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Constant-Partially Accelerated Life Tests for Three-Parameter Distribution: Bayes Inference using Progressive Type-II Censoring

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Abstract: This article explores accelerated life test from constant-stress test based on progressive type-II censoring. We consider that the lifetime of items under use condition follows the three-parameter inverted generalized linear exponential distribution. To estimate the distribution parameters and the acceleration factor, we employ the maximum likelihood method. The Gibbs sampler with the Metropolis-Hastings algorithm is applied to generate the Markov chain Monte Carlo samples from the posterior functions to approximate the Bayes estimation using several loss functions and to establish the symmetric credible interval for the parameters and the acceleration factor. A real data and simulated data are analyzed for more illustration. A simulation study is presented to compare the obtained estimates based on mean square error and average absolute bias.

Keywords: Progressive type-II; Gibbs sampler; Partially accelerate life test; Maximum likelihood estimation; Inverted generalized linear exponential distribution; Metropolis-Hastings algorithm

1 Introduction

Censoring plays a vital role in statistical analysis for life tests. In type-II censoring scheme, which has been extremely prominent, the lifetime test is terminated when pre-specified number r out of n items has fails. The analysis of type-II censoring scheme with various distributions was discussed by[1,2,3]. The progressive type-II censoring scheme (PT-IICS) is considered as a generalization of the type-II censoring scheme. It has been extensively researched as it saves the cost and time of the experimental test. Recently, several authors [4,5,6,7,8,9] have studied the analysis of PT-IICS under different lifetime distributions.

Due to the continuous improvement in the manufacturing design, it is gradually difficult to obtain data on the life of highly reliable products at test time in normal conditions. As such lifetime test in normal conditions all-round costly and time consuming. As a result, accelerated life tests (ALTs) are chosen for use in manufacturing to obtain enough failure data, within a short time frame, necessary to infer its relationship to external stress variables. In ALTs, test items are tested only in accelerated conditions, namely, higher than normal pressure levels, to induce early failure. The data collected in such accelerated conditions is then extrapolated by a physically appropriate statistical model to estimate the lifetime distribution under normal use conditions. According to [10], there are mainly three ALT methods. The first method is called the constant-stress ALT, the stress is kept at a constant level throughout the life of the test products [11,12,13, 14]. The second one is referred to as progressive-stress ALT, the stress applied to a test product, which is continuously increasing in time [15]. The third is the step-stress ALT, in which the test condition changes at a given time or upon the occurrence of a specified number of failures. The step-stress ALT has been studied by several authors [16,17]. When the acceleration factor cannot be assumed as a known value, the partially accelerated life test (PALT) will be a good choice to perform the life test. In PALTs, items are tested at both accelerated and use conditions. Also, there are three major stress

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types of PALTs: constant-stress, step-stress and progressive-stress. When a test involves two levels of stress with the first one being the normal level and at a specific time point the stress will be change, it is referred to as a step-stress PALT. Several authors have dealt with this type of ALT, including [18,19]. The constant-stress PALT (CSPALT) (which is the main topic of this paper) runs each item at either use or accelerated condition only, see [20,21,22,23].

One of the most well- known distributions for fitting real data in reliability and medical studies is linear exponential distribution (LED). The principal objective of proposing any new distribution is that it can be fitted better than the well-known distributions under different applications to real data sets of varying study areas. Thats why several studies developed the LED. [24] presented the generalized linear failure distribution and the transmuted LED is studied by [25]. [26] suggested a new generalization known as the generalized LED. The distributions with an unimodal hazard rate function (HRF) have a pivotal role in many practical experiments. Recently, [27] proposed an extension of the generalized LED with unimodal HRF known as inverted generalized LED (IGLED) and [28] studied the parameter estimation and optimal censoring for this distribution under progressive first failure.

They considered IGLED as a generalization of the inverted exponential distribution (IED), inverse Weibull distribution (IWD) and inverse Rayleigh distribution (IRD). Several important statistical properties were established by them. They also constructed a several explicit forms for the HRF, RHRF, mean residual life (MRL) time, mean waiting time (MWT), the variance residual life time (VRL) and the variance of the reversed residual life (VRRL) and studied the behavior of them. Some measures of income inequality using IGLED distribution are also studied. Furthermore, the IGLED is compared with IWD, IRD, IED, generalized IWD, log-normal distribution and inverted Gaussian distribution using different real data sets. It was obvious that the IGLED fits all data sets better than the other distributions.

The probability density function (PDF) and the cumulative distribution function (CDF) of IGLED with parameter vector $\Phi = (\lambda, \mu, \theta)$ are given by

$$f_1(t;\Phi) = \theta\, e^{-(\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta}}\, (\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta-1}\, (\frac{\lambda}{t^2}+\frac{\mu}{t^3}), \qquad \lambda>0,\ \mu>0,\ \theta>0,\ t>0, \tag{1}$$

and

$$F_1(t;\Phi) = e^{-(\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta}}, \qquad t>0, \tag{2}$$

respectively.
The SF and HRF are given by:

$$S_1(t;\Phi) = 1 - e^{-(\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta}}, \tag{3}$$

and

$$h_1(t;\Phi) = \frac{\theta\, e^{-(\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta}}\, (\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta-1}\, (\frac{\lambda}{t^2}+\frac{\mu}{t^3})}{1 - e^{-(\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta}}}, \tag{4}$$

respectively.

2 Basic Assumptions and Model Description

2.1 Basic assumptions

– Under normal condition, the lifetimes of test units follow the IGLED and are independent and identically distributed.
– Under the acceleration condition, the HRF of test unit can be given by $h_2(t) = \gamma\, h_1(t)$, where $\gamma > 1$ is the acceleration factor. Then the PDF, CDF, SF and HRF can be written as

$$f_2(t;\lambda,\mu,\theta,\gamma) = \theta\,\gamma\left(1 - e^{-(\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta}}\right)^{\gamma-1} e^{-(\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta}}\, (\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta-1}\, (\frac{\lambda}{t^2}+\frac{\mu}{t^3}), \tag{5}$$

$$F_2(t;\lambda,\mu,\theta,\gamma) = 1 - \left(1 - e^{-(\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta}}\right)^{\gamma}, \tag{6}$$

$$S_2(t;\lambda,\mu,\theta,\gamma) = \left(1 - e^{-(\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta}}\right)^{\gamma}, \tag{7}$$

and

$$h_2(t;\lambda,\mu,\theta,\gamma) = \frac{\gamma\,\theta\, e^{-(\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta}}\,(\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta-1}\,(\frac{\lambda}{t^2}+\frac{\mu}{t^3})}{1\,-\,e^{-(\frac{\lambda}{t}+\frac{\mu}{2\,t^2})^{\theta}}}. \tag{8}$$

2.2 Model description

Upon using the CSPALT, the total size of units are divided into two groups: n_1 units for use condition and n_2 units for accelerated condition. Furthermore, let the lifetime T_{ji}, $i = 1,...,n_j$, $j = 1,2$ be two PT-IIC samples from IGLED. At the time of the first-failure T_{j1}, R_{j1} items are randomly withdrawn from the remaining $n_j - 1$ surviving items. At the second-failure T_{j2}, R_{j2} items from the remaining $n_j - 2 - R_{j1}$ items are randomly withdrawn. The test continues until the m_j th failure T_{jm_j} at which time all the remaining $R_{jm_j} = n_j - mj - \sum_{k=1}^{m_j-1} R_{jk}$ items are withdrawn for $j = 1,2$. Also, let R_{ji} be fixed prior and $m_j < n_j$.

