Journal of Statistics Applications & Probability
*An International Journal*

http://dx.doi.org/10.18576/jsap/100319

# Cumulative Exposure Model under Frechet Distribution under Data Type II Censoring with Simulation Study

*Sh. A. M. Mubarak* *

The High Institute of Engineering and Technology, Ministry of Higher Education, El-Minia, Egypt

Received: 1 Feb. 2020, Revised: 12 Jun. 2020, Accepted: 16 Aug. 2020.
Published online: 1 Nov. 2021.

**Abstract:** In many problems of life-testing, the test process may require an unacceptable long time period for its completion, if the test is simply carried out under specified standard stress conditions. In such problem, It is generally possible to run the life test under higher stresses than the specified standard, in order to accelerate life-testing. In this article, a cumulative exposure model description under Frechet distribution is introduced. In addition, "the maximum likelihood method" is used to obtain the estimators of the unknown parameters from Frechet distribution, with respect to a cumulative exposure model under data Type II censoring. The performance of the findings in the article is showed by demonstrating some numerical illustrations through Monte Carlo simulation. Finally, tables to illustrate all the discussed inference methods are presented.

**Keywords:** Frechet distribution; Cumulative exposure model; Step-stress model; Type-II censoring data; Accelerated life testing ; Maximum likelihood estimation.

## 1 Introduction

Manufacturers are now under strong pressure to produce modern, highertechnology goods rapidly, while improving prod uctivity. This has stimulated the advancement of techniques such as simultaneous engineering and facilitated the broad use of engineered product and process improvement experiments.

The need for further upfront testing of the materials, components, and systems was increased by the criteria for greater r eliability. This is in linewith the new theory of quality for the development of highreliability products: achieving high rel iability by enhancing design and production processes; moving away from inspection (or screening) dependency to achi eve high reliability, see Meeker.For several decades, engineers in the manufacturing industry have used Accelerated Life Test (ALT) experiments. The object of ALT experiments is to quickly obtain reliable information. According to Bai et al. [2] and Nelson [3], one way of applying stress to the test is a step-stress scheme which allows the stress setting of a unit to be changed at pre-specified times or upon the occurrence of a fixed number of failures. This scheme is called step stress accelerated life test (SSALT). To implement the SSALT, first low stress is applied to all products. If a product endures the stress (does not fail) we apply a higher stress, if only one change of the stress level is done, then it is called a simple step-stress accelerated life test.

The aim of the SSALT experiment is to estimate the prediction of percentile life or reliability by choosing the optimal ti me to increase the amount of stress that results in the most precise estimate. The main aim is to choose the times to adjus t the level of stress in such a way that the variance of the estimator of the above parameters is minimised under the level of natural stress, see Khamis [4] and Fard[5]. The step-stress procedure was first introduced, with the cumulative exposure model, by Nelson [6]. Furthermore, Miller and Nelson [7] provided the optimum simple stress plans for the accelerated life testing.

The Frechet distribution was named after the French mathematician Maurice Frechet (1878–1973). It is also known as the Extreme Value Type II distribution. It has the cumulative distribution function specified by

$$F(t) = exp\left\{-\left(\frac{t}{\theta}\right)^{-\alpha}\right\} \;,\; t > 0, \alpha > 0 \;\; and \;\; \theta > 0. \quad (1)$$

The corresponding probability density function, is

*Corresponding author e-mail: Shaimaa.Mubarak@mhiet.edu.eg

$$f(t) = \left(\frac{\alpha}{\theta}\right)\left(\frac{t}{\theta}\right)^{-(\alpha+1)} exp\left\{-\left(\frac{t}{\theta}\right)^{-\alpha}\right\}, \tag{2}$$

where $\alpha$ is a shape parameter and $\theta$ is a scale parameter. In engineering applications shape parameter is usually greater than 2. Frechet [8] considered the Frechet distribution, which was proposed to model extreme events such as annually maximum one-day rainfalls and river discharges. This distribution has found wide application in extreme value theory. Further details about the Frechet distribution can be found in Kotz and Nadarajah [9]. Mead et al. [10] defined and studied a new generalization of the Frechet distribution called the beta exponential Frechet distribution. Abd- Elfattah, et al. [11] consider the estimation of the unknown parameters of the generalized Frechet distribution. Kamran et al. [12] develop the Bayesian estimators in the context of reference priors for the two-parameter Frechet distribution. Wagner Barreto-Souza et al. [13] derive Some Results for beta Frechet distribution. D. Gary Harlow [14] consider the applications of the Frechet distribution function. While Krishna [15] considered the applications of Marshall–Olkin Frechet distribution. Nadarajah et al. [16] considered the exponentiated Frechet distribution.

## 2 Cumulative Exposure Model Description

### *2.1 Notation*

ALT Acceerated life testing.
SST Step stess testing.
$S_i$ Stress levels.
pdf Probability density function.
cdf Cumulative distribution function.
$G(t)$ Cumulative exposure distribution function.
$g(t)$ Probability exposure density function.
$n$ Identical units under an initial stress level $s_0$.
$t_{i:n}$ The ordered failure times of the $i$ unit under test.
$\tau_1$ The time before which the stress level is changed from $S_0$ to $S_1$.
$\tau_i$ The time before which the stress level is changed from $S_{i-1}$ to $S_i$.
$t_{r:n}$ The time when the $r^{th}$ failure occurs the experiment is terminated.
$N_i$ Number of units that fail before time $\tau_i$ at stress level $S_{i-1}$.
$\alpha$ The shape parameter of the Frechet distribution
$\theta_i$ The scale parameters of the Frechet distribution.

### *2.2 Assumptions*

Supposing that the cumulative distribution function of the lifetime (of the items) under life testing is $F(x)$, and probability density function $f(x)$, the failure rate function of $F(x)$ will be

$$\lambda(x) = \frac{f(x)}{1-F(x)}, \tag{3}$$

and the cumulative failure rate function be

$$\Lambda(x) = \int_0^x \lambda(t)dt, \tag{4}$$

Then we get $F(x)$ as follows,

$$F(x) = 1 - exp(-\Lambda(x)), \tag{5}$$

Now, suppose that, for a particular m-step stress pattern, step i ($1 \le i \le m$) runs at stress $S_i$, that (starts) at time $\tau_{i-1}$ and ends at time $\tau_i$. The cumulative distribution function of time to failure for units run at the stress $S_i$ is denoted by $F_i(t)$. Let $G(t)$ be the cumulative distribution function of time to failure under a particular step-stress pattern. Then the cdf $G(t)$ of time to failure under a particular step-stress pattern can be expressed mathematically as follows:
The population cumulative fraction of specimen failing by time $t$ in first stress level $S_0$ is

$$G(t) = F_1(t), \quad 0 \le t < \tau_1 . \tag{6}$$

The population cumulative fraction of sample failing by time $t$ in second stress level $S_1$ is

$$G(t) = F_2[(t-\tau_1)+u_1], \quad \tau_1 \le t < \tau_2, \tag{7}$$

where $u_1$, the equivalent starting time, is the solution of

$$F_2(u_1) = F_1(\tau_1). \tag{8}$$

Similarly, in stress level $S_2$,

$$G(t) = F_3[(t-\tau_2)+u_2], \quad \tau_2 \le t < \tau_3, \tag{9}$$

with equivalent starting time $u_2$, being the solution of

$$F_3(u_2) = F_2[(\tau_2 - \tau_1) + u_1]. \tag{10}$$

In general, in i-stress level $S_i$,

$$G(t) = F_i[(t - \tau_{i-1}) + u_{i-1}], \quad \tau_{i-1} \le t < \tau_i, \tag{11}$$

with equivalent starting time $u_{i-1}$, being the solution of

$$F_i(u_{i-1}) = F_{i-1}[(\tau_{i-1} - \tau_{i-2}) + u_{i-2}]. \tag{12}$$

It is required to know the probability density function of the failure time of a unit under a particular m-step stress pattern to carry out statistical inference in order to determine the maximum likelihood estimators and the optimal $\tau_i$. Suppose that the pdf of time to failure for units run at a constant stress $S_i$ is $g_i(t)$.

