)^2} + \frac{U\,\Delta T}{2\,(\Delta x)}\ .$

Equation (34) represents a formula for $C_{i,j+1}$ at the $(i, j+1)^{th}$ mesh point in terms of known values along the j^{th} time row (Anderson [20]).

For case study one : the initial and boundary conditions (9-11) for one-dimensional case $(\chi = x)$, for point source concentration of uniform input condition $(\beta = 0)$, can be written in the finite difference form as:

$$\mathrm{C}_{i,0} = 0\ , \qquad x \geq 0,\ \ \mathrm{T} = 0, \tag{35}$$

$$C_{0,j} = C_0\,, \qquad x = 0,\ T > 0, \tag{36}$$

$$C_{N,j} = C_{N-1,j}\,, \quad x \to \infty,\ T \geq 0, \tag{37}$$

where $N = \frac{x_\infty}{\Delta x}$ is the grid dimension in the x direction and x_∞ is the distance from $x = 0$ in the direction x at which $\frac{\partial C}{\partial x} \to 0$.

For case study two : The initial and boundary conditions (15-17) for one-dimensional case $(\chi = x)$, can be written in the finite difference form as:

$$C_{i,0} = C_1\, e^{\frac{x_i}{k}}, \qquad x \geq 0\,,\ T = 0, \tag{38}$$

$$C_{0,j} = 0\,, \qquad x = 0\,,\ T > 0, \tag{39}$$

$$C_{N,j} = C_{N-1,j}\,, \quad x \to \infty,\ T \geq 0. \tag{40}$$

6 Results and discussions

The solution given by equation (27) is illustrated in figures (1) and (2) for the values $0 \leq x \leq 1$ m, $0 \leq y \leq 1$ m, $R = 1, n = 1\,(day^{-1}), C_0 = 0.1\,(\text{kg m}^{-3}), v_0 = 0.095\,(m\,day^{-1}), \text{D}_{\text{x}_0} = 1.05\,(\text{m}^2 day^{-1}), \text{D}_{\text{y}_0} = 0.105\,(\text{m}^2 day^{-1}), \gamma_0 = 0.4\,(day^{-1})$ and $\beta = 1$. Figure (1), shows the variation of $C(x,y,T)$ with time for the values $T = 1, 3($ day $)$ and $u_0 = 0.95\,(m\,day^{-1})$. From figure (1), it is clear that:

1- C increases as T increases at any point of the domain $0 \leq x \leq 1$ and $0 \leq y \leq 1$. This is due to the fact that at any point (x,y), the accumulation of the pollutant increases especially for small values of u_0 and v_0. This result agrees with that obtained by Yadav et al. [4] and Dimain et al. [21].

2- For any cross-section $y =$ constant, as x increases, C decreases. This result agrees with that obtained by Andallah and Khatun [22] and Yaday and Kumar [23].

3- As expected the maximum value of C is at the origin $(0,0)$, while the minimum value of C is at the point $(1,1)$. This result agrees with that obtained by Yadav and Kumar [23].

4- At the origin $(0,0)$, for $T = 1$, the value of C is 0.06, which is less than the value of $C_0 = 0.1$. This is due to the presence of the two positive terms in the left-hand side of equation (10). Hence the value of the term $-D\frac{\partial C}{\partial \chi} = 0.0392\,(\text{kg m}^{-2} day^{-1})$.

5- Numerical studies and figure (1) in general, show that the decrease of C in the range $0 \leq x \leq 1$ is much greater than the corresponding decrease in the range $0 \leq y \leq 1$. This is due to the fact that $u_0 \gg v_0$.

Figure (2), shows the variation of C with u_0 for the values $u_0 = 0.95, u_0 = 2.5\,(m\,day^{-1})$ and $T = 1(day)$. From figure (2) it is clear that: at any fixed point $(x,y), C$ increases as u_0 increases, this is due to the fact that the releasing water at the origin $(x = y = 0)$ is more polluted than the water of the river.