Suppose that the failure times are from two continuous populations with PDFs and CDFs given in (1), (5), (2), and (6), then the likelihood function for the two PT-IIC samples $T_{j1:m_j:n_j} < T_{j2:m_j:n_j} < ... < T_{jm_j:m_j:n_j}$ can be written as

$$L(\lambda,\mu,\theta,\gamma|\mathbf{t}) = \prod_{j=1}^{2} C_j \prod_{i=1}^{m_j} f_j(t_{ji:m_j:n_j})\,\left(\,S_j(t_{ji:m_j:n_j})\right)^{R_{ji}}, \tag{9}$$

where $C_j = n_j(n_j - R_{j1} - 1)(n_j - R_{j1} - R_{j2} - 1)...(n_j - m_j - \sum_{k=1}^{m_j-1} R_{jk})$.

3 Maximum Likelihood Estimation and Fisher Information Matrix

In this section, the maximum Likelihood (ML) estimators of the parameters and the accelerated factor under PT-IICS are discussed for IGLED and their asymptotic variance covariance matrix are derived.

Upon inserting (1), (3), (5), and (7) in (9), the likelihood function under PT-IIC data for IGLED can be obtained as:

$$L(\lambda,\mu,\theta,\gamma|\mathbf{t}) = \prod_{j=1}^{2} C_j\,\theta^{m_j}\,\gamma^{(j-1)m_j}\prod_{i=1}^{m_j} e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta-1} \times\left(\frac{\lambda}{t_{ji}^2}+\frac{\mu}{t_{ji}^3}\right)\left(1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\right)^{\gamma^{(j-1)}(R_{ji}+1)-1}. \tag{10}$$

Let us define the log-likelihood function $\ell = \log\, L(\lambda,\mu,\theta,\gamma|\mathbf{t})$ ignoring the constant terms as:

$$\begin{aligned}\ell(\lambda,\mu,\theta,\gamma|\mathbf{t}) =& (m_1+m_2)\log\theta + m_2\log\gamma - \sum_{i=1}^{m_1}(\frac{\lambda}{t_{1i}}+\frac{\mu}{2\,t_{1i}^2})^{\theta} + (\theta-1)\sum_{i=1}^{m_1}\log(\frac{\lambda}{t_{1i}}+\frac{\mu}{2\,t_{1i}^2})\\ &+\sum_{i=1}^{m_1}\log(\frac{\lambda}{t_{1i}^2}+\frac{\mu}{t_{1i}^3}) + \sum_{i=1}^{m_1} R_{1i}\,\log\left(1-e^{-(\frac{\lambda}{t_{1i}}+\frac{\mu}{2\,t_{1i}^2})^{\theta}}\right)\\ &-\sum_{i=1}^{m_2}(\frac{\lambda}{t_{2i}}+\frac{\mu}{2\,t_{2i}^2})^{\theta} + (\theta-1)\sum_{i=1}^{m_2}\log(\frac{\lambda}{t_{2i}}+\frac{\mu}{2\,t_{2i}^2})\\ &+\sum_{i=1}^{m_2}\log(\frac{\lambda}{t_{2i}^2}+\frac{\mu}{t_{2i}^3}) + \sum_{i=1}^{m_2}(\gamma\,(R_{2i}+1)-1)\,\log\left(1-e^{-(\frac{\lambda}{t_{2i}}+\frac{\mu}{2\,t_{2i}^2})^{\theta}}\right).\end{aligned} \tag{11}$$

The normal equations of the unknown parameters and the accelerated factor can be given as:

$$\begin{aligned}\frac{\partial\ell}{\partial\lambda} =& \sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{\theta\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta-1}}{t_{ji}}\left(\frac{(\gamma^{j-1}\,(R_{ji}+1)-1)\,e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}}{1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}}-1\right)\\ &+\sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{(\theta-1)}{t_{ji}\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)} + \sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{1}{t_{ji}^2\left(\frac{\lambda}{t_{ji}^2}+\frac{\mu}{t_{ji}^3}\right)} = 0,\end{aligned} \tag{12}$$

$$\frac{\partial \ell}{\partial \mu}=\sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{\theta\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta-1}}{2\,t_{ji}^2}\left(\frac{\left(\gamma^{j-1}\left(R_{ji}+1\right)-1\right)e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}}{1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}}-1\right)$$
$$+\sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{(\theta-1)}{2\,t_{ji}^2\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)}+\sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{1}{t_{ji}^3\left(\frac{\lambda}{t_{ji}^2}+\frac{\mu}{t_{ji}^3}\right)}=0, \tag{13}$$

$$\frac{\partial \ell}{\partial \theta}=\frac{m_1+m_2}{\theta}+\sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{\left(\gamma^{j-1}\left(R_{ji}+1\right)-1\right)e^{-\left(\frac{\mu}{2\,t_{ji}^2}+\frac{\lambda}{t_{ji}}\right)^{\theta}}\left(\frac{\mu}{2\,t_{ji}^2}+\frac{\lambda}{t_{ji}}\right)^{\theta}\log\left(\frac{\mu}{2\,t_{ji}^2}+\frac{\lambda}{t_{ji}}\right)}{1-e^{-\left(\frac{\mu}{2\,t_{ji}^2}+\frac{\lambda}{t_{ji}}\right)^{\theta}}}$$
$$-\sum_{j=1}^{2}\sum_{i=1}^{m_j}\left(\frac{\mu}{2\,t_{ji}^2}+\frac{\lambda}{t_{ji}}\right)^{\theta}\log\left(\frac{\mu}{2\,t_{ji}^2}+\frac{\lambda}{t_{ji}}\right)+\sum_{j=1}^{2}\sum_{i=1}^{m_j}\log\left(\frac{\mu}{2\,t_{ji}^2}+\frac{\lambda}{t_{ji}}\right)=0, \tag{14}$$

and

$$\frac{\partial \ell}{\partial \gamma}=\frac{m_2}{\gamma}+\sum_{i=1}^{m_2}(1+R_{2i})\log\left(1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\right) \tag{15}$$

The ML estimators of the unknown parameters and the accelerated factor can be computed by solving the normal equations. It is obvious that the normal equations cannot be solved analytically and the numerical method is indispensable. The ML estimators of λ, μ, θ and γ can be denoted as $\hat{\lambda}$, $\hat{\mu}$, $\hat{\theta}$ and $\hat{\gamma}$. For this purpose, $Mathematica$11.3 was used.