The pdf for cumulative exposure model is

$$g(t) = \begin{cases} f_1(t) & ,0 \le t < \tau_1, \\ f_2[(t-\tau_1)+u_1] & ,\tau_1 \le t < \tau_2, \\ f_3[(t-\tau_2)+u_2] & ,\tau_2 \le t < \tau_3, \\ \cdot & \\ \cdot & \\ \cdot & \\ f_m[(t-\tau_{m-1})+u_{m-1}] & ,\tau_{m-1} \le t < \infty. \end{cases} \tag{13}$$

then, we can say,

$$g(t) = \frac{dG(t)}{dt}. \tag{14}$$

The failure rate function for the cumulative exposure model at step $i$ $(i = 1,2,\dots,m)$ is

$$\lambda_0(t) = \lambda_i[(t - \tau_{i-1}) + u_{i-1}], \quad \tau_{i-1} \le t < \tau_i, \tag{15}$$

or, we can say

$$\lambda_0(t) = \frac{g(t)}{1-G(t)}. \tag{16}$$

## 3 Cumulative Exposure Model Description under Frechet Distribution

In this section the cumulative exposure model is deduced under Frechet distribution (see, Miller and Nelson [7]). The failure rate function is

$$\lambda(t) = \frac{\left(\frac{\alpha}{\theta}\right)\left(\frac{t}{\theta}\right)^{-(\alpha+1)} exp\left\{-\left(\frac{t}{\theta}\right)^{-\alpha}\right\}}{1-exp\left\{-\left(\frac{t}{\theta}\right)^{-\alpha}\right\}}. \tag{17}$$

The cdf function of time to failure under a particular step-stress pattern can be expressed mathematically as follows:

The population cumulative fraction of specimen failing by time $t$ in stress level $S_0$ is

$$G(t) = F_1(t) = exp\left\{-\left(\frac{t}{\theta_1}\right)^{-\alpha}\right\}, \quad 0 \le t < \tau_1. \tag{18}$$

The population cumulative fraction of specimen failing by time $t$ in stress level $S_1$ is

$$G(t) = F_2[(t-\tau_1)+u_1] = exp\left\{-\left(\frac{t-\tau_1+u_1}{\theta_2}\right)^{-\alpha}\right\}, \quad \tau_1 \le t < \tau_2, \tag{19}$$

where $u_1$, the equivalent starting time, is the solution of

$$F_2(u_1) = F_1(\tau_1) \quad \Rightarrow \quad exp\left\{-\left(\frac{u_1}{\theta_2}\right)^{-\alpha}\right\} = exp\left\{-\left(\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}. \tag{20}$$

Then, we get

$$\left(\frac{u_1}{\theta_2}\right)^{-\alpha} = \left(\frac{\tau_1}{\theta_1}\right)^{-\alpha} \quad \Rightarrow \quad \frac{u_1}{\theta_2} = \frac{\tau_1}{\theta_1} \quad \Rightarrow \quad u_1 = \left(\frac{\theta_2}{\theta_1}\right)\tau_1. \tag{21}$$

Then, we can rewrite $G(t)$ in stress level $S_1$ as follows

$$G(t) = exp\left\{-\left(\frac{t-\tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}, \quad \tau_1 \le t < \tau_2, \tag{22}$$

The population cumulative fraction of specimen failing by time $t$ in stress level $S_2$ is

$$G(t) = F_3[(t-\tau_2)+u_2] =, \quad \tau_2 \le t < \tau_3, \tag{23}$$

with equivalent starting time $u_2$ being the solution of

$$F_3(u_2) = F_2[(\tau_2-\tau_1)+u_1] \;\Rightarrow exp\left\{-\left(\frac{u_2}{\theta_3}\right)^{-\alpha}\right\} = exp\left\{-\left(\frac{(\tau_2-\tau_1)+u_1}{\theta_2}\right)^{-\alpha}\right\}, \tag{24}$$

$$\frac{u_2}{\theta_3} = \frac{(\tau_2-\tau_1)+u_1}{\theta_2} \quad \Rightarrow \quad \frac{u_2}{\theta_3} = \frac{(\tau_2-\tau_1)+\left(\frac{\theta_2}{\theta_1}\right)\tau_1}{\theta_2}, \tag{25}$$

then, we get

$$u_2 = \left(\frac{(\tau_2-\tau_1)}{\theta_2} + \frac{\tau_1}{\theta_1}\right)\theta_3\,, \tag{26}$$

Then, we can rewrite $G(t)$ in stress level $S_2$ as follows:

$$G(t) = exp\left\{-\left(\frac{t-\tau_2}{\theta_3} + \frac{\tau_2-\tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}, \quad \tau_2 \le t < \tau_3. \tag{27}$$

The population cumulative fraction of specimen failing by time $t$ in stress level $S_3$ is

$$G(t) = F_4[(t-\tau_3) + u_3] = exp\left\{-\left(\frac{(t-\tau_3)+u_3}{\theta_4}\right)^{-\alpha}\right\}, \quad \tau_3 \le t < \tau_4, \tag{28}$$

with equivalent stating time $u_3$ being the solution of

$$F_4(u_3) = F_3[(\tau_3-\tau_2) + u_2], \tag{29}$$

$$exp\left\{-\left(\frac{u_3}{\theta_4}\right)^{-\alpha}\right\} = exp\left\{-\left(\frac{(\tau_3-\tau_2)+u_2}{\theta_3}\right)^{-\alpha}\right\} \Rightarrow \frac{u_3}{\theta_4} = \frac{(\tau_3-\tau_2)+u_2}{\theta_3}, \tag{30}$$

Where, $u_2 = \left(\frac{(\tau_2-\tau_1)}{\theta_2} + \frac{\tau_1}{\theta_1}\right)\theta_3$ , then we rewrite Eq. (30) as follows:

$$\frac{u_3}{\theta_4} = \frac{(\tau_3-\tau_2)+\left(\frac{(\tau_2-\tau_1)}{\theta_2}+\frac{\tau_1}{\theta_1}\right)\theta_3}{\theta_3}, \tag{31}$$

then, we get

$$u_3 = \left(\frac{\tau_3-\tau_2}{\theta_3} + \frac{\tau_2-\tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}\right)\theta_4, \tag{32}$$

Then, we can rewrite $G(t)$ in stress level $S_3$ as the follows:

$$G(t) = exp\left\{-\left(\frac{t-\tau_3}{\theta_4} + \frac{\tau_3-\tau_2}{\theta_3} + \frac{\tau_2-\tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}, \quad \tau_3 \le t < \tau_4. \tag{33}$$

The population cumulative fraction of specimen failing by time $t$ in stress level $S_m$ is
In general, in step $m$,

$$G(t) = F_{m+1}[(t-\tau_m) + u_m] = exp\left\{-\left(\frac{(t-\tau_m)+u_m}{\theta_{m+1}}\right)^{-\alpha}\right\}, \quad \tau_m \le t < \infty, \tag{34}$$

with equivalent stating time $u_m$ being the solution of

$$F_{m+1}(u_m) = F_m[(\tau_m-\tau_{m-1}) + u_{m-1}], \tag{35}$$

Then, we obtain that,

$$u_m = \left(\frac{\tau_m-\tau_{m-1}}{\theta_m} + \cdots + \frac{\tau_3-\tau_2}{\theta_3} + \frac{\tau_2-\tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}\right)\theta_{m+1}, \tag{36}$$

$$u_m = \left(\sum_{\substack{i=1\\ j=i+1}}^{m} \frac{\tau_i-\tau_{i-1}}{\theta_{j-1}}\right)\theta_{m+1}, \tag{37}$$

Then, we can rewrite $G(t)$ in stress level $S_m$ as the follows:

$$G(t) = exp\left\{-\left(\frac{t-\tau_m}{\theta_{m+1}} + \sum_{\substack{i=1\\ j=i+1}}^{m} \frac{\tau_i-\tau_{i-1}}{\theta_{j-1}}\right)^{-\alpha}\right\}, \quad \tau_m \le t < \infty, \tag{38}$$

Where, $i = 0, 1, 2, \dots, m, \ \ and \ \ j = i+1$,
Then, we can say, the cumulative exposure distribution function (cedf) for Frechet distribution as follows:

$$G(t) = \begin{cases} G_1(t) & ,0 \le t < \tau_1, \\ G_2(t) & ,\tau_1 \le t < \tau_2, \\ G_3(t) & ,\tau_2 \le t < \tau_3, \\ G_4(t) & ,\tau_3 \le t < \tau_4, \\ & . \\ & . \\ & . \\ G_{m+1}(t) & ,\tau_m \le t < \infty, \end{cases} \tag{39}$$

where

$$G_1(t) = exp\left\{-\left(\frac{t}{\theta_1}\right)^{-\alpha}\right\}, \quad 0 \le t < \tau_1, \tag{40}$$

$$G_2(t) = exp\left\{-\left(\frac{t-\tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}, \quad \tau_1 \le t < \tau_2, \tag{41}$$

$$G_3(t) = exp\left\{-\left(\frac{t-\tau_2}{\theta_3} + \frac{\tau_2-\tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}, \quad \tau_2 \le t < \tau_3. \tag{42}$$

$$G_4(t) = exp\left\{-\left(\frac{t-\tau_3}{\theta_4} + \frac{\tau_3-\tau_2}{\theta_3} + \frac{\tau_2-\tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}, \quad \tau_3 \le t < \tau_4, \tag{43}$$