The solution given by equation (32) is illustrated in figures (3-5) for the values $0 \leq x \leq 10$ m, $0 \leq y \leq 10$ m, $t = 1.5\,(day), R = 1, n = 1(day^{-1}), C_1 = 0.2\,(\text{kg m}^{-3}), v_0 = 0.095\,(\text{m day}^{-1}), \text{D}_{\text{x}_0} = 1.05\,(\text{m}^2 day^{-1}), \text{D}_{\text{y}_0} = 0.105\,(\text{m}^2 day^{-1}), \gamma_0 = 0.4\,(day^{-1})$ and $k = 1\,(day^{-1})$. Let the cross-section area of the river at $\chi = 0$ be A, then the flux of the water (the volume of water crossing A every day) will be $Q = A\,U$.Consequently increasing the value of U means increasing the value of Q. Let the zone of clean water measured from barrage $(\chi = 0)$ in the direction of the flow be denoted by χ_0. Let the maximum value of C be denoted by C_m and the corresponding value of χ associated with C_m be χ_m. Figures (3-5) show the variation of $C(x,y,t)$ with u_0 for the values $\text{u}_0 = 0.95, 5$ and $10\,(m\,day^{-1})$ respectively. From figures (3-5) and numerical results, it is clear that: C_m and χ_0 increase as u_0 increases, hence figures (3-5) emphasize the fact that the zone of clean water measured form $\chi = 0$ in the direction of the flow increases as the quantity of the clean water Q entering the cross- section A increases. This result agrees with that obtained by Saleh et al. [10] and Hadhouda and Hassan [18].

The solution given by equation (32) in the one-dimensional case $(D_{y_0} = 0, \chi = x,\ \ D = D_{x_0}$ and $U = u_0)$ is illustrated in figure (6) for the values $\text{t} = 0.1, 0.2$ and $0.4\,(day), R = 1,\ 0 \leq x \leq 10$ m, $C_1 = 0.2\,(\text{kg m}^{-3}), \text{D} = 1.05\,(\text{m}^2 day^{-1}), \text{n} = 1\,(day^{-1}), \text{k} = 1\,(day^{-1}), U = 2\,(m\,day^{-1})$ and $\gamma_0 = 0.4\,(day^{-1})$. From figure (6), it is clear that: as t increases the value of C decreases along the river this is due to the fact that at any point the accumulation of the clean water increases.

Numerical solution of equation (34) using explicit finite difference method with the initial and boundary conditions (35-37) is given in figure (7), for $t = 0.02, 0.04$ and 0.06 (day). The input data are: $0 \leq x \leq 1$ m, $C_0 = 0.1\,(\text{kg m}^{-3}), \gamma_0 = 0.4\,(day^{-1}), u_0 = 0.95\,(\text{m day}^{-1}), n = 1\,(day^{-1}), D_{x_0} = 1.05\,(\text{m}^2 day^{-1})$ and $R = 1$. In the numerical calculations, the step lengths $\Delta x = 0.1(\text{ m})$ and $\Delta T = 0.002$ (day) have been used to achieve the stability of the finite difference scheme [24]. From figure (7), it is clear that the pollutant concentration increases as the time increases, this is due to the fact that the releasing water at the origin $(x = y = 0)$ is polluted. This result agrees with that obtained by Kumar et al. [3] and Dimian et al. [21]. To test the accuracy of the numerical solution, a comparison between the analytical solution given by equation (31) and the numerical solution given from equation (34), associated with initial and boundary conditions (35-37) is made. Also, it is clear that the explicit finite difference method is effective and accurate for solving the advection-diffusion equation for point source concentration of uniform input condition, which is especially important when arbitrary initial and boundary conditions are required.

Numerical solution of equation (34) using explicit finite difference method with the initial and boundary conditions (38-40) is given in figure (6), for the values t = 0.1, 0.2 and 0.4(*day*), $R = 1$, $0 \le x \le 10$ m, $C_1 = 0.2\left(\mathrm{kg\,m^{-3}}\right)$, $D = 1.05\left(\mathrm{m}^2 day^{-1}\right)$, $n = 1\left(day^{-1}\right)$, $k = 1\left(day^{-1}\right)$, $U = 2\left(m\, day^{-1}\right)$ and $\gamma_0 = 0.4\left(day^{-1}\right)$. In the numerical calculation, the step lengths $\Delta x = 1(m)$ and $\Delta T = 0.06(day)$, have been used to achieve the stability of the finite difference scheme [24]. To test the accuracy of the numerical solution, a comparison between the analytical solution given by equation (32) and the numerical solution given from equation (34), associated with initial and boundary conditions (38-40) is made and illustrated in figure (6). From figure (6) it is clear that the explicit finite difference method is effective and accurate for solving advection-diffusion equation for point source concentration of uniform input condition, which is especially important when arbitrary initial and boundary conditions are required.