Since the computation of Fisher information matrix (given by taking the expectation of the second partial derivative of (11)) is very difficult, it seems appropriate to approximate these expected values by their ML estimates. Then, the asymptotic variance-covariance matrix is given as [see, [29]];

$$I^{-1}=\begin{pmatrix} Var(\hat{\lambda}) & Cov(\hat{\lambda},\hat{\mu}) & Cov(\hat{\lambda},\hat{\theta}) & Cov(\hat{\lambda},\hat{\gamma}) \\ Cov(\hat{\mu},\hat{\lambda}) & Var(\hat{\mu}) & Cov(\hat{\mu},\hat{\theta}) & Cov(\hat{\mu},\hat{\gamma}) \\ Cov(\hat{\theta},\hat{\lambda}) & Cov(\hat{\theta},\hat{\mu}) & Var(\hat{\theta}) & Cov(\hat{\theta},\hat{\gamma}) \\ Cov(\hat{\gamma},\hat{\lambda}) & Cov(\hat{\gamma},\hat{\mu}) & Cov(\hat{\theta},\hat{\gamma}) & Var(\hat{\gamma}) \end{pmatrix}=\begin{pmatrix} -\ell_{\lambda\lambda} & -\ell_{\lambda\mu} & -\ell_{\lambda\theta} & -\ell_{\lambda\gamma} \\ -\ell_{\mu\lambda} & -\ell_{\mu\mu} & -\ell_{\mu\theta} & -\ell_{\mu\gamma} \\ -\ell_{\theta\lambda} & -\ell_{\theta\mu} & -\ell_{\theta\theta} & -\ell_{\theta\gamma} \\ -\ell_{\gamma\lambda} & -\ell_{\gamma\mu} & -\ell_{\gamma\theta} & -\ell_{\gamma\gamma} \end{pmatrix}^{-1}_{(\hat{\lambda},\hat{\mu},\hat{\theta},\hat{\gamma})}, \tag{16}$$

where $\ell_{\Phi_i\Phi_j}=\frac{\partial^2\ell}{\partial\Phi_i\partial\Phi_j}$, $i,j=1,2,3,4$. Accordingly, the approximate confidence intervals (ACIs) based on the asymptotic variance-covariance matrix for the parameters λ, μ and θ and the accelerated factor γ are given respectively as:

$$\hat{\lambda}\pm z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\lambda})},\ \hat{\mu}\pm z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\mu})},\ \hat{\theta}\pm z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\theta})},\text{ and }\hat{\gamma}\pm z_{\frac{\alpha}{2}}\sqrt{Var(\hat{\gamma})},$$

where $z_{\frac{\alpha}{2}}$ is the percentile of the standard normal distribution with right tail probability $\frac{\alpha}{2}$.

4 Bayesian Estimation

This section describes the Bayes estimation which used to estimate the unknown parameters and the accelerated factor of IGLED under CPALT using PT-IICS. For this purpose, several loss functions are proposed like squared error loss function (SELF) and balanced squared error loss function (BSELF). For more details see, [8,30].

It is assumed that the priors of unknown parameters are independent. The priors of the parameters θ and γ can be assumed as a gamma distributions with parameters (a_1,b_1) for θ and (a_2,b_2) for γ and can be written as

$$\pi(\theta)\propto\theta^{a_1-1}e^{-b_1\,\theta},\qquad a_1>0,b_1>0, \tag{17}$$

and

$$\pi(\gamma)\propto\gamma^{a_2-1}e^{-b_2\,\gamma},\qquad a_2>0,b_2>0. \tag{18}$$

Furthermore, the priors of parameters λ and μ can be considered as log-concave distributions (uniform distribution). They can be written as

$$\begin{cases} \pi(\lambda) \propto 1;\ \lambda > 0, \\ \pi(\mu) \propto 1;\ \mu > 0, \end{cases} \tag{19}$$

for simplification, see [31]. Then the joint prior distribution can be written as:

$$\pi(\lambda,\mu,\theta) \propto \theta^{a_1-1}\, e^{-b_1\,\theta}\, \gamma^{a_2-1} e^{-b_2\,\gamma}. \tag{20}$$

Across using Equations (10) and (20), one can write the joint posterior distribution of λ,μ,θ, and γ as

$$\pi^*(\lambda,\mu,\theta,\gamma|\mathbf{t}) \propto \theta^{m_1+m_2+a_1-1}\, e^{-b_1\,\theta}\, \gamma^{m_2+a_2-1} e^{-b_2\,\gamma} \prod_{j=1}^{2}\prod_{i=1}^{m_j} e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}} \left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta-1}\left(\frac{\lambda}{t_{ji}^2}+\frac{\mu}{t_{ji}^3}\right)\left(1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\right)^{\gamma^{(j-1)}(R_{ji}+1)-1}. \tag{21}$$

Therefore, to compute the Bayes estimators of any function $\omega(\lambda,\mu,\theta,\gamma)$, the posterior expected value must be calculated as

$$E_\pi(\omega(\lambda,\mu,\theta,\gamma)|\mathbf{x}) = \frac{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \omega(\lambda,\mu,\theta,\gamma)\, L(\lambda,\mu,\theta,\gamma|\mathbf{t})\, \pi(\lambda,\mu,\theta,\gamma)\, d\lambda\, d\mu\, d\theta\, d\gamma}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L(\lambda,\mu,\theta,\gamma|\mathbf{t})\, \pi(\lambda,\mu,\theta,\gamma)\, d\lambda\, d\mu\, d\theta\, d\gamma}. \tag{22}$$

Equation (22) cannot be solved analytically, so the MCMC with Gibbs sampling is applied to approximate it.