In general,

$$G_{m+1}(t) = exp\left\{-\left(\frac{t-\tau_m}{\theta_{m+1}} + \sum_{\substack{i=1\\ j=i+1}}^{m} \frac{\tau_i-\tau_{i-1}}{\theta_{j-1}}\right)^{-\alpha}\right\}, \quad \tau_m \le t < \infty,, \tag{44}$$

Where $\tau_0 = 0$.
The simple step stress model is a special case from the cumulative exposure model so we can define, the cumulative

exposure distribution function $G(t)$ for Frechet distribution at two stress level as follows:

$$G(t) = \begin{cases} G_1(t) & ,0 \le t < \tau_1, \\ G_2(t) & ,\tau_1 \le t < \tau_2. \end{cases} \quad (45)$$

The corresponding probability expsure density function $g(t)$ is:

$$g(t) = \begin{cases} g_1(t) & ,0 \le t < \tau_1, \\ g_2(t) & ,\tau_1 \le t < \tau_2, \\ g_3(t) & ,\tau_2 \le t < \tau_3, \\ g_4(t) & ,\tau_3 \le t < \tau_4, \\ & . \\ & . \\ & . \\ g_{m+1}(t) & ,\tau_m \le t < \infty, \end{cases}, \quad (46)$$

Where

$$g_1(t) = \left(\frac{\alpha}{\theta_1}\right)\left(\frac{t}{\theta_1}\right)^{-(\alpha+1)} * exp\left\{-\left(\frac{t}{\theta_1}\right)^{-\alpha}\right\}, \quad (47)$$

$$g_2(t) = \left(\frac{\alpha}{\theta_2}\right)\left(\frac{t-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-(\alpha+1)} * exp\left\{-\left(\frac{t-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}, \quad (48)$$

$$g_3(t) = \left(\frac{\alpha}{\theta_3}\right)\left(\frac{t-\tau_2}{\theta_3}+\frac{\tau_2-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-(\alpha+1)} * exp\left\{-\left(\frac{t-\tau_2}{\theta_3}+\frac{\tau_2-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}, \quad (49)$$

$$g_4(t) = \left(\frac{\alpha}{\theta_4}\right)\left(\frac{t-\tau_3}{\theta_4}+\frac{\tau_3-\tau_2}{\theta_3}+\frac{\tau_2-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-(\alpha+1)} * exp\left\{-\left(\frac{t-\tau_3}{\theta_4}+\frac{\tau_3-\tau_2}{\theta_3}+\frac{\tau_2-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\},$$

In general,

$$g_{m+1}(t) = \left(\frac{\alpha}{\theta_{m+1}}\right)\left(\frac{t-\tau_m}{\theta_{m+1}}+\sum_{\substack{i=1\\j=i+1}}^{m}\frac{\tau_i-\tau_{i-1}}{\theta_{j-1}}\right)^{-(\alpha+1)} * exp\left\{-\left(\frac{t-\tau_m}{\theta_{m+1}}+\sum_{\substack{i=1\\j=i+1}}^{m}\frac{\tau_i-\tau_{i-1}}{\theta_{j-1}}\right)^{-\alpha}\right\},$$

(50) then, we can rewrite pedf for simple step stress as follows:

$$g(t) = \begin{cases} g_1(t) & ,0 \le t < \tau_1, \\ g_2(t) & ,\tau_1 \le t < \tau_2. \end{cases} \quad (51)$$

For a simple step-stress model, we have $n$ identical units under an initial stress level $S_0$. The stress level is changed to $S_1$ at time $\tau_1$.

The failure rate function $\lambda_0(t)$ under cumulative exposure model for Frechet distribution is

$$\lambda_0(t) = \begin{cases} \lambda_1(t) & ,0 \le t < \tau_1, \\ \lambda_2(t) & ,\tau_1 \le t < \tau_2, \\ \lambda_3(t) & ,\tau_2 \le t < \tau_3, \\ \lambda_4(t) & ,\tau_3 \le t < \tau_4, \\ & . \\ & . \\ & . \\ \lambda_{m+1}(t) & ,\tau_m \le t < \infty, \end{cases}, \quad (52)$$

where

$$\lambda_1(t) = \frac{\left(\frac{\alpha}{\theta_1}\right)\left(\frac{t}{\theta_1}\right)^{-(\alpha+1)} * exp\left\{-\left(\frac{t}{\theta_1}\right)^{-\alpha}\right\}}{1-exp\left\{-\left(\frac{t}{\theta}\right)^{-\alpha}\right\}}, \quad (53)$$

$$\lambda_2(t) = \frac{\left(\frac{\alpha}{\theta_2}\right)\left(\frac{t-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-(\alpha+1)} * exp\left\{-\left(\frac{t-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}}{1-exp\left\{-\left(\frac{t-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}}, \quad (54)$$

$$\lambda_3(t) = \frac{\left(\frac{\alpha}{\theta_3}\right)\left(\frac{t-\tau_2}{\theta_3}+\frac{\tau_2-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-(\alpha+1)} * exp\left\{-\left(\frac{t-\tau_2}{\theta_3}+\frac{\tau_2-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}}{1-exp\left\{-\left(\frac{t-\tau_2}{\theta_3}+\frac{\tau_2-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}}, \quad (55)$$

$$\lambda_4(t)=\frac{\left(\frac{\alpha}{\theta_4}\right)\left(\frac{t-\tau_3}{\theta_4}+\frac{\tau_3-\tau_2}{\theta_3}+\frac{\tau_2-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-(\alpha+1)}*exp\left\{-\left(\frac{t-\tau_3}{\theta_4}+\frac{\tau_3-\tau_2}{\theta_3}+\frac{\tau_2-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}}{1-exp\left\{-\left(\frac{t-\tau_3}{\theta_4}+\frac{\tau_3-\tau_2}{\theta_3}+\frac{\tau_2-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}}, \tag{56}$$

In general,

$$\lambda_{m+1}(t)=\frac{\left(\frac{\alpha}{\theta_{m+1}}\right)\left(\frac{t-\tau_m}{\theta_{m+1}}+\sum_{\substack{i=1\\j=i+1}}^{m}\frac{\tau_i-\tau_{i-1}}{\theta_{j-1}}\right)^{-(\alpha+1)}*exp\left\{-\left(\frac{t-\tau_m}{\theta_{m+1}}+\sum_{\substack{i=1\\j=i+1}}^{m}\frac{\tau_i-\tau_{i-1}}{\theta_{j-1}}\right)^{-\alpha}\right\}}{1-exp\left\{-\left(\frac{t-\tau_m}{\theta_{m+1}}+\sum_{\substack{i=1\\j=i+1}}^{m}\frac{\tau_i-\tau_{i-1}}{\theta_{j-1}}\right)^{-\alpha}\right\}}, \tag{57}$$

Then, we can rewrite pedf for simple step stress as follows

$$\lambda(t)=\begin{cases}\lambda_1(t) & ,0\le t<\tau_1,\\ \lambda_2(t) & ,\tau_1\le t<\tau_2.\end{cases} \tag{58}$$

## 4 Maximum Likelihood Estimation of Frechet Distribution Parameters under Simple Step Stress Model

The maximum likelihood method is used to obtain the estimators of the unknown parameters from Frechet distribution with respect to simple step stress model under data Type II censoring . The simple step stress model is special case from the cumulative exposure model. From the cumulative exposure distribution function by Eq. (45) and the corresponding probability exposure denisty function Eq. (51), we obtain the likelihood function of $\theta_1$ ,$\theta_2$ and $\alpha$ based on the Type-II censored sample as follows,

$$L(\theta_1,\theta_2,\alpha|t_{i:n})=\frac{n!}{(n-r)!}\left[\prod_{i=1}^{r}g(t_{i:n})\right]*\left[1-G(t_{r:n})\right]^{n-r},$$
$$,0<t_{1:n}<\ldots<t_{N_1:n}<\tau_1<t_{N_1+1}<\ldots<t_{r:n}<\infty, \tag{59}$$

where $r=N_1+N_2$ and $t$ is the vector of observed Type-II censored data. Therfore the likelihood function becomes,