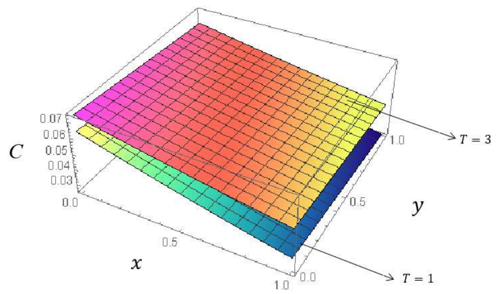

Fig. 1: The variation of $C(x,y,t)$ along the river with time for the values $T = 1, T = 3, R = 1$, $D_{x_0} = 1.05, u_0 = 0.95, D_{y_0} = 0.105, v_0 = 0.095$, $\gamma_0 = 0.4, \beta = 1, C_o = 0.1$ and $n = 1$.

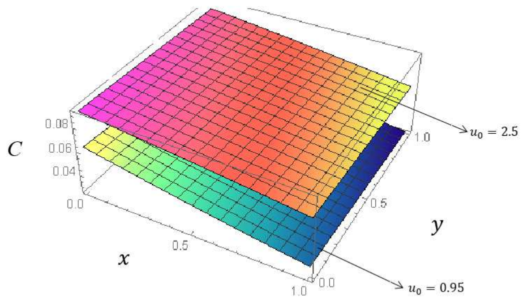

Fig. 2: The variation of $C(x,y,t)$ along the river with u_0 for the values $u_0 = 0.95, u_0 = 2.5, T = 1$, $D_{x_0} = 1.05, D_{y_0} = 0.105, v_0 = 0.095$, $\gamma_0 = 0.4, \beta = 1, C_0 = 0.1, R = 1$ and $n = 1$.

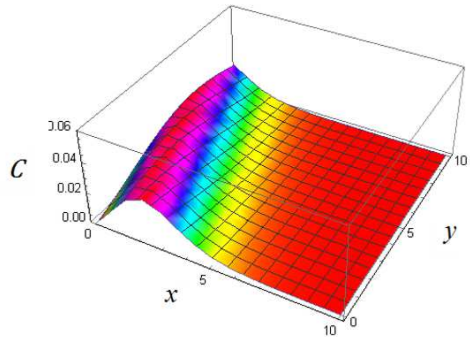

Fig. 3: The variation of $C(x,y,t)$ along the river for $u_0 = 0.95, R = 1, t = 1.5, D_{x_0} = 1.05, D_{y_0} = 0.105$, $v_0 = 0.095, \gamma_0 = 0.4, C_1 = 0.2, n = 1$ and $k = 1$.

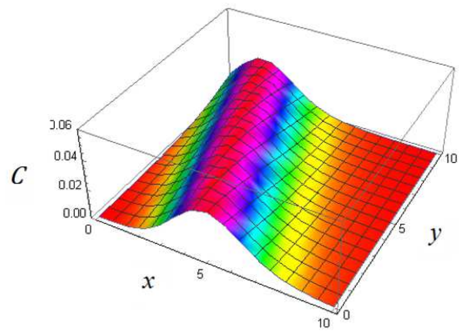

Fig. 4: The variation of $C(x,y,t)$ along the river for $u_0 = 5, R = 1, t = 1.5, D_{x_0} = 1.05, D_{y_0} = 0.105$, $v_0 = 0.095, \gamma_0 = 0.4, C_1 = 0.2, n = 1$ and $k = 1$.

7 Conclusions

Analytical and numerical solutions are obtained for the two-dimensional dispersion equation through the semi-infinite homogeneous river or porous medium. The concentration of increasing nature (mixed type or third type) is considered at the origin. Both the dispersion coefficients, the velocity components and, the first-order decay term are considered exponentially decreasing functions of time. We take two cases, the first case: initially, the river is solute free. Second case: at the origin, at any time, the source of pollution is removed. A comparison between analytical solution and numerical solution is made. The different effects of the parameters controlling the pollutant dispersion along the river at any time are studied separately with the help of figures. For a real situation, our simple model can provide decision support for planning restrictions to be imposed on farming and urban practices. We deduced that the zone of clean water measured from $\chi = 0$ in the direction of the flow increases as the quantity of the clean water Q entering the cross-section A increases.