4.1 Gibbs sampling technique

From Equation (21), the full conditional probability distribution for λ, μ, θ, and γ can be written as

$$v_1(\lambda|\mu,\theta,\gamma,\mathbf{t}) \propto \prod_{j=1}^{2}\prod_{i=1}^{m_j} e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}} \left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta-1}\left(\frac{\lambda}{t_{ji}^2}+\frac{\mu}{t_{ji}^3}\right)\left(1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\right)^{\gamma^{(j-1)}(R_{ji}+1)-1}, \tag{23}$$

$$v_2(\mu|\lambda,\theta,\gamma,\mathbf{t}) \propto \prod_{j=1}^{2}\prod_{i=1}^{m_j} e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}} \left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta-1}\left(\frac{\lambda}{t_{ji}^2}+\frac{\mu}{t_{ji}^3}\right)\left(1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\right)^{\gamma^{(j-1)}(R_{ji}+1)-1}, \tag{24}$$

$$v_3(\theta|\mu,\lambda,\gamma,\mathbf{t}) \propto \theta^{m_1+m_2+a_1-1}\, e^{-b_1\,\theta} \prod_{j=1}^{2}\prod_{i=1}^{m_j} e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}} \left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}\left(1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\right)^{\gamma^{(j-1)}(R_{ji}+1)-1}, \tag{25}$$

and

$$v_4(\gamma|\mu,\theta,\lambda,\mathbf{t}) \propto \Gamma\left[m_2+a_2, b_2-\sum_{i=1}^{m_2}(R_{2i}+1)\left(1-e^{-\left(\frac{\lambda}{t_{2i}}+\frac{\mu}{2\,t_{2i}^2}\right)^{\theta}}\right)\right] \tag{26}$$

Lemma 1: *$\nu_1(\lambda|\mu,\theta,\gamma,\boldsymbol{t})$ and $\nu_2(\mu|\lambda,\theta,\gamma,\boldsymbol{t})$ are a log-concave functions under $\theta \geq 1$.*

Proof 1: *See Appendix 1.*

Lemma 2: *$\nu_3(\theta|\mu,\lambda,\gamma,\boldsymbol{x})$ is a log-concave function.*

Proof 2: *See Appendix 2.*

Lemma 3: *$\nu_4(\gamma|\mu,\theta,\lambda,\boldsymbol{t})$ is a gamma distribution.*

Proof 3: *The result is clearly shown from the equation (26).*

It is clear from Figures (1), (2), and (3) that the full conditional posterior distribution for λ, μ, and θ take the same shape of the normal distribution. Upon using the idea in [32] with the fact that the full conditional posterior distribution for λ, μ, and θ take the same shape of the normal distribution, the MCMC technique can be used to generate samples. For this purpose, the M-H algorithm with the Gibbs sampling is a commonly used strategy in MCMC method.

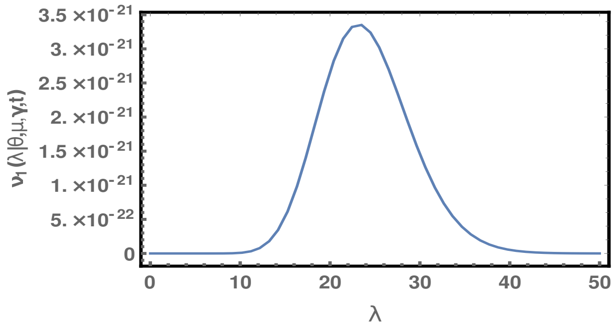

Fig. 1: The posterior density function of λ

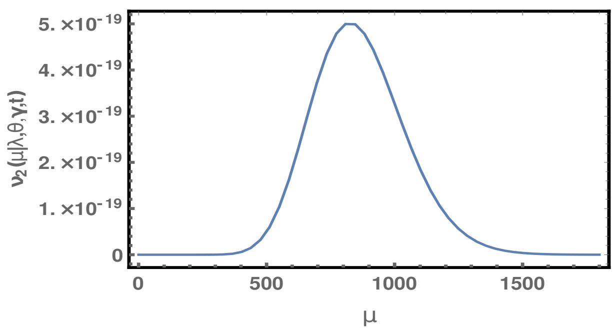

Fig. 2: The posterior density function of μ

The following procedures illustrates the steps for this strategy.

1. Start by guessing initial value of λ, μ, θ and γ say λ_0, μ_0, θ_0 and γ_0.
2. Set j=1.
3. Generate γ_j from $\Gamma\left[m_2 + a_2, b_2 - \sum_{i=1}^{m_2}(R_{2i}+1)\left(1 - e^{-\left(\frac{\lambda}{t_{2i}} + \frac{\mu}{2\,t_{2i}^2}\right)^{\theta}}\right)\right]$.
4. Due to the idea in [32], generate λ_j from $\nu_1(\lambda_{j-1}|\mu_{j-1},\theta_{j-1},\gamma_j,\mathbf{t})$, μ_j from $\nu_2(\mu_{j-1}|\lambda_j,\theta_{j-1},\gamma_j,\mathbf{t})$, and θ_j from $\nu_3(\theta_{j-1}|\mu_j,\lambda_j,\gamma_j,\mathbf{t})$, with the normal proposal distributions $N[\lambda_{j-1},Var(\hat{\lambda})]$, $N[\mu_{j-1},Var(\hat{\mu})]$, and $N[\theta_{j-1},Var(\hat{\theta})]$ where $Var(.)$ is the variance of parameter obtained from the asymptotic variance-covariance matrix.
   (a) Generate a proposal λ^* from $N[\lambda_{j-1},Var(\hat{\lambda})]$, μ^* from $N[\mu_{j-1},Var(\hat{\mu})]$, and θ^* from $N[\theta_{j-1},Var(\hat{\theta})]$.

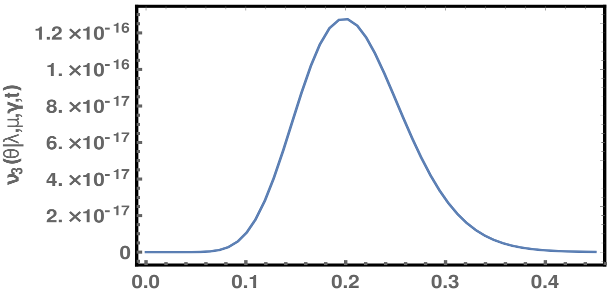

Fig. 3: The posterior density function of θ

(b)Compute the acceptance probabilities

$$\rho(\lambda_{j-1},\lambda^{*}) = \min[1, \frac{\nu_1(\lambda^{*}|\mu_{j-1},\theta_{j-1},\gamma_j,\mathbf{t})}{\nu_1(\lambda_{j-1}|\mu_{j-1},\theta_{j-1},\gamma_j,\mathbf{t})}],$$

$$\rho(\mu_{j-1},\mu^{*}) = \min[1, \frac{\nu_2(\mu^{*}|\lambda_j,\theta_{j-1},\gamma_j,\mathbf{t})}{\nu_2(\mu_{j-1}|\lambda_j,\theta_{j-1},\gamma_j,\mathbf{t})}],$$

$$\rho(\theta_{j-1},\theta^{*}) = \min[1, \frac{\nu_3(\theta^{*}|\lambda_j,\mu_j,\gamma_j,\mathbf{t})}{\nu_3(\theta_{j-1}|\lambda_j,\mu_j,\gamma_j,\mathbf{t})}].$$

(c)Generate u_1, u_2, and u_3 from uniform (0,1) distribution.
(d)If $u_1 < \rho(\lambda_{j-1},\lambda^{*})$, then set $\lambda^{*} = \lambda_j$, else set $\lambda_j = \lambda_{j-1}$.
(e)If $u_2 < \rho(\mu_{j-1},\mu^{*})$, then set $\mu^{*} = \mu_j$, else set $\mu_j = \mu_{j-1}$.
(f)If $u_3 < \rho(\theta_{j-1},\theta^{*})$, then set $\theta^{*} = \theta_j$, else set $\theta_j = \theta_{j-1}$.