1- If $N_1=r$ and $N_2=0$, the likelihood function becomes

$$L(\theta_1,\theta_2,\alpha|t_{i:n})=\frac{n!}{(n-r)!}\left[\prod_{i=1}^{r}g_1(t_{i:n})\right]\times\left[1-G_1(t_{r:n})\right]^{n-r} \qquad ,0<t_{1:n}<\ldots<t_{r:n}<\tau_1<\infty, \tag{60}$$

$$L(\theta_1,\theta_2,\alpha|t_{i:n})=\frac{n!}{(n-r)!}\left[\prod_{i=1}^{r}\left(\frac{\alpha}{\theta_1}\right)\left(\frac{t_{i:n}}{\theta_1}\right)^{-(\alpha+1)}\times exp\left\{-\left(\frac{t_{i:n}}{\theta_1}\right)^{-\alpha}\right\}\right]\times\left[1-exp\left\{-\left(\frac{t_{r:n}}{\theta_1}\right)^{-\alpha}\right\}\right]^{n-r},$$
$$,0<t_{1:n}<\ldots<t_{r:n}<\tau_1<\infty, \tag{61}$$

2- If $N_1=0$ and $N_2=r$, the likelihood function becomes

$$L(\theta_1,\theta_2,\alpha|t_{i:n})=\frac{n!}{(n-r)!}\left[\prod_{i=1}^{r}g_2(t_{i:n})\right]\times\left[1-G_2(t_{r:n})\right]^{n-r} \qquad ,\tau_1\le t_{1:n}<\cdots<t_{r:n}<\infty, \tag{62}$$

$$L(\theta_1,\theta_2,\alpha|t_{i:n})=\frac{n!}{(n-r)!}\left[\prod_{i=1}^{r}\left(\frac{\alpha}{\theta_2}\right)\left(\frac{t_{i:n}-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-(\alpha+1)}\times exp\left\{-\left(\frac{t_{i:n}-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}\right]$$
$$\times\left[1-exp\left\{-\left(\frac{t_{r:n}-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}\right]^{n-r} \qquad ,\tau_1\le t_{1:n}<\ldots<t_{r:n}<\infty. \tag{63}$$

3- If $1\le N_1\le r-1$, the likelihood function becomes

$$L(\theta_1,\theta_2,\alpha|t_{i:n})=\frac{n!}{(n-r)!}\left[\prod_{i=1}^{N_1}g_1(t_{i:n})\right]\times\left[\prod_{i=N_1+1}^{r}g_2(t_{i:n})\right]\times\left[1-G_2(t_{r:n})\right]^{n-r},$$
$$,t_{1:n}<\ldots<t_{N_1:n}<\tau_1\le t_{N_1+1:n}<\ldots<t_{r:n}<\infty, \tag{64}$$

$$L(\theta_1,\theta_2,\alpha|t_{i:n})=\frac{n!}{(n-r)!}\left[\prod_{i=1}^{N_1}\left(\frac{\alpha}{\theta_1}\right)\left(\frac{t_{i:n}}{\theta_1}\right)^{-(\alpha+1)}\times exp\left\{-\left(\frac{t_{i:n}}{\theta_1}\right)^{-\alpha}\right\}\right]\times\left[1-exp\left\{-\left(\frac{t_{r:n}-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}\right]^{n-r}$$
$$\times\left[\prod_{i=N_1+1}^{r}\left(\frac{\alpha}{\theta_2}\right)\left(\frac{t_{i:n}-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-(\alpha+1)}\times exp\left\{-\left(\frac{t_{i:n}-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}\right],$$
$$,t_{1:n}<\ldots<t_{N_1:n}<\tau_1\le t_{N_1+1:n}<\ldots<t_{r:n}<\infty. \tag{65}$$

The log-likelihood function may then be written as

$$\pounds=\ln L(\theta_1,\theta_2,\alpha|t_{i:n})=\ln\frac{n!}{(n-r)!}+N_1\ln\left(\frac{\alpha}{\theta_1}\right)+N_2\ln\left(\frac{\alpha}{\theta_2}\right)-(\alpha+1)\sum_{i=1}^{N_1}\ln\left(\frac{t_{i:n}}{\theta_1}\right)-\sum_{i=1}^{N_1}\left(\frac{t_{i:n}}{\theta_1}\right)^{-\alpha}$$
$$-\sum_{i=N_1+1}^{r}\left(\frac{t_{i:n}-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}-(\alpha+1)\sum_{i=N_1+1}^{r}\ln\left(\frac{t_{i:n}-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)$$
$$+(n-r)\ln\left[1-exp\left\{-\left(\frac{t_{r:n}-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\}\right],$$
$$,t_{1:n}<\ldots<t_{N_1:n}<\tau_1\le t_{N_1+1:n}<\ldots<t_{r:n}<\infty. \tag{66}$$

Then, we obtain the estimators of $\theta_1$, $\theta_2$ and $\alpha$ are estimated of by differentiating Eq. (66) with respect to $\theta_1$ ,$\theta_2$ and $\alpha$ respectively and equating to zero, in this case we have

$$\frac{\partial \pounds}{\partial \alpha}=\frac{N_1+N_2}{\alpha}-\sum_{i=1}^{N_1}\ln\left(\frac{t_{i:n}}{\theta_1}\right)+\sum_{i=1}^{N_1}\left(\frac{t_{i:n}}{\theta_1}\right)^{-\alpha}\ln\left(\frac{t_{i:n}}{\theta_1}\right)+\sum_{i=1}^{N_1}A_{i:n}^{-\alpha}\ln(A_{i:n})$$
$$-\sum_{i=N_1+1}^{r}\ln(A_{i:n})-(n-r)B_{r:n}^{-\alpha}\ln(B_{r:n})\left[\frac{e^{-B_{r:n}^{-\alpha}}}{1-e^{-B_{r:n}^{-\alpha}}}\right]=0\,, \quad (67)$$

$$\frac{\partial \pounds}{\partial \theta_1}=\frac{N_1\alpha}{\theta_1}-\sum_{i=1}^{N_1}\left[\left(\frac{\alpha t_{i:n}}{\theta_1^2}\right)\left(\frac{t_{i:n}}{\theta_1}\right)^{-(\alpha+1)}\right]+(\alpha+1)\sum_{i=N_1+1}^{r}\left[\frac{\tau_1}{\theta_1^2}\left(\frac{1}{A_{i:n}}\right)\right]-\sum_{i=N_1+1}^{r}\left[\left(\frac{\alpha\tau_1}{\theta_1^2}\right)\left(\frac{1}{A_{i:n}}\right)^{(\alpha+1)}\right]$$
$$+\left(\frac{\alpha(n-r)\tau_1}{\theta_1^2}\right)\left(\frac{1}{B_{r:n}}\right)^{(\alpha+1)}\left[\frac{e^{-B_{r:n}^{-\alpha}}}{1-e^{-B_{r:n}^{-\alpha}}}\right]=0\,, \quad (68)$$

$$\frac{\partial \pounds}{\partial \theta_2}=-\frac{N_2}{\theta_2}+(\alpha+1)\sum_{i=N_1+1}^{r}\left(\frac{t_{i:n}-\tau_1}{\theta_2^2}\right)\left(\frac{1}{A_{i:n}}\right)-\sum_{i=N_1+1}^{r}\left[\frac{\alpha t_{i:n}}{\theta_2^2}\left(\frac{1}{A_{i:n}}\right)^{(\alpha+1)}-\frac{\alpha\tau_1}{\theta_2^2}\left(\frac{1}{A_{i:n}}\right)^{(\alpha+1)}\right]$$
$$+\left(\frac{\alpha(n-r)(t_{r:n}-\tau_1)}{\theta_2^2}\right)\left(\frac{1}{B_{r:n}}\right)^{(\alpha+1)}\left[\frac{e^{-B_{r:n}^{-\alpha}}}{1-e^{-B_{r:n}^{-\alpha}}}\right]=0\,. \quad (69)$$

Where $\underline{\theta}=(\theta_1,\theta_2,\alpha)$, $\hat{\underline{\theta}}=(\hat{\theta}_1,\hat{\theta}_2,\hat{\alpha})$, $A_{i:n}=\frac{t_{i:n}-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}$ , and $B_{r:n}=\frac{t_{r:n}-\tau_1}{\theta_2}+\frac{\tau_1}{\theta_1}$.