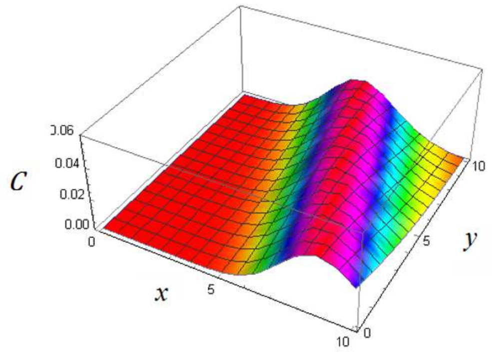

Fig. 5: The variation of $C(x,y,t)$ along the river for $u_0 = 10, R = 1, t = 1.5,\quad \mathrm{D}_{x_0} = 1.05, \mathrm{D}_{y_0} = 0.105, v_0 = 0.095, \gamma_0 = 0.4, C_1 = 0.2, n = 1$ and $k = 1$.

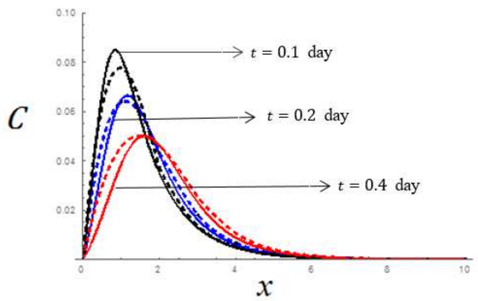

Fig. 6: Comparison between the analytical solution (equation (32)) and the numerical solution (equation (34)) for t $= 0.1, 0.2$ and $0.4, R = 1, C_1 = 0.2$, D $= 1.05$, $n = 1$, k $= 1$, $U = 2$ and $\gamma_0 = 0.4$. (lines represent analytical solution).

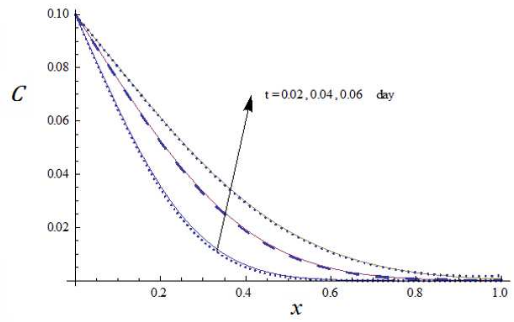

Fig. 7: Comparison between the analytical solution (equation((31))) and numerical solution (equation(34))) for $t = 0.02, 0.04$ and 0.06, $R = 1$, $\mathrm{D}_{x_0} = 1.05$, $u_0 = 2$, $\gamma_0 = 0.4$, $n = 1$ and $C_0 = 0.1$. (lines represent analytical solution)

Acknowledgements

This project was supported financially by the Academy of Scientific Research and Technology (ASRT), Egypt, Grant No (6366), (ASRT) is the second affiliation of this paper. We would like to thank Dr. Sarah Tawfik and Philopatir B. Raafat as they helped us in this research with their constructive scientific discussions with us.

References

[1] B. Pimpunchat, W.L. Sweatman, W. Triampo, G.C. Wake, A. Parshotam, Modelling river pollution and removal by aeration, In: MODSIM 2007, International Congress on Modelling and Simulation. Land, Water and Environmental Management: Integrated Systems for Sustainability,2431-2437,(2007).

[2] A. Kumar, A. Jaiswal and R.R. Yadav, One-dimensional Solute transport for uniform and varying pulse type input point source with temporally dependent coefficients in longitudinal semi-infinite homogeneous porous domain, International journal of mathematics and Scientific computing,1,56-66(2011).

[3] A. Kumar, D.K. Jaiswal and N. kumar, Analytical solutions of one-dimensional advection-diffusion equation with variable coefficients in a finite domain, Journal of earth system science,118 ,539-549(2009).