5.Set j=j+1.
6.Reiterate Steps 3-5 NG times to get $(\lambda_j,\mu_j,\theta_j,\gamma_j)$, $(j = 1,2,...,NG)$
7.Then the approximate Bayes estimates of λ, μ, θ, and γ under several loss functions are given as:
Under SELF, the approximate Bayes estimates of λ, μ, θ, and γ are respectively given by

$$\hat{\lambda}_S = \frac{1}{NG-M}\sum_{j=M+1}^{NG}\lambda_j,$$

$$\hat{\mu}_S = \frac{1}{NG-M}\sum_{j=M+1}^{NG}\mu_j,$$

$$\hat{\theta}_S = \frac{1}{NG-M}\sum_{j=M+1}^{NG}\theta_j,$$

and

$$\hat{\gamma}_S = \frac{1}{NG-M}\sum_{j=M+1}^{NG}\gamma_j.$$

Under BSELF, the approximate Bayes estimates of λ, μ, θ, and γ are respectively given by

$$\hat{\lambda}_{BS} = w\,\hat{\lambda} + (1-w)\frac{1}{NG-M}\sum_{j=M+1}^{NG}\lambda_j,$$

$$\hat{\mu}_{BS} = w\,\hat{\mu} + (1-w)\frac{1}{NG-M}\sum_{j=M+1}^{NG}\mu_j,$$

$$\hat{\theta}_{BS} = w\,\hat{\theta} + (1-w)\frac{1}{NG-M}\sum_{j=M+1}^{NG}\theta_j,$$

and

$$\hat{\gamma}_{BS} = w\,\hat{\gamma} + (1-w)\,\frac{1}{NG-M}\sum_{j=M+1}^{NG}\gamma_j, \qquad w\in[0,1].$$

For building the credible interval (CRI) of λ, μ, θ, and γ under the MCMC with the Gibbs sampling technique, an idea given by [33] is considered. Sort λ_j, μ_j, θ_j, and γ_j in ascending order. One can see that the $100(1-\alpha)\%$ CRI for λ, μ, θ, and γ can be given respectively by

$$(\lambda_j,\lambda_{j+[(1-\alpha)NG-M]}),\ (\mu_j,\mu_{j+[(1-\alpha)NG-M]})\ ,\ (\theta_j,\theta_{j+[(1-\alpha)NG-M]})\ \text{and}\ (\gamma_j,\gamma_{j+[(1-\alpha)NG-M]})$$

where j selected which gives the smallest length for intervals and $[o]$ is the largest integer less than or equal to o.

5 Numerical Computations

To illustrate the computation of methods presented in the previous sections, a real life data and a simulated data are presented.

5.1 Real data analysis

The failure data in hours of gold-aluminum bonds in three encapsulating resins for integrated circuits under temperature-accelerated life test which was presented in [10], is considered. In this section, these data under the first resin is slightly modified for illustrating our problem. The temperatures 213^oC and 231^oC are considered as use and accelerated temperatures. First, the ML estimates is used under complete data to check the validity of the IGLED distribution to fit the data set for use and accelerated temperatures. The Kolmogorov-Smirnov (K-S) distance and the corresponding P-value is obtained for use and accelerated temperatures. The results are summarized in Table (2). From Table (2), the IGLED provide a good fit to the data sets.

Table 1: The failure data in hours under use and accelerated temperatures

Use temperature (213)	33.8 , 34.8 , 24.2, 20.5 , 22.5 , 18.8, 18.2 , 24.2
Accelerated temperature (231)	14.2, 14.6 , 14.8, 14.8 , 16.2 , 16.7 , 18.9

Now, the PT-IIC samples are generated from the use and accelerated temperatures as given in Table (3). The estimation methods, which are given in Section 3 and Section 4, are used to obtain the estimates of the unknown parameters of the IGLED distribution and the accelerate factor using the use and accelerated temperatures. The estimates based on real data sets under different methods of estimation are tabulated in Table (4).

Table 2: The ML estimates of parameters for IGLED, the K-S values and the associated P-values under use and accelerated temperatures

Data set	Estimates	$K-S$	P-value
Use temperature (213)	$\lambda = 15.3206$, $\mu = 264.286$, $\theta = 4.4382$	0.1714	0.9728
Accelerated temperature (231)	$\lambda = 8.8975$, $\mu = 184.015$, $\theta = 10.9213$	0.2872	0.6106

Table 3: PT-IICSs using the use and accelerated temperatures with $(n_1,n_2) = (8,7)$ and $(m_1,m_2) = (6,5)$

Temperature	CS	PT-IICSs
213	R_1 = {0*5,2}	{18.2, 20.5, 22.5, 24.2, 24.2, 33.8, 34.8 }
231	R_2 = {2, 0*4 }	{14.2, 14.8, 16.2, 16.7, 18.9 }

Table 4: The ML and Bayes estimates, AICs and CRIs for $(\lambda,\mu,\theta,\gamma)$ using real data

Estimates	ML	ACI	SELF	BSELF	CRI
$\hat{\lambda}$	14.5958	{0,166.564}	14.3502	14.473	{0.686,27.3994}
$\hat{\mu}$	394.827	{0.34,7400.99}	594.704	494.765	{5.8113,1543.86}
$\hat{\theta}$	2.4838	{0,14.3818}	2.2462	2.36498	{1.1877,4.1299}
$\hat{\gamma}$	20.9811	{0,55.525}	29.1865	28.7599	{4.631,145.996}

5.2 Simulated data analysis

The two PT-IIC samples are generated by adapting the procedure used by [34] from IGLED with the parameters $(\lambda,\mu,\theta,\gamma)=(0.2,0.02,2,12.5)$ of sizes of two samples $n_1=40$ and $n_2=30$ and two observed samples $m_1=20$ and $m_2=15$ with two censoring schemes (CSs) $R_1=\{0,1,0,1,0,1,0,1,0,1,0,1,0,1,0,$
$1,0,1,0,1,0,1,0,1,0,1,0\}$ and $R_2=\{5,0,0,0,0,0,0,5,0,0,0,0,0,0,5\}$. The two simulated PT-IIC samples can be given as:
For use condition
$t_{1i}=\{0.128301,0.146732,0.152845,0.161519,0.162132,0.181885,0.191791,0.196716,0.206666,0.212564,$
$0.232836,0.239599,0.254043,0.261641,0.299884,0.356677,0.384122,0.398528,0.509063,0.514587\}$
For accelerate condition
$t_{2i}=\{0.111878,0.114752,0.12297,0.131734,0.13253,0.137284,0.138394,0.139216,0.140129,0.149177,$
$0.149652,0.152815,0.156706,0.157131,0.158206\}$.