Since the closed form solution to nonlinear equations Eq. (67- 69) is very hard to obtain. Therefore, we use the MATLAB program to solve the previous nonlinear equations simultaneously to obtain $\hat{\theta}_1,\hat{\theta}_2$ and $\hat{\alpha}$, see Tables (a-1, b-1).

## 5 Inverse Fisher information Matrix and Confidence Interval for Frechet Distribution under Type-II Censored

The asymptotic variance and covariance matrix of maximum likelihood estimates are given by the elements of the inverse of Fisher information matrix as follows,

$$I_{ij}(\underline{\theta})\cong E\left\{-\frac{\partial^2\pounds}{\partial\theta_i\partial\theta_j}\right\}. \quad (70)$$

Unfortunately, the exact mathematical expressions for the previous expectation are very difficult to obtain. Therefore, Fisher information matrix is given by

$$I_{ij}(\underline{\theta})\cong\left\{-\frac{\partial^2\pounds}{\partial\theta_i\partial\theta_j}\right\}, \quad (71)$$

which is obtained by approximating the expectation on operation $E$ and replaced $\theta_1,\theta_2$ and $\alpha$ with $\hat{\theta}_1,\hat{\theta}_2$ and $\hat{\alpha}$ respectively, Cohen [17]. The asymptotic variance covariance matrix $F$ for the maximum likelihood estimates can be written as follows,

$$F^{-1}=\begin{pmatrix}-\frac{\partial^2\pounds}{\partial\alpha^2} & -\frac{\partial^2\pounds}{\partial\alpha\partial\theta_1} & -\frac{\partial^2\pounds}{\partial\alpha\partial\theta_2}\\ -\frac{\partial^2\pounds}{\partial\theta_1\partial\alpha} & -\frac{\partial^2\pounds}{\partial\theta_1^2} & -\frac{\partial^2\pounds}{\partial\theta_1\partial\theta_2}\\ -\frac{\partial^2\pounds}{\partial\theta_2\partial\alpha} & -\frac{\partial^2\pounds}{\partial\theta_2\partial\theta_1} & -\frac{\partial^2\pounds}{\partial\theta_2^2}\end{pmatrix}^{-1}. \quad (72)$$

The elements of matrix $F$ are given as the following:

$$\frac{\partial^2\pounds}{\partial\theta_1^2}=-\frac{\alpha N_1}{\theta_1^2}-\left(\frac{(n-r)\alpha^2\tau_1^2}{\theta_1^4}\right)\left[\frac{e^{-2B_{r:n}^{-\alpha}}B_{r:n}^{-2-2\alpha}}{\left(1-e^{-B_{r:n}^{-\alpha}}\right)^2}\right]+\left(\frac{(n-r)\alpha\tau_1^2}{\theta_1^4}\right)\left[\frac{e^{-B_{r:n}^{-\alpha}}B_{r:n}^{-2-\alpha}}{\left(1-e^{-B_{r:n}^{-\alpha}}\right)}\right]$$
$$-\left(\frac{2(n-r)\alpha\tau_1}{\theta_1^3}\right)\left[\frac{e^{-B_{r:n}^{-\alpha}}B_{r:n}^{-1-\alpha}}{\left(1-e^{-B_{r:n}^{-\alpha}}\right)}\right]-\sum_{i=1}^{N_1}\left[-\frac{2\alpha t_{i:n}}{\theta_1^3}\left(\frac{t_{i:n}}{\theta_1}\right)^{-1-\alpha}+\left(\frac{\alpha t_{i:n}}{\theta_1^4}\right)\left((1+\alpha)t_{i:n}\left(\frac{t_{i:n}}{\theta_1}\right)^{-2-\alpha}\right)\right]$$
$$-(1+\alpha)\sum_{i=1+N_1}^{r}\left[\frac{2\tau_1}{\theta_1^3A_{i:n}}-\frac{\tau_1^2}{\theta_1^4A_{i:n}^2}\right]-\sum_{i=1+N_1}^{r}\left[-\frac{2\alpha\tau_1}{\theta_1^3}A_{i:n}^{-1-\alpha}+\left(\frac{\alpha\tau_1}{\theta_1^4}\right)(1+\alpha)\tau_1A_{i:n}^{-2-\alpha}\right], \quad (73)$$

$$\frac{\partial^2\pounds}{\partial\theta_2^2}=\frac{N_2}{\theta_2^2}-\frac{(n-r)\alpha^2(t_{r:n}-\tau_1)^2}{\theta_2^4}\left[\frac{e^{-2B_{r:n}^{-\alpha}}B_{r:n}^{-2-2\alpha}}{\left(1-e^{-B_{r:n}^{-\alpha}}\right)^2}\right]+\frac{(n-r)\alpha(t_{r:n}-\tau_1)^2}{\theta_2^4}\left[\frac{e^{-B_{r:n}^{-\alpha}}B_{r:n}^{-2-\alpha}}{\left(1-e^{-B_{r:n}^{-\alpha}}\right)}\right]$$
$$-\frac{2(n-r)\alpha(t_{r:n}-\tau_1)}{\theta_2^3}\left[\frac{e^{-B_{r:n}^{-\alpha}}B_{r:n}^{-1-\alpha}}{\left(1-e^{-B_{r:n}^{-\alpha}}\right)}\right]-(1+\alpha)\sum_{i=1+N_1}^{r}\left[\frac{2(t_{i:n}-\tau_1)}{\theta_2^3A_{i:n}}-\left(\frac{(t_{i:n}-\tau_1)^2}{\theta_2^4A_{i:n}^2}\right)\right]$$
$$-\sum_{i=1+N_1}^{r}\left[\frac{2\alpha(\tau_1-t_{i:n})A_{i:n}^{-1-\alpha}}{\theta_2^3}+\frac{\alpha t_{i:n}(1+\alpha)(t_{i:n}-\tau_1)A_{i:n}^{-2-\alpha}}{\theta_2^4}-\frac{\alpha\tau_1(1+\alpha)(t_{i:n}-\tau_1)A_{i:n}^{-2-\alpha}}{\theta_2^4}\right], \quad (74)$$

$$\frac{\partial^2\pounds}{\partial\alpha^2}=-\frac{N_1}{\alpha^2}-\frac{N_2}{\alpha^2}-\frac{e^{-2B_{r:n}^{-\alpha}}(n-r)\ln[B_{r:n}]^2B_{r:n}^{-2\alpha}}{\left(1-e^{-(B_{r:n})^{-\alpha}}\right)^2}-\frac{e^{-B_{r:n}^{-\alpha}}(n-r)\ln[B_{r:n}]^2B_{r:n}^{-2\alpha}}{1-e^{-(B_{r:n})^{-\alpha}}}$$
$$+\frac{e^{-B_{r:n}^{-\alpha}}(n-r)\ln[B_{r:n}]^2B_{r:n}^{-\alpha}}{1-e^{-B_{r:n}^{-\alpha}}}-\sum_{i=1}^{N}\ln\left[\frac{t_{i:n}}{\theta_1}\right]^2\left(\frac{t_{i:n}}{\theta_1}\right)^{-\alpha}-\sum_{i=1+N_1}^{r}\ln[A_{i:n}]^2A_{i:n}^{-\alpha}, \quad (75)$$

$$\frac{\partial^2 \pounds}{\partial\theta_1\partial\theta_2} = -\frac{(n-r)\alpha^2(t_{r:n}-\tau_1)\tau_1}{\theta_1^2\theta_2^2}\left[\frac{e^{-2B_{r:n}^{-\alpha}}B_{r:n}^{-2-2\alpha}}{\left(1-e^{-B_{r:n}^{-\alpha}}\right)^2}\right] - \frac{(n-r)\alpha^2(t_{r:n}-\tau_1)\tau_1}{\theta_1^2\theta_2^2}\left[\frac{e^{-B_{r:n}^{-\alpha}}B_{r:n}^{-2-2\alpha}}{1-B_{r:n}^{-\alpha}}\right]$$
$$+\frac{(n-r)(1+\alpha)\alpha(t_{r:n}-\tau_1)\tau_1}{\theta_1^2\theta_2^2}\left[\frac{e^{-B_{r:n}^{-\alpha}}B_{r:n}^{-2-\alpha}}{1-e^{-B_{r:n}^{-\alpha}}}\right] + (1+\alpha)\sum_{i=1+N_1}^{r}\frac{\tau_1}{\theta_1^2}\left(\frac{t_{i:n}-\tau_1}{\theta_2^2(A_{i:n})^2}\right)$$
$$-\sum_{i=1+N_1}^{r}\frac{\alpha\tau_1}{\theta_1^2}\left[\frac{(t_{i:n}-\tau_1)(A_{i:n})^{-2-\alpha}}{\theta_2^2} + \frac{\alpha(t_{i:n}-\tau_1)(A_{i:n})^{-2-\alpha}}{\theta_2^2}\right], \tag{76}$$

