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SPC Techniques using M/M/c Queueing Model

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Abstract: In a manufacturing process, the lots are submitted for inspection and the items come in a queue discipline. The queueing system may be characterized by complex input process, service time distribution, number of quality inspectors, buffer size or a waiting place which is limited. These queues help to maintain a manufacturing process in a regular interval. The factors that can be controlled include; the arrival process, control process and number of manufacturers, number of system places and number of lots to be entering and leaving the system after control. The objective of this paper is to introduce a control chart using $M/M/c$ queueing discipline to study and monitor the production system.

Keywords: Statistical Process Control, Queueing disciplines, $M/M/c$ queueing.

1 Introduction

The need for statistical process control (SPC) arises from variability occurs in manufacturing processes and the fact that no two manufactured items are exactly alike. When the random causes are alone present, we say that the process is "in control". On the other hand, when assignable causes are active, the process is "out of control".

During the production process, the lots are submitted for inspection and the items come in a queue discipline. The queueing system may be characterized by complex input process, service time distribution, number of quality inspectors, and buffer size or a waiting place which is limited.

In this paper, queue discipline and queueing model are introduced to monitor the production process. One can examine every item in a queue, including the arrival process, control process (service), number of controllers, number of system places and number of lots to be inspected. In statistical process control, the group of lots arriving for inspection, wait for inspection in a queue and then leave the system after inspections.

Many authors have contributed towards queueing theory, but few applied it into quality control practice. Therefore, an attempt has been made to develop a quality and efficiency domain for a production process with an $M/M/c$ queueing system. This is a generalization of the traditional queue, provided that the inspection is provided independently by c inspectors operating. This change is natural, because if the average arrival rate is higher than the inspection rate, the system will not be stable, which will cause the increasing of the number of inspectors. In this case, however, one can have parallel inspectors and are interested in distributing the first inspection completion.

The main advantage of combining the queueing theory and quality control concept is to minimize the number of inspectors and the waiting time of the products during the process of inspection and hence the decision on the lots; whether to accept it or reject it. This is a novel method which combines the queueing and statistical process control theory to control the product's quality.

The objective of this paper is to make a control chart using $M/M/c$ queueing discipline in order to identify the assignable causes if it is present in the system.

The paper is organized as follows: Section 2 provides the $M/M/c$ model description, Section 3 focuses on the results relating to mean and variance for $M/M/c$ Queueing model in Process Control. Control limits for the $M/M/c$ Queueing Model are provided in section 4. A numerical illustration is given in Section 5, and the conclusions are provided in Section 6.

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2 Description of M/M/c Queueing Process

In such queueing system, the rate equality principle is applied, that is, the rate of entering lots and the rate of leaving lots at a particular state are equal for every state. Thus, the arrivals occur from an infinite source by following a Poisson process with parameter λ, where the interarrival times and service times are considered as an independent exponentially distributed with mean $(1/\lambda)$ and μ respectively with c inspectors. Therefore, the queue discipline is First Come First Serve (FCFS), with the utilization factor $\rho = (\lambda/\mu)$.

Even though several literatures exist on control charts, the following conditions are assumed and the results are presented in the next section.

i. The production process should be stable and steady

ii. The production process should be able to deliver products in lots

iii. The control is done by the statistical quality controller

iv. Lots arrive according to the M/M/c queueing discipline.

Assuming that the arrival occurs from an infinite source in accordance with Poisson process with rate λ, and parallel inspection channels c $(1 \leq c \leq \infty)$ which having the manufacturing times are independently and exponentially distributed with parameter; say, μ. If there are n units in the manufacturing system, and are less than c, then, all n channels are busy and the completion interval between two consecutive inspections, that is, being the minimum of n independently and identically exponential random variables each with rate μ, is again exponential with rate $n\mu$. If in the manufacturing system the units $n \geq c$, then all the inspectors are busy and the interval between two consecutive inspection completions is also exponential with rate $c\mu$. As a result, a birth and death process is created, providing a constant arrival with rate λ and state- dependent inspection rate

$$\mu_n = \begin{cases} n\,\mu; & n = 0, 1, \ldots, c \\ c\mu; & n = c+1, c+2, \ldots \end{cases}$$

The queue discipline is FCFS (First Come First Serve), then the utilization factor becomes $\rho = \left(\lambda / c\mu \right)$.