[4] R.R. Yadav, D.K. Jaiswal, H.K. Yadav and Gulrana, Temporally dependent dispersion through semi-infinite homogeneous porous media: An analytical solution, IJRRAS,6,158-164(2011).

[5] F.N. Ibrahim, M.F. Dimain and A.S. Wadi, Remediation of pollution in a river by unsteady aeration with arbitrary initial and boundary conditions, J. hydro.,525,793-797(2015).

[6] R.R. Yadav and L.K. Kumar, Solute transport for pulse type input point source along temporally and spatially dependent flow, Pollution,5,53-70(2019).

[7] R. Kumar, A. Chatterjee, M.K.Singh and V.P. Singh, Study of solute dispersion with source/sink impact in semi- infinite porous medium, Pollution,6,87-98(2020).

[8] S. Suryani, J. Kusuma, N. Ilyas, M. Bahri and M.I. Azis, A boundary element method solution to spatially variable coefficients diffusion convection equation of anisotropic media, Journal of Physics: Conference Series,1341(6),062018(2019).

[9] N. Manitcharoen and B. Pimpunchat, Analytical and numerical solutions of pollution concentration with uniformly and exponentially increasing forms of sources, J. of Applied Mathematics, 2020,1-9(2020).

[10] A. Saleh, F.N. Ibrahim and M.Kh. Hadhouda, Remediation of pollution in a river by releasing clean water, Inf. Sci. Lett.,11,127-133(2022).

[11] S.D. Kurakbayeva, Zh.R. Umarova, G.A. Besbayev and S.B. Botayeva1, Modeling and Development of a Software Complex for Calculating Pollution Propagation in a Water Bodies, Appl. Math. Inf. Sci.,9,1699-1708(2015).

[12] A. Allahem, S. Boulaaras and S. Ghanem, A New Mathematical Model and its Application in the Pollution of Air and Water: An Application of Virtual Experience in Qassim Province in Kingdom of Saudi Arabia, Appl. Math. Inf. Sci.,11,1615-1624(2017).

[13] V. Miguel, L. Motta Bittencurt Luciano M. and N. Valdes Juan E., On a Generalized Laplace Transform, Appl. Math. Inf. Sci.,15,667-675(2021).
[14] A.F. Scheidegger, General theory of dispersion in porous media, Journal of General Theory, 66,3273-3278(1961).
[15] E.H. Ebach, R. White, Mixing of fluids flowing through beds of packed Soils, Journal of Amer. Inst. Chem. Eng.,4, 161-164(1958).
[16] J. Crank, The mathematics of diffusion, Oxford Univ. Press, London UK, second edition(1975).
[17] D.k. Jaiswal, A. kumar, N. Kumar and R.R. Yadav, Analytical solutions for temporally and spatially dependent solute dispersion of pulse type input concentration in one dimensional semi-infinite media, Journal of Hydro-environment research,2, 254-263(2009).
[18] M.Kh. Hadhouda, Z.S. Hassan, Mathematical model for unsteady remediation of river pollution by aeration, Inf. Sci. Lett.,11,323-329(2022).
[19] R.W. Cleary and D.D. Adrian, Analytical solution of the convective-dispersive equation for cation adsorption in soils.,Soil sci. soc. Am. Proc.,37,197-199(1973).
[20] J.D.Jr. Anderson, Computational Fluid Dynamics, McGraw-Hill, New York,128-165,(1995).
[21] M.F. Dimain, A.S. Wadi and F.N. Ibrahim, The effect of added pollutant along a river on the pollutant concentration described by one-dimensional advection diffusion equation, International Journal of Engineering Science and Technology,5,1662-1671(2013).
[22] L.S. Andallah and M.R. Khatun, Numerical solution of advection-diffusion equation using finite difference schemes, Bangladesh J. Sci. Ind. Res.,55,15-22(2020).
[23] R.R. Yadav and L.K. Kumar, One-dimensional spatially dependent solute transport in semi-infinite porous media: an analytical solution, International Journal of Engineering, Science and Technology,9,20-27(2017).
[24] S. Savovic and A. Djordjevich, Finite difference solution of the one-dimensional advection- diffusion equation with variable coefficients in semi-infinite media, International Journal of Heat and Mass Transfer,55,4291-4294(2012).