$$\frac{\partial^2 \pounds}{\partial\theta_1\partial\alpha} = \frac{N_1}{\theta_1} + \frac{(n-r)\alpha\tau_1}{\theta_1^2}\left[\frac{e^{-2(B_{r:n})^{-\alpha}}\ln(B_{r:n})(B_{r:n})^{-1-2\alpha}}{\left(1-e^{-(B_{r:n})^{-\alpha}}\right)^2}\right] + \frac{(n-r)\alpha\tau_1}{\theta_1^2}\left[\frac{e^{-(B_{r:n})^{-\alpha}}\ln(B_{r:n})(B_{r:n})^{-1-2\alpha}}{1-e^{-(B_{r:n})^{-\alpha}}}\right]$$
$$+\frac{(n-r)\tau_1}{\theta_1^2}\left[\frac{e^{-(B_{r:n})^{-\alpha}}(B_{r:n})^{-1-\alpha}}{\left(1-e^{-(B_{r:n})^{-\alpha}}\right)}\right] - \frac{(n-r)\alpha\tau_1}{\theta_1^2}\left[\frac{e^{-(B_{r:n})^{-\alpha}}\ln(B_{r:n})(B_{r:n})^{-1-\alpha}}{1-e^{-(B_{r:n})^{-\alpha}}}\right]$$
$$-\sum_{i=1}^{N}\left[-\frac{\alpha\ln\left[\frac{t_i}{\theta_1}\right]t_i\left(\frac{t_i}{\theta_1}\right)^{-1-\alpha}}{\theta_1^2} + \frac{\left(\frac{t_i}{\theta_1}\right)^{-\alpha}}{\theta_1}\right] + \sum_{i=1+N_1}^{r}\frac{\tau_1}{\theta_1^2(A_{i:n})} - \sum_{i=1+N_1}^{r}\left[\frac{(1-\alpha\ln(A_{i:n}))\tau_1(A_{i:n})^{-1-\alpha}}{\theta_1^2}\right], \tag{77}$$

$$\frac{\partial^2 \pounds}{\partial\theta_2\partial\alpha} = \frac{(n-r)\alpha(t_r-\tau_1)}{\theta_2^2}\left[\frac{e^{-2(B_{r:n})^{-\alpha}}\ln[B_{r:n}](B_{r:n})^{-1-2\alpha}}{\left(1-e^{-(B_{r:n})^{-\alpha}}\right)^2} + \frac{e^{-(B_{r:n})^{-\alpha}}\ln[B_{r:n}](B_{r:n})^{-1-2\alpha}}{\left(1-e^{-(B_{r:n})^{-\alpha}}\right)}\right]$$
$$+\frac{e^{-(B_{r:n})^{-\alpha}}(n-r)(t_r-\tau_1)(B_{r:n})^{-1-\alpha}}{\left(1-e^{-(B_{r:n})^{-\alpha}}\right)\theta_2^2} - \frac{e^{-(B_{r:n})^{-\alpha}}(n-r)\alpha\ln[B_{r:n}](t_r-\tau_1)(B_{r:n})^{-1-\alpha}}{\left(1-e^{-(B_{r:n})^{-\alpha}}\right)\theta_2^2}$$
$$+\sum_{i=1+N_1}^{r}\left[\frac{t_i-\tau_1}{\theta_2^2(A_{i:n})}\right] - \sum_{i=1+N_1}^{r}\left[\frac{(t_i-\tau_1)(A_{i:n})^{-1-\alpha}}{\theta_2^2} - \ln[A_{i:n}]\left(\frac{\alpha(t_i-\tau_1)(A_{i:n})^{-1-\alpha}}{\theta_2^2}\right)\right], \tag{78}$$

where $A_{i:n} = \frac{t_{i:n}-\tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}$ , and $B_{r:n} = \frac{t_{r:n}-\tau_1}{\theta_2} + \frac{\tau_1}{\theta_1}$.

Consequently, the maximum likelihood estimators $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\alpha}$ for $\theta_1, \theta_2$ and $\alpha$ have an asymptotic variance covariance matrix defined by inverting the Fisher information matrix $F$ and substituting $\underline{\hat{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha})$, for $\underline{\theta} = (\theta_1, \theta_2, \alpha)$, see Tables (a-2, b-2).

Now, we construct confidence intervals (CIs) for the unknown parameters $\theta_1, \theta_2$ and $\alpha$ for larger sample sizes under data Type II-censored. Let the unknown parameters $\theta$ and assume that $L_\theta = L_\theta(t_{1:n}, \ldots, t_{n:n})$ and $U_\theta = U_\theta(t_{1:n}, \ldots, t_{n:n})$ are functions of the sample data $t_{1:n}, \ldots, t_{n:n}$, then a confidence interval for a population parameter $\theta$ is given by

$$P[L_\theta \le \theta \le U_\theta] = \delta, \tag{79}$$

where $L_\theta$ and $U_\theta$ are lower and upper confidence limits which enclosed $\theta$ with probability $\delta$. The interval $[L_\theta, U_\theta]$ is called a two sided $100\delta\%$ confidence interval for $\theta$. For large sample size, the maximum likelihood estimations, under appropriate regularity conditions, are consistent and asymptotically normally distributed. Therefore, the two-sided approximate $100\,\delta\%$ confidence limits for the maximum likelihood estimate $\hat{\theta}$ of a population parameters $\theta$ can be constructed, such that,

$$P\left[-z \le \frac{\hat{\theta}-\theta}{\sigma(\hat{\theta})} \le z\right] = \delta, \tag{80}$$

where, $z$ is the $\left[\frac{100(1-\delta)}{2}\right]$ standard normal percentile. Therefore, the two sided approximate $100\,\delta\%$ confidence limits for a population parameter $\theta$ can be obtained, such that,

$$P\left[\hat{\theta} - z\sigma(\hat{\theta}) \le \theta \le \hat{\theta} + z\sigma(\hat{\theta})\right] \cong \delta \tag{81}$$

Then , the approximate confidence limits for $\theta_1, \theta_2$ and $\alpha$ will be constructed using Eq. (81) with confidence levels 95% and 99%, see Tables (a-3, b-3).

## 6 Sumulation Study

In this section, we describe the algorithm to obtain the Type-II censored sample, and the estimators $\hat{\theta}_1, \hat{\theta}_2, \hat{\alpha}$. In addition, the absolute relative bias (RABias), mean square error (MSE) and relative error (RE).

***Step 1:*** Given $\tau_1$, and initial values for the parameters $\theta_1, \theta_2$ and $\alpha$.

***Step 2:*** Based on $n, r, \tau_1, \theta_1, \theta_2$ and $\alpha$, we generate a random sample of size $n$ from Uniform (0,1) distribution, and obtain the order statistics $U_{1:n}, \ldots, U_{n:n}$ such that,

$$\{U_{1:n} < \ldots < U_{n:n}\}. \tag{82}$$

***Step 3:***Find $N_1$ such that

$$U_{N_1:n} \le exp\left\{-\left(\frac{\tau_1}{\theta_1}\right)^{-\alpha}\right\} < U_{N_1+1:n}. \tag{83}$$

Solving the above inequality to get the failure times of units from unit 1 to unit $N_1$. The failure times under Frechet distribution of units is given as follows:

for $1 \le i \le N_1$ we set

$$t_{i:n} = \frac{\theta_1}{[\ln(U_{i:n})^{-1}]^{\frac{1}{\alpha}}}, \tag{84}$$

for $1 \le j \le r - N_1$, we set

$$t_{N_1+j:n} = \frac{\theta_2}{\left[\ln(U_{N_1+j:n})^{-1}\right]^{\frac{1}{\alpha}}} - \frac{\theta_2}{\theta_1}\tau_1 + \tau_1 . \tag{85}$$

***Step 4:*** Based on $n, r, N_1, \tau_1$, and order observations

$$\{t_{1:n}, \dots, t_{N_1:n}, t_{N_1+1:n}, \dots, t_{r:n}\}, \tag{86}$$

where $r = N_1 + N_2$

- **A1:** If $N_1 = r$ and $N_2 = 0$, then, we solve the system of nonlinear equations get from Eq. (61) to obtain the unknown parameters $\hat{\theta}_1$ and $\hat{\alpha}$. **Go to step 5**.
- **A2:** If $N_1 = 0$ and $N_2 = r$, then, we solve the system of nonlinear equations get from Eq. (63) to obtain the unknown parameters $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\alpha}$. **Go to step 5**.
- **A3:** If $1 \le N_1 \le r - 1$, then, we solve the system of nonlinear equations (65) to obtain the unknown parameters $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\alpha}$. **Go to step 5**.