Assume that steady state exists, then using rate up equals to rate down principle, that is

$$\lambda_n \, p_n = \mu_{n+1} \, p_{n+1}\,; \qquad n = 0, 1, \ldots$$

$$p_n = \prod_{k=0}^{n-1} \frac{\lambda_k}{\mu_{k+1}} p_0\,; \qquad n = 1, 2, \ldots \tag{1}$$

By the normalizing condition

$$\sum_{n=0}^{\infty} p_n = 1,$$

one gets,

$$p_0 = \frac{1}{1 + \sum_{n=1}^{\infty} \prod_{k=0}^{n-1} \frac{\lambda_k}{\mu_{k+1}}} \tag{2}$$

Substituting the values of λ_n and μ_n in equations (1) and (2), one can obtain

$$p_n = \begin{cases} \dfrac{\lambda\ \lambda \cdots \lambda}{(\mu)(2\mu)\cdots(n\mu)} p_0 = \dfrac{\lambda^n}{n!\,\mu^n}\, p_0; & n = 1, 2, \cdots, c \\[2ex] \dfrac{(\lambda)\ (\lambda)\cdots(to\ n\ \ factors)}{(\mu)\ (2\mu)\cdots(c\,\mu)\cdots(to[n-c]\,factors)} p_0 = \dfrac{\lambda^n}{c^{n-c}\ c!\,\mu^n}\, p_0; & n = c, c+1, c+2, \cdots \end{cases} \tag{3}$$

To find p0, the $\sum_{n=c}^{\infty} \left(\lambda/c\mu\right)^n$ must be convergent, which will imply that $\rho < 1$. Therefore,

$$p_0 = \left[\frac{\left(\lambda/\mu\right)^c}{c!(1-\rho)} + \sum_{n=0}^{c-1} \frac{\left(\lambda/\mu\right)^n}{n!} \right]^{-1}$$

Since the arrivals of products follow a Poisson law the distribution of the system at arrival instants equals to the distribution at random moments, hence the probability that an arriving of product has to wait in the manufacturing system is

$$P(N \ge c) = \sum_{n=c}^{\infty} p_n = \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!\,(1-\rho)}$$

This probability is frequently used in different practical problems, for example in telephone systems, call centers. It is also a very famous formula which is referred to as Erlang's C formula and it is denoted $C\left(c, \frac{\lambda}{\mu}\right)$.

3 Mean and Variance of M/M/c Queueing in Process Control

Let *N*be the number of lots in the process and *W* be the waiting time in the process. Then, in the steady state condition, one can have the mean,

$$E[N] = E[B] + E[Q]$$

where$E[Q]$ is the expected queue size, and can be obtained using equation (3) as

$$E[Q] = \sum_{n=c}^{\infty} (n-c)\, p_n = \sum_{n=c}^{\infty} (n-c)\, \frac{\left(\frac{\lambda}{\mu}\right)^n}{c!\, c^{n-c}}\ p_0$$

After simplifications,

$$E[Q] = \left[\frac{\left(\frac{\lambda}{\mu}\right)^c \rho}{c!(1-\rho)^2} \right] p_0 \tag{4}$$

$$Var[Q] = \frac{\rho \left[\dfrac{\left(\lambda/\mu\right)^c}{c!(1-\rho)} \right] \left\{ 1 + \rho - \rho \left[\dfrac{\left(\lambda/\mu\right)^c}{c!(1-\rho)} \right] \right\}}{(1-\rho)^2}$$

And $E[B]$ is the expected number of busy inspectors, which can also be obtained using equation (3) as

$$E[B] = \sum_{n=0}^{c-1} n\, p_n + \sum_{n=c}^{\infty} c\, p_n = \frac{\lambda}{\mu} \left[\sum_{n=1}^{c-1} \frac{\left(\frac{\lambda}{\mu}\right)^{n-1}}{(n-1)!} + \frac{\left(\frac{\lambda}{\mu}\right)^{c-1}}{(c-1)!(1-\rho)} \right] p_0 = \frac{\lambda}{\mu} \left[\sum_{n=1}^{c-1} \frac{\left(\frac{\lambda}{\mu}\right)^m}{m!} + \frac{\left(\frac{\lambda}{\mu}\right)^c}{c!(1-\rho)} \right] p_0$$