Under the two simulated PT-IIC samples, the ML estimates of the parameters and accelerated factor can be computed. Furthermore, the Bayesian estimation can be calculated for the parameters and accelerated factor. For this purpose, we generate $NG=20000$ MCMC samples and eliminate the first $M=2000$ values as burn-in. Also, the non-informative priors are considered with a weight function $w=0.5$. The results of this subsection are summarized in Table (5)

Table 5: The ML and Bayes estimates, AICs and CRIs for $(\lambda,\mu,\theta,\gamma)$ using simulated data

Estimates	ML estimates	ACI	SELF	BSELF	CRI
$\hat{\lambda}$	0.1692	{0,0.5056}	0.2115	0.1903	{0.0807,0.2768}
$\hat{\mu}$	0.0259	{0,0.1738}	0.0145	0.0202	{0.0001,0.0753}
$\hat{\theta}$	2.0225	{0,4.5097}	2.2066	2.1145	{1.306,2.9011}
$\hat{\gamma}$	13.5096	{0,27.2825}	19.6187	16.564	{6.2748,50.4194}

6 Simulation Study

The principal reason for this section is to compare the estimators of the parameters by utilizing mean square error (MSE) and average absolute bias (AAB). For varying values of n_1 and n_2 (number of two samples) and m_1 and m_2 (two observed data) with different censoring schemes R_1 and R_2, a large number $N=1000$ of PT-IIC samples are generated from IGLED. Take the values of hyper parameters as $(a_1,b_1)=(0.0,0.0)$ and $(a_2,b_2)=(0.0,0.0)$ for non-informative prior and the true values of λ, μ, θ and γ as $(\lambda,\mu,\theta,\gamma)=(0.6,0.3,1.3,1.6)$ with $w=0.5$. To carry out the numerical study, the following steps are required:

1. With different choice of n_1, n_2, m_1, m_2, R_1 and R_2, the PT-IIC samples are generated by adapting the procedure used by [34] from IGLED with distribution function $F(x)$.
2. The ML estimates of the unknown parameters and accelerated factor are computed using the results given in Section (3).

3.Across using the results obtained in Section (4), the Bayes estimates of the unknown parameters and accelerated factor are calculated. For this purpose, we generate $NG = 20000$ MCMC samples and eliminate the first $M = 2000$ values as burn-in.
4.Repeat Steps $1-3$, $N = 1000$ times.
5.Calculate the MSE and AAB of Φ.

The results obtained from the numerical study are presented in Tables (6-9). The most interesting aspect of these tables is the comparison between different methods based on MSE and AAB of all estimates. It is evident from these tables that:

1.The MSE and AAB decrease for all estimates when n_1, n_2, m_1 and m_2 increase.
2.The MSE and AAB of almost cases based on Bayes estimates are better than MSE and AAB based on MLEs.
3.The MSE and AAB based on Bayes estimates under SELF is the best one for all cases.

Table 6: The AAB and the MSE of ML and Bayes estimates of parameters λ and μ under different PT-IICSs. ($n_1 = n_2 = n$ and $m_1 = m_2 = m$)

(n,m)	CS		λ			μ		
			MLE	SELF	BSELF	MLE	SELF	BSELF
$(40,30)$	$(15,0*29)$	MSE	0.0304	0.0121	0.0177	0.0465	0.0120	0.0209
		AAB	0.1410	0.0868	0.1056	0.1731	0.0868	0.01107
	$(0*29,15)$	MSE	0.0188	0.0109	0.0125	0.0285	0.0119	0.0144
		AAB	0.1116	0.0821	0.0895	0.1388	0.0883	0.0979
	$(0*14,15,0*15)$	MSE	0.0283	0.0107	0.0163	0.0444	0.0123	0.0211
		AAB	0.1353	0.0820	0.1022	0.1674	0.0893	0.1125
$(45,40)$	$(5,0*39)$	MSE	0.0286	0.0109	0.0163	0.0470	0.0091	0.0206
		AAB	0.1389	0.0831	0.1046	0.1744	0.0737	0.1129
	$(0*39,5)$	MSE	0.0281	0.0109	0.0157	0.0466	0.0083	0.0194
		AAB	0.1394	0.0820	0.1024	0.1755	0.0693	0.1104
	$(0*19,5,0*20)$	MSE	0.0281	0.0103	0.0157	0.0468	0.0086	0.0200
		AAB	0.1347	0.0805	0.1002	0.1702	0.0704	0.1084
$(75,55)$	$(20,0*54)$	MSE	0.0268	0.0093	0.0145	0.0448	0.0063	0.0182
		AAB	0.1331	0.0769	0.0977	0.1686	0.0609	0.1063
	$(0*54,20)$	MSE	0.0175	0.0081	0.0096	0.0329	0.0051	0.0121
		AAB	0.1093	0.0731	0.0788	0.1507	0.0544	0.0971
	$(0*27,20,0*27)$	MSE	0.0257	0.0088	0.0140	0.0464	0.0072	0.0197
		AAB	0.1290	0.0740	0.0942	0.1740	0.0651	0.1123
$(75,65)$	$(10,0*64)$	MSE	0.0260	0.0091	0.0142	0.0434	0.0059	0.0177
		AAB	0.1336	0.0769	0.0975	0.1699	0.0612	0.1084
	$(0*64,10)$	MSE	0.0210	0.0081	0.0107	0.0389	0.0045	0.0140
		AAB	0.1210	0.0718	0.0848	0.1649	0.0527	0.0978
	$(0*32,10,0*32)$	MSE	0.0227	0.0080	0.0122	0.0405	0.0058	0.0165
		AAB	0.1221	0.0716	0.0892	0.1628	0.0602	0.1028

7 Conclusion

In this paper, the problem of estimating the unknown parameters, and the accelerated factor under PT-IICS in CSPALT was studied. For classical estimation the MLEs and ACIs were computed. Furthermore, the Bayes estimates using PT-IICS was considered. It cannot be given in explicit form, so Gibbs sampling technique with MCMC has been used to compute the Bayes estimators and constructed the CRIs. A real data set and a simulated data were analyzed for illustrative purposes. Also, a simulation study was presented to compare the proposed methods. For future work, these methods can be extended for other censoring schemes.