***Step 5:*** Repeat steps (2) to (4) $H$ times and arrange all $\hat{\theta}_1, \hat{\theta}_2$ and $\hat{\alpha}$ in ascending order to obtain

$$\left\{\hat{\theta}_1^{[1]}, \dots, \hat{\theta}_1^{[H]}\right\}, \quad \left\{\hat{\theta}_2^{[1]}, \dots, \hat{\theta}_2^{[H]}\right\} \quad and \quad \left\{\hat{\alpha}^{[1]}, \dots, \hat{\alpha}^{[H]}\right\} \tag{87}$$

Then, we get the estimators as follows,

$$\hat{\theta}_1^{\,*} = \frac{1}{H}\sum_{h=1}^{H} \hat{\theta}_1^{[h]}, \quad \hat{\theta}_2^{\,*} = \frac{1}{H}\sum_{h=1}^{H} \hat{\theta}_2^{[h]} \quad and \quad \hat{\alpha}^* = \frac{1}{H}\sum_{h=1}^{H} \hat{\alpha}^{[h]} \tag{88}$$

Substituting the values of parameters $\hat{\theta}_1^{\,*}$, $\hat{\theta}_2^{\,*}$ and $\hat{\alpha}^*$, the absolute relative bias (RABias), mean square error (MSE) and relative error (RE) are obtained. Furthermore, the asymptotic variance and covariance matrix and two-sided confidence intervals of the estimators are obtained.

Table a-1: The RABais, MSE and RE of the parameter based on Type II censoring from Frechet distribution ($r \cong 75\%n$ and $\tau_1 = 10$).

| $(n)$ | | $\theta$ | $\hat{\theta}$ | RABias | MSE | RE |
|---|---|---|---|---|---|---|
| (25) | Case (2.5, 2 ,1.5) $r \cong 75\%n$ and $\tau_1 = 10$ | $\theta_1$ | 2.986038 | 0.194415 | 0.255767 | 0.202294 |
| | | $\theta_2$ | 2.082963 | 0.041481 | 0.059205 | 0.121661 |
| | | $\alpha$ | 1.235096 | 0.176603 | 0.073779 | 0.181082 |
| (50) | | $\theta_1$ | 2.947478 | 0.447478 | 0.219545 | 0.187423 |
| | | $\theta_2$ | 2.022144 | 0.011072 | 0.041355 | 0.101680 |
| | | $\alpha$ | 1.167002 | 0.221998 | 0.111863 | 0.222973 |
| (75) | | $\theta_1$ | 2.935701 | 0.174280 | 0.200745 | 0.179218 |
| | | $\theta_2$ | 1.935100 | 0.032450 | 0.084205 | 0.145090 |
| | | $\alpha$ | 1.186251 | 0.209166 | 0.099839 | 0.210649 |
| (100) | | $\theta_1$ | 2.927132 | 0.170853 | 0.190418 | 0.174548 |
| | | $\theta_2$ | 1.878951 | 0.060525 | 0.116857 | 0.170922 |
| | | $\alpha$ | 1.187907 | 0.208062 | 0.100805 | 0.211665 |
| (125) | | $\theta_1$ | 2.905252 | 0.162101 | 0.174537 | 0.167110 |
| | | $\theta_2$ | 1.858857 | 0.070571 | 0.066891 | 0.129316 |
| | | $\alpha$ | 1.183475 | 0.211017 | 0.101222 | 0.212102 |

Table a-2: Asymptotic variance and covariance of estimates based on Type II censoring data from Frechet distribution ($r \cong 75\%n$ and $\tau_1 = 10$).

| $(n)$ | | $\hat{\theta}$ | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\hat{\alpha}$ |
|---|---|---|---|---|---|
| (25) | Case (2.5, 2 ,1.5) $r \cong 75\%n$ and $\tau_1 = 10$ | $\hat{\theta}_1$ | -0.1103 | -0.0325 | -0.2377 |
| | | $\hat{\theta}_2$ | - | 0.2941 | 0.1110 |
| | | $\hat{\alpha}$ | - | - | 4.4042 |
| (50) | | $\hat{\theta}_1$ | -0.0357 | -0.0314 | -0.0929 |
| | | $\hat{\theta}_2$ | - | 0.1804 | 0.0403 |
| | | $\hat{\alpha}$ | - | - | 3.2681 |
| (75) | | $\hat{\theta}_1$ | -0.0271 | -0.0196 | -0.0648 |
| | | $\hat{\theta}_2$ | - | 0.1124 | 0.0260 |
| | | $\hat{\alpha}$ | - | - | 3.0735 |
| (100) | | $\hat{\theta}_1$ | -0.0202 | -0.0144 | -0.0455 |
| | | $\hat{\theta}_2$ | - | 0.0838 | 0.0189 |

| | | | | | |
|---|---|---|---|---|---|
| | | $\hat{\alpha}$ | - | - | 2.9475 |
| (125) | | $\hat{\theta}_1$ | -0.0146 | -0.0121 | -0.0342 |
| | | $\hat{\theta}_2$ | - | 0.0676 | 0.0147 |
| | | $\hat{\alpha}$ | - | - | 2.0121 |

Table a-3: Confidence bounds of the estimates at confidence level based on Type II censoring data 95% and 99% ($r \cong 75\%n$ and $\tau_1 = 10$).

| $(n)$ | | $\theta$ | (L, U) bound 1 | (L, U) bound 2 |
|---|---|---|---|---|
| (25) | Case (2.5, 2 ,1.5) $r \cong 75\%n$ and $\tau_1 = 10$ | $\theta_1$ | (2.712100, 3.259976) | (2.625446, 3.346630) |
| | | $\theta_2$ | (1.634630, 2.531295) | (1.492810, 2.673115) |
| | | $\alpha$ | (1.117414, 1.352778) | (1.080188, 1.390004) |
| (50) | | $\theta_1$ | (2.675122, 3.219833) | (2.588969, 3.305986) |
| | | $\theta_2$ | (1.625930, 2.418358) | (1.500597, 2.543691) |
| | | $\alpha$ | (1.105790, 1.228215) | (1.086427, 1.247578) |
| (75) | | $\theta_1$ | (2.730982, 3.140419) | (2.666224, 3.205177) |
| | | $\theta_2$ | (1.380753, 2.489448) | (1.205399, 2.664802) |
| | | $\alpha$ | (1.112894, 1.259607) | (1.089690, 1.282812) |
| (100) | | $\theta_1$ | (2.752093, 3.102172) | (2.696723, 3.157542) |
| | | $\theta_2$ | (1.252351, 2.505550) | (1.054141, 2.703760) |
| | | $\alpha$ | (1.073572, 1.302243) | (1.037404, 1.338410) |
| (125) | | $\theta_1$ | (2.706261, 3.104244) | (2.643315, 3.167190) |
| | | $\theta_2$ | (1.434077, 2.283637) | (1.299708, 2.418006) |
| | | $\alpha$ | (1.120470, 1.246479) | (1.100540, 1.266409) |

Table b-1: The RABais, MSE and RE of the parameter based on Type II censoring from Frechet distribution ($r \cong 95\%n$ and $\tau_1 = 10$).