After simplifications, $E[B] = \left(\lambda/\mu\right)$ (5)

From equation (4) and (5), $E[N]$ can be expressed as

$$E[N]=\left(\frac{\lambda}{\mu}\right)+\left[\frac{\left(\frac{\lambda}{\mu}\right)^{c}\rho}{c!(1-\rho)^{2}}\right]p_0$$

Then, $$Var[N]=\frac{\rho\left[\frac{\left(\lambda/\mu\right)^{c}}{c!(1-\rho)}\right]\left\{1+\rho-\rho\left[\frac{\left(\lambda/\mu\right)^{c}}{c!(1-\rho)}\right]\right\}}{(1-\rho)^{2}}+\rho\left[1+\frac{\left(\lambda/\mu\right)^{c}}{c!(1-\rho)}\right]$$

From Little's formula [5]; $L=\lambda W$, One can obtain, after simplifications, the average waiting time in the manufacturing system and is given by

$$E(W)=\frac{E(N)}{\lambda}=\frac{\left(\frac{\lambda}{\mu}\right)+\left[\frac{\left(\frac{\lambda}{\mu}\right)^{c}\rho}{c!(1-\rho)^{2}}\right]p_0}{\lambda}$$

Therefore, the distribution of the number of items in the queue, that is, N_Q can be found from number of items N and more details can be found in [2]. Thus,

$$E[N_Q]=E[N]-\rho=\left\{\left(\frac{\lambda}{\mu}\right)+\left[\frac{\left(\frac{\lambda}{\mu}\right)^{c}\rho}{c!(1-\rho)^{2}}\right]p_0\right\}-\rho \text{ and}$$

$$E[W_Q]=\frac{E[N_Q]}{\lambda}=\frac{\rho(1+\rho^{2})}{\lambda(1-\rho^{2})}$$

4 Control limits for the M/M/c Queueing System

A typical control chart has control limit sets and values such that if the process is in control nearly all points will be within the upper control limit (UCL) and the lower control limit (LCL). Therefore, the control limits for M/M/c queueing model are given by:

$$UCL=\left(\frac{\lambda}{\mu}\right)+\left[\frac{\left(\frac{\lambda}{\mu}\right)^{c}\rho}{c!(1-\rho)^{2}}\right]p_0+3\sqrt{\frac{\rho\left[\frac{\left(\lambda/\mu\right)^{c}}{c!(1-\rho)}\right]\left\{1+\rho-\rho\left[\frac{\left(\lambda/\mu\right)^{c}}{c!(1-\rho)}\right]\right\}}{(1-\rho)^{2}}+\rho\left[1+\frac{\left(\lambda/\mu\right)^{c}}{c!(1-\rho)}\right]}$$

$$Centre\,line\,(CL)=\left(\frac{\lambda}{\mu}\right)+\left[\frac{\left(\frac{\lambda}{\mu}\right)^{c}\rho}{c!(1-\rho)^{2}}\right]p_0$$

$$LCL=\left(\frac{\lambda}{\mu}\right)+\left[\frac{\left(\frac{\lambda}{\mu}\right)^{c}\rho}{c!(1-\rho)^{2}}\right]p_0-3\sqrt{\frac{\rho\left[\frac{\left(\lambda/\mu\right)^{c}}{c!(1-\rho)}\right]\left\{1+\rho-\rho\left[\frac{\left(\lambda/\mu\right)^{c}}{c!(1-\rho)}\right]\right\}}{(1-\rho)^{2}}+\rho\left[1+\frac{\left(\lambda/\mu\right)^{c}}{c!(1-\rho)}\right]}$$

where, $\rho=\left(\lambda/c\mu\right)$

5 Numerical Illustrations

A numerical study has been carried out to analyze the performance of the queue in the manufacturing system and control limits with reference to the different inter arrival times and production times with parameters λ and μ, respectively.

The following Table 1 gives the control limits, the averages of waiting time $E[W]$ in the manufacturing system, the expected size of items in the queue $E[N_Q]$ and expected waiting time in the queue $E[W_Q]$ with different timing of inter arrival and production times λ and μ, respectively.