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Table 7: The AAB and the MSE of ML and Bayes estimates of parameters θ and γ under different PT-IICSs. ($n_1 = n_2 = n$ and $m_1 = m_2 = m$)

(n,m)	CS		θ			γ		
			MLE	SELF	BSELF	MLE	SELF	BSELF
(40,30)	$(15,0*29)$	MSE	0.0366	0.0081	0.0167	0.2035	0.2573	0.2220
		AAB	0.1533	0.0707	0.1045	0.3316	0.3538	0.3373
	$(0*29,15)$	MSE	0.0283	0.0080	0.0139	0.2514	0.3299	0.2816
		AAB	0.1327	0.0669	0.0928	0.3650	0.3952	0.3736
	$(0*14,15,0*15)$	MSE	0.0326	0.0083	0.0156	0.2131	0.2594	0.2294
		AAB	0.1472	0.0893	0.1018	0.3343	0.3435	0.3326
(45,40)	$(5,0*39)$	MSE	0.0335	0.0088	0.0164	0.1739	0.2023	0.1832
		AAB	0.1487	0.0739	0.1039	0.3125	0.3155	0.3104
	$(0*39,5)$	MSE	0.0316	0.0083	0.0147	0.1652	0.1955	0.1762
		AAB	0.1445	0.0700	0.0988	0.3139	0.3251	0.3162
	$(0*19,5,0*20)$	MSE	0.0371	0.0095	0.0180	0.1578	0.1772	0.1632
		AAB	0.1552	0.0740	0.1075	0.2986	0.3017	0.2965
(75,55)	$(20,0*54)$	MSE	0.0322	0.0093	0.0162	0.1143	0.1279	0.1191
		AAB	0.1470	0.0759	0.1038	0.2580	0.2616	0.2579
	$(0*54,20)$	MSE	0.0276	0.0089	0.0132	0.1281	0.1491	0.1363
		AAB	0.1366	0.0717	0.0940	0.2784	0.2844	0.2809
	$(0*27,20,0*27)$	MSE	0.0331	0.0094	0.0166	0.1209	0.1296	0.1233
		AAB	0.1485	0.0744	0.1035	0.2577	0.2564	0.2553
(75,65)	$(10,0*64)$	MSE	0.0305	0.0102	0.0159	0.0831	0.0875	0.0840
		AAB	0.1408	0.0789	0.1020	0.2244	0.2234	0.2223
	$(0*64,10)$	MSE	0.0253	0.0094	0.0124	0.0940	0.1077	0.01986
		AAB	0.1305	0.0753	0.0903	0.2387	0.2468	0.2413
	$(0*32,10,0*32)$	MSE	0.0305	0.0097	0.0157	0.0863	0.0923	0.0878
		AAB	0.1412	0.0776	0.1010	0.2290	0.2301	0.2284

Table 8: The AAB and the MSE of ML and Bayes estimates of parameters λ and μ under different PT-IICSs.

(n_1,m_1)	R_1		λ			μ		
(n_2,m_2)	R_2		MLE	SELF	BSELF	MLE	SELF	BSELF
(50,30)	$(20,0*29)$	MSE	0.0325	0.0133	0.0186	0.0498	0.0141	0.0236
(35,20)	$(15,0*19)$	AAB	0.1454	0.0885	0.1079	0.1779	0.0955	0.1170
	$(0*29,20)$	MSE	0.0163	0.01113	0.0114	0.0279	0.0241	0.0189
	$(0*19,15)$	AAB	0.1015	0.0857	0.0857	0.1340	0.1321	0.1126
	$(0*15,20,0*14)$	MSE	0.0280	0.0108	0.0157	0.0459	0.0203	0.0242
	$(0*9,15,0*10)$	AAB	0.1346	0.0831	0.1009	0.1715	0.1219	0.1219
(50,40)	$(10,0*39)$	MSE	0.0286	0.0105	0.0155	0.0501	0.0101	0.0223
(35,30)	$(5,0*29)$	AAB	0.1397	0.0812	0.1010	0.1778	0.0782	0.1150
	$(0*39,10)$	MSE	0.0221	0.0101	0.0128	0.0387	0.0149	0.0194
	$(0*29,5)$	AAB	0.1232	0.0792	0.0912	0.1631	0.1014	0.1145
	$(0*20,10,0*19)$	MSE	0.0295	0.0112	0.0167	0.0512	0.0147	0.0248
	$(0*15,5,0*14)$	AAB	0.1401	0.0828	0.1046	0.1801	0.0963	0.1203
(75,50)	$(25,0*49)$	MSE	0.0281	0.0110	00157	0.0472	0.0072	0.0197
(55,40)	$(15,0*39)$	AAB	0.1360	0.0835	0.1012	0.1732	0.0651	0.1102
	$(0*49,25)$	MSE	0.0162	0.0088	0.0098	0.0307	0.0136	0.0156
	$(0*39,15)$	AAB	0.1033	0.0748	0.0797	0.1450	0.0956	0.1031
	$(0*24,25,0*25)$	MSE	0.0264	0.0086	0.0143	0.0460	0.0109	0.0209
	$(0*20,15,0*19)$	AAB	0.1342	0.0753	0.0980	0.1741	0.0825	0.1123
(75,65)	$(10,0*64)$	MSE	0.0267	0.0106	0.0145	0.0474	0.0063	0.0197
(55,50)	$(5,0*49)$	AAB	0.1328	0.0822	0.0980	0.1728	0.0607	0.1112
	$(0*64,10)$	MSE	0.0230	0.0083	0.0122	0.0405	0.0080	0.0179
	$(0*49,5)$	AAB	0.1280	0.0715	0.0914	0.1660	0.0713	0.1098
	$(0*32,10,0*32)$	MSE	0.0237	0.0087	0.0126	0.0457	0.0072	0.0195
	$(0*25,5,0*24)$	AAB	0.1249	0.0754	0.0914	0.1720	0.0656	0.1106

Table 9: The AAB and the MSE of ML and Bayes estimates of parameters θ and γ under different PT-IICSs.