| $(n)$ | | $\theta$ | $\hat{\theta}$ | RABias | MSE | RE |
|---|---|---|---|---|---|---|
| (25) | Case (2.5, 2 ,1.5) $r \cong 95\%n$ and $\tau_1 = 10$ | $\theta_1$ | 2.423428 | 0.030629 | 0.082577 | 0.114945 |
| | | $\theta_2$ | 2.103334 | 0.051667 | 0.078293 | 0.139905 |
| | | $\alpha$ | 1.520450 | 0.013633 | 0.034554 | 0.123925 |
| (50) | | $\theta_1$ | 2.454549 | 0.018180 | 0.049582 | 0.089068 |
| | | $\theta_2$ | 1.913356 | 0.043322 | 0.089118 | 0.149263 |
| | | $\alpha$ | 1.530352 | 0.020234 | 0.020592 | 0.095665 |
| (75) | | $\theta_1$ | 2.470489 | 0.011804 | 0.041845 | 0.081824 |
| | | $\theta_2$ | 2.020295 | 0.010148 | 0.080728 | 0.142063 |
| | | $\alpha$ | 1.467296 | 0.021803 | 0.024430 | 0.104200 |
| (100) | | $\theta_1$ | 2.576434 | 0.030574 | 0.025293 | 0.063615 |
| | | $\theta_2$ | 1.881799 | 0.059101 | 0.079928 | 0.141358 |
| | | $\alpha$ | 1.500267 | 0.000178 | 0.017880 | 0.089143 |
| (125) | | $\theta_1$ | 2.508792 | 0.003517 | 0.018813 | 0.054864 |
| | | $\theta_2$ | 1.891526 | 0.054237 | 0.082183 | 0.143338 |
| | | $\alpha$ | 1.545032 | 0.030021 | 0.016620 | 0.085945 |

Table b-2: Asymptotic variance and covariance of estimates based on Type II censoring data from Frechet distribution ($r \cong 95\%n$ and $\tau_1 = 10$).

| $(n)$ | | $\hat{\theta}$ | $\hat{\theta}_1$ | $\hat{\theta}_2$ | $\hat{\alpha}$ |
|---|---|---|---|---|---|
| (25) | Case (2.5, 2 ,1.5) $r \cong 95\%n$ and $\tau_1 = 10$ | $\hat{\theta}_1$ | -0.2316 | -0.0009 | -0.4292 |
| | | $\hat{\theta}_2$ | - | 0.1238 | 0.0882 |
| | | $\hat{\alpha}$ | - | - | 2.4199 |
| (50) | | $\hat{\theta}_1$ | -0.1169 | -0.0014 | -0.2110 |
| | | $\hat{\theta}_2$ | - | 0.0609 | 0.0405 |
| | | $\hat{\alpha}$ | - | - | 0.8929 |
| (75) | | $\hat{\theta}_1$ | -0.0592 | -0.0069 | -0.1218 |
| | | $\hat{\theta}_2$ | - | 0.0473 | 0.0217 |
| | | $\hat{\alpha}$ | - | - | 0.7144 |
| (100) | | $\hat{\theta}_1$ | -0.0656 | -0.0013 | -0.1202 |
| | | $\hat{\theta}_2$ | - | 0.0341 | 0.0197 |
| | | $\hat{\alpha}$ | - | - | 0.4352 |
| (125) | | $\hat{\theta}_1$ | -0.0552 | -0.0006 | -0.1022 |
| | | $\hat{\theta}_2$ | - | 0.0242 | 0.0157 |
| | | $\hat{\alpha}$ | - | - | 0.4115 |

Table b-3: Confidence bounds of the estimates at confidence level based on Type II censoring data 95% and 99% ($r \cong 95\%n$ and $\tau_1 = 10$).

| $(n)$ | | $\theta$ | (L, U) bound 1 | (L, U) bound 2 |
|---|---|---|---|---|
| (25) | Case (2.5, 2 ,1.5) $r \cong 95\%n$ and $\tau_1 = 10$ | $\theta_1$ | (1.880560, 2.966295) | (1.708837, 3.138019) |
| | | $\theta_2$ | (1.593677, 2.612992) | (1.432458, 2.774210) |
| | | $\alpha$ | (1.158322, 1.882577) | (1.043772, 1.997128) |
| (50) | | $\theta_1$ | (2.027306, 2.881792) | (1.892158, 3.016940) |
| | | $\theta_2$ | (1.353430, 2.473282) | (1.176311, 2.650401) |
| | | $\alpha$ | (1.255459, 1.805244) | (1.168504, 1.892200) |
| (75) | | $\theta_1$ | (2.073745, 2.867234) | (1.948244, 2.992735) |
| | | $\theta_2$ | (1.464829, 2.575761) | (1.289121, 2.751470) |
| | | $\alpha$ | (1.167728, 1.766863) | (1.072967, 1.861625) |
| (100) | | $\theta_1$ | (2.303082, 2.849787) | (2.216613, 2.936255) |
| | | $\theta_2$ | (1.378431, 2.385166) | (1.219202, 2.544395) |
| | | $\alpha$ | (1.238187, 1.762347) | (1.155284, 1.845250) |
| (125) | | $\theta_1$ | (2.240512, 2.777073) | (2.155648, 2.861937) |
| | | $\theta_2$ | (1.371419, 2.411634) | (1.206895, 2.576158) |
| | | $\alpha$ | (1.308269, 1.781794) | (1.233375, 1.856689) |

## 6 Conclusion

The cumulative exposure model under Frechet distribution is very important in the industry world because this model is used to accelerate failure and this helps to measure validity. This article present derivation of: the probability density function, the cumulative distribution function and the failure rate function under the cumulative exposure model for Frechet distribution. From this; the simple step-stress model under the same distribution was concluded. It was concluded that; the cumulative exposure model of Frechet distribution is used for all types of data and also for a mixture of data. In this paper, a simple step-stress model was consided, with two stress levels from Frechet distribution when the data are Type II censored. Also procedure was proposed for constructing the estimator $\hat{\theta}$ beside RABies, MSE and RE of it (see tables (a-1, b-1). Moreover, the variance and covariance matrix, see tables (a-2, b-2), and the confidence interval (Cls) for $\theta$ at 95% and 99%, see tables (a-3, b-3). Since, we chose the value of parameters $\theta$ to be $(2.5, 2, 1.5)$, for different $(n)$, we take $r = 75\%\, n$ and $95\%\, n$ the best results were $r = 95\%n$. The parameters $\theta$ increases the estimates have smaller MSE and RE, as the sample size increases.

### Conflict of Interest

The authors declare that they have no conflict of interest.

## References

[1] Meeker, W.Q. and Escobar, L.A., "Statistical methods for reliability data", A Wiley-Inter science publication. (1998).

[2] Bai, D. S.; Kim, M. S. and Lee, S.H., "Optimum simple step-stress accelerated life tests with censoring", IEEE Transactions on Reliability., **38**, 528–532(1989).

[3] Nelson, W.B., "Accelerated Life Testing: Statistical Models, Data Analysis and Test Plans", John Wiley and sons, New York. (1990).

[4] Khamis, I.H., "Optimum M-step design with K stress variables", Communications in Statistics: Simulation & Computation., **26(**4**),** 1301–1313(1997).

[5] Fard, N. and Li, C., "Optimal simple step stress accelerated life test design for reliability prediction", Journal of Statistical Planning and Inference.,**139,** 1799–1808(2009).

[6] Nelson, W.B., "Accelerated life testing-step-stress models and data analyses", IEEE Transactions on Reliability., **29**, 103-108(1980).

[7] Miller, R. and Nelson, W.B., "Optimum simple step-stress plans for accelerated life testing", IEEE Transactions on Reliability., **32**, 59-65(1983).

[8] Frechet, M., "Sur la Loi des Erreurs d'Observation", Bulletin de la Societe Mathematique de Moscou., **33**, 5-8(1924).

[9] Kotz, S. and Nadarajah, S., “Extreme Value Distributions: Theory and Applications”, Imperial College Press, London., (2000).

[10] M. E. Mead, Ahmed Z. A_fy, G. G., Hamedani, Indranil Ghosh, “The beta exponential Frechet distribution with applications”, Austrian Journal of Statistics.,**46**, 41-63(2017).

[11] Abd- Elfattah, A.M and Omima, A.M, “Estimation of the unknown parameters of the generalized Frechet distribution”, Journal of Applied Sciences Research., **5(10),** 1398-1408(2009).

[12] Kamran Abbas and Yincal Tang, “Analysis of Frechet distribution using reference priors”, Communications in Statistics—Theory and Methods., **44**, 2945–2956(2015).

[13] Wagner Barreto-Souza, Gaussm. Cordeiro, and Alexandre B. Simas, “Some Results for beta Frechet distribution”, Communications in Statistics—Theory and Methods., **40**, 798–811(2011).

[14] D. Gary Harlow, “Applications of the Frechet distribution function”, Int. J. of Materials & Product Technology ., **17**, Nos 5/6(2002).

[15] E. Krishna, K. K. Jose, and Miroslav M. Ristić, “Applications of Marshall–Olkin Fréchet Distribution”, Communication in Statistics- Simulation and Computation., **42(1)**,76-89(2012).

[16] S.Nadarajah and S.Kotz, “The exponentiated Fréchet distribution”, Inter stat Electron., J, pp.1–7(2003).