Table 1: Control limits for M/M/c queueing model.

λ	μ	c	ρ	$E(N)$	LCL	UCL	$E(W_Q)$	$E(W)$	$E(N_Q)$
1	3.5	3	0.095	0.29	0	3.88	0.00034	0.29	0.00034
	4.5		0.095	0.22	0	3.81	0.00034	0.22	0.00034
	5.5		0.06	0.18	0	3.55	0.00006	0.18	0.00006
	6.5		0.05	0.15	0	3.45	0.00003	0.15	0.00003
	7.5		0.04	0.13	0	3.37	0.00002	0.13	0.00002
	8.5		0.04	0.12	0	3.36	0.00001	0.12	0.00001
	9.5		0.035	0.105	0	3.32	0.00001	0.105	0.00001
	10		0.03	0.1	0	3.28	0.00001	0.1	0.00001
	20		0.02	0.05	0	3.17	0	0.05	0
	3.5	5	0.06	0.29	0	3.66	0	0.29	0
	4.5		0.04	0.22	0	3.46	0	0.22	0
	5.5		0.04	0.18	0	3.42	0	0.18	0
	6.5		0.03	0.15	0	3.33	0	0.15	0
	7.5		0.02	0.13	0	3.25	0	0.12	0
	8.5		0.02	0.12	0	3.24	0	0.12	0
	9.5		0.02	0.105	0	3.23	0	0.105	0
	10		0.02	0.1	0	3.22	0	0.1	0
	20		0.01	0.05	0	3.11	0	0.05	0
3	3.5	7	0.12	0.86	0	4.61	0	0.29	0
	4.5		0.095	0.67	0	4.26	0	0.22	0
	5.5		0.08	0.545	0	4.04	0	0.18	0
	6.5		0.07	0.4	0	3.83	0	0.15	0
	7.5		0.06	0.4	0	3.77	0	0.13	0
	8.5		0.05	0.35	0	3.65	0	0.12	0
	9.5		0.045	0.32	0	3.59	0	0.105	0
	10		0.04	0.3	0	3.54	0	0.1	0
	20		0.02	0.15	0	3.27	0	0.05	0
	5.5	3	0.6	2.38	0	7.32	0.05	0.24	0.56
	6.5		0.51	1.8	0	8.37	0.03	0.18	0.26
	7.5		0.4	1.5	0	7.36	0.01	0.15	0.14
	8.5		0.39	1.26	0	7.13	0.01	0.13	0.09
	9.5		0.35	1.11	0	6.64	0.005	0.11	0.05

10	10		0.33	1.04	0	6.40	0.004	0.1	0.04
	20		0.17	0.5	0	4.60	0.0003	0.05	0.003
	3.5	5	0.01	3.13	0.07	6.19	0.03	0.003	0.27
	4.5		0.44	2.29	0	8.48	0.01	0.23	0.07
	5.5		0.36	1.84	0	7.49	0.002	0.18	0.02
	6.5		0.31	1.55	0	6.77	0.001	0.15	0.01
	7.5		0.27	1.34	0	6.22	0.0004	0.13	0.004
	8.5		0.23	1.18	0	5.73	0.0002	0.12	0.002
	9.5		0.21	1.05	0	5.45	0.0001	0.1	0.001
	10		0.2	1	0	5.32	0.0001	0.1	0.001
	20		0.1	0.5	0	4.12	0	0.05	0.00002
	3.5	7	0.41	2.88	0	8.92	0.002	0.29	0.02
	4.5		0.32	2.23	0	7.55	0.0004	0.22	0.004
	5.5		0.26	1.82	0	6.62	0.0001	0.18	0.0001
	6.5		0.22	1.54	0	6.01	0.00003	0.15	0.0003
	7.5		0.19	1.33	0	5.58	0.00001	0.13	0.0001
	8.5		0.17	1.18	0	5.28	0	0.12	0.00005
	9.5		0.15	1.05	0	5.01	0	0.1	0.00002
	10		0.14	1	0	4.89	0	0.1	0.00001
	20		0.07	0.5	0	3.93	0	0.05	0

From the above table, one can observe that

(i) Whenever the arrival rate of products increases with a fixed production rate, the control limits deviate from the center line, the expected waiting time in the manufacturing process decreases, as well as the expected number and waiting of items in the queue also decrease

(ii) Whenever the arrival rate of products is fixed with an increasing of production time, the deviation of control limits decreases, and this can lead to out-of-control situations. Similarly, the expected waiting time in the manufacturing process decreases, as well as the expected number and waiting of items in the queue also decrease.