(n_1,m_1) (n_2,m_2)	R_1 R_2		θ MLE	θ SELF	θ BSELF	γ MLE	γ SELF	γ BSELF
$(50,30)$	$(20,0*29)$	MSE	0.037	0.0073	0.0161	0.3175	0.5369	0.4036
$(35,20)$	$(15,0*19)$	AAB	0.1556	0.0699	0.1048	0.3965	0.4928	0.4281
	$(0*29,20)$	MSE	0.0292	0.0059	0.0124	0.3602	0.6411	0.4746
	$(0*19,15)$	AAB	0.1362	0.0654	0.0907	0.4177	0.5440	0.4650
	$(0*15,20,0*14)$	MSE	0.0370	0.0064	0.0148	0.2500	0.4008	0.3077
	$(0*9,15,0*10)$	AAB	0.1547	0.0678	0.1003	0.3621	0.4397	0.3864
$(50,40)$	$(10,0*39)$	MSE	0.0330	0.0075	0.0153	0.1656	0.2524	0.1971
$(35,30)$	$(5,0*29)$	AAB	0.1479	0.0695	0.1017	0.3070	0.3621	0.3253
	$(0*39,10)$	MSE	0.0287	0.0062	0.0126	0.2242	0.3411	0.2718
	$(0*29,5)$	AAB	0.1372	0.0645	0.0924	0.3521	0.4185	0.3778
	$(0*20,10,0*19)$	MSE	0.0339	0.0066	0.0147	0.2178	0.3169	0.2565
	$(0*15,5,0*14)$	AAB	0.1503	0.0654	0.1001	0.3391	0.3884	0.3557
$(75,50)$	$(25,0*49)$	MSE	0.0320	0.0070	0.0147	0.1453	0.2059	0.1679
$(55,40)$	$(15,0*39)$	AAB	0.1447	0.0682	0.0990	0.2874	0.3297	0.3022
	$(0*49,25)$	MSE	0.0258	0.0052	0.0109	0.1733	0.2378	0.1986
	$(0*39,15)$	AAB	0.1299	0.0599	0.0857	0.3030	0.3371	0.3144
	$(0*24,25,0*25)$	MSE	0.0348	0.0069	0.0154	0.1317	0.1700	0.1447
	$(0*20,15,0*19)$	AAB	0.1523	0.0666	0.1013	0.2788	0.2972	0.2824
$(75,65)$	$(10,0*64)$	MSE	0.0296	0.0078	0.0145	0.1072	0.1552	0.1247
$(55,50)$	$(5,0*49)$	AAB	0.1392	0.0703	0.0970	0.2552	0.2934	0.2678
	$(0*64,10)$	MSE	0.0268	0.0057	0.0121	0.0993	0.1393	0.1133
	$(0*49,5)$	AAB	0.1333	0.0601	0.0906	0.2428	0.2719	0.2521
	$(0*32,10,0*32)$	MSE	0.0318	0.0066	0.0147	0.1045	0.1507	0.1215
	$(0*25,5,0*24)$	AAB	0.1471	0.0662	0.1008	0.2474	0.2858	0.2612

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Appendix 1*To prove Lemma 1, we want to prove that* $\frac{\partial^2 \log(\nu_1(\lambda|\mu,\theta,\gamma,\boldsymbol{t}))}{\partial\lambda^2}$ *and* $\frac{\partial^2 \log(\nu_2(\mu|\lambda,\theta,\gamma,\boldsymbol{t}))}{\partial\mu^2}$ *are negative. The second derivative of* $\log(\nu_1(\lambda|\mu,\theta,\gamma,\boldsymbol{x}))$ *w.r.t.* λ *can be written as*

$$
\begin{aligned}
\frac{\partial^2 \log(\nu_1(\lambda|\mu,\theta,\gamma,\boldsymbol{t}))}{\partial\lambda^2} =& -\sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{\theta\,(\theta-1)}{t_{ji}^2}\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta-2} - \sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{1}{t_{ji}^4\left(\frac{\lambda}{t_{ji}^2}+\frac{\mu}{t_{ji}^3}\right)^2} \\
&-(\theta-1)\sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{1}{t_{ji}^2\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^2} + \sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{\theta\,(\gamma^{j-1}(R_{ji}+1)-1)\,e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta-2}}{t_{ji}^2\left(1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\right)^2} \\
&\times\left[-\left(1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\right)-\theta\left(e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}+\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}-1\right)\right],
\end{aligned}
$$

and the second derivative of $\log(v_2(\mu|\lambda,\theta,\gamma,\boldsymbol{t}))$ *w.r.t.* μ *can be written as*

$$\frac{\partial^2 \log(v_2(\mu|\lambda,\theta,\gamma,\boldsymbol{t}))}{\partial\mu^2} = -\sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{\theta\,(\theta-1)}{4\,t_{ji}^4}\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta-2} - \sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{1}{t_{ji}^6\left(\frac{\lambda}{t_{ji}^2}+\frac{\mu}{t_{ji}^3}\right)^2}$$

$$-(\theta-1)\sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{1}{4\,t_{ji}^4\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^2} + \sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{\theta\,(\gamma^{j-1}(R_{ji}+1)-1)\,e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta-2}}{4\,t_{ji}^4\left(1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\right)^2}$$

$$\times\left[-\left(1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\right)-\theta\left(e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}+\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}-1\right)\right].$$

Since $\left(e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}+\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}\right)>1$ *for* $\lambda>0,\ \mu>0,\ \theta>0$ *and* $t_{ji}>0, i=1,\ldots,m$ *and* $j=1,2$*, then the result is satisfied.*

Appendix 2*To prove Lemma 2, we want to prove that* $\frac{\partial^2\log(v_3(\theta|\mu,\lambda,\gamma,\boldsymbol{t}))}{\partial\theta^2}$ *is negative. The second derivative of* $\log(v_3(\theta|\mu,\lambda,\gamma,\boldsymbol{t}))$ *w.r.t.* θ *can be written as*

$$\frac{\partial^2\log(v_3(\theta|\mu,\lambda,\gamma,\boldsymbol{t}))}{\partial\theta^2} = \frac{-1}{\theta^2}(m_1+m_2+a_1-1) - \sum_{j=1}^{2}\sum_{i=1}^{m_j}\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}\log^2\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)$$

$$+\sum_{j=1}^{2}\sum_{i=1}^{m_j}\frac{(\gamma^{j-1}(R_{ji}+1)-1)\,e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta-2}\log^2\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)}{\left(1-e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}\right)^2}$$

$$\times\left[-\left(e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}+\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}-1\right)\right]$$

Since $\left(e^{-\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}}+\left(\frac{\lambda}{t_{ji}}+\frac{\mu}{2\,t_{ji}^2}\right)^{\theta}\right)>1$ *for* $\lambda>0,\ \mu>0,\ \theta>0$ *and* $t_{ji}>0, i=1,\ldots,m$ *and* $j=1,2$*, then the result is satisfied.*

Conflict of interest: The authors declare that they have no conflict of interest.

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