(iii) Whenever the number of inspector increases, the expected number and waiting of items in the queue also decrease and tends to zero.

Further, one can see that,

(i) The UCL and LCL should be symmetric around the Center Line. However, in the case where CL is less than 9, the LCL becomes negative. In this case, the LCL can be rounded to zero and, more details can be found in [3] and [6].

(ii) $\rho < 1$ is a necessary condition for the existence of steady states, more details can be found in [1] and [4].

The below Fig. 1 shows the control chart using M/M/c queueing model for special case λ = 10, μ = 3.5 with five manufacturing machines.

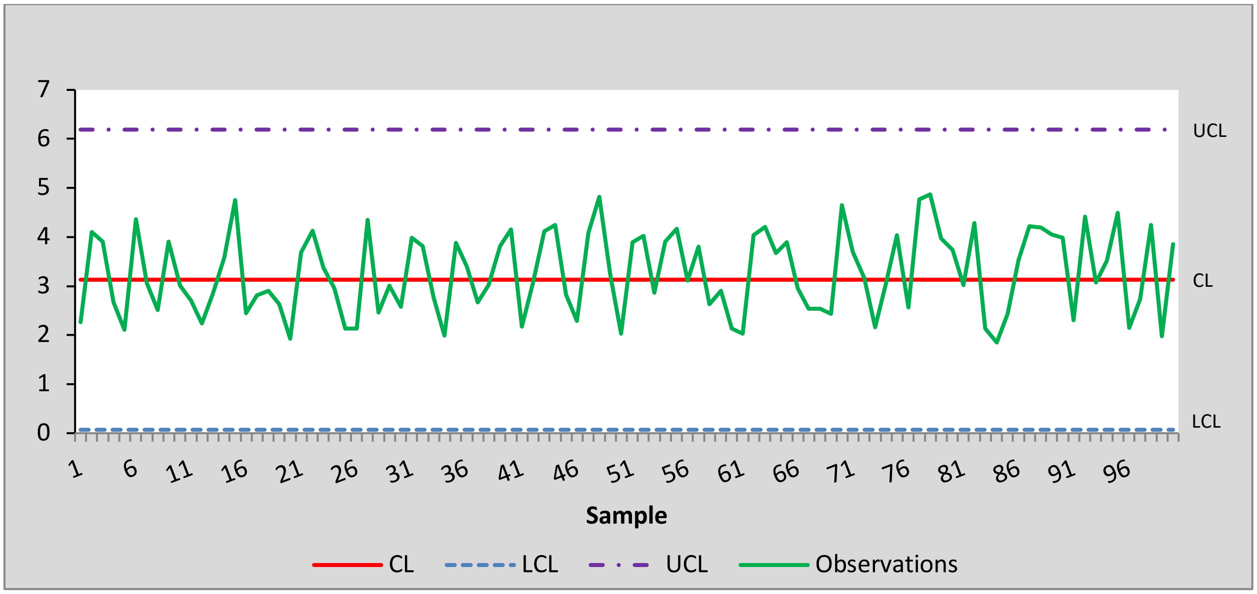

Fig.1: Control Chart for M/M/c Queueing Model.

6 Conclusions

In this paper, control limits have been developed for the production process by assuming the M/M/c queueing process, and illustrated. The control chart using M/M/c queueing model for special case $\lambda = 10$, $\mu = 3.5$ with number of manufacturing machines (inspectors) $c = 5$ has been also monitored with a simulated production data where the process control shows to be under control. It is however advisable to keep the medium arrival rates of products with a medium production rate during manufacturing so as to keep the process in control. If the arrival rate of products is higher and the production rate is low, the number of inspectors has to be increased so as to keep the process in control and to avoid the traffic intensity in the manufacturing system.

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