Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/110103

Estimating South Africa's Growth Risk using GARCH-Type Models and Heavy-Tailed Distributions

Retius Chifurira and Knowledge Chinhamu*

School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X54001, Durban 4000, South Africa

Received: 19 Jun. 2020, Revised: 17 Aug. 2020, Accepted: 17 Oct. 2020
Published online: 1 Jan. 2022

Abstract: The daily returns from financial market variables, such as stock indices, exhibit empirical distributions that are often heavy or semi-heavy or more Gaussian-like tailed. Estimating value-at-risk (VaR) and other risk measures such as conditional VaR (expected shortfall) depend highly on the distributional characteristics of the stock returns. The main objective of this study is to investigate the relative performance of the generalized hyperbolic skew Student-*t* and Pearson type-IV distributions governing the generalized autoregressive conditional heteroscedasticity (GARCH) innovations in estimation of the VaR for the daily returns from the FTSE/JSE growth index (J280). The results show that the ARMA(1,1)-EGARCH(1,1) model with a generalized hyperbolic skew Student-*t* distribution governing the innovations outperforms the competing models at estimating the VaR at a 95% level. Results also show that the ARMA(1,1)-EGARCH(1,1) model with a Pearson type-IV distribution governing the innovations outperforms the other competing models at estimating the VaR at all levels for the long position. This study recommends that the ARMA(1,1)-EGARCH(1,1) model with generalized hyperbolic skew Student-*t* and Pearson type-IV distributions be used in the modeling of daily returns from stock indices.

Keywords: FTSE/JSE Growth index, GARCH-type models, heavy-tailed distributions, value-at-risk.

1 Introduction

The FTSE/JSE growth index is designed to reflect portfolios focusing on earnings and revenue growth, weighted towards those companies with identifiable growth characteristics, providing investors with a comprehensive measure of performance of the South African stock market. The growth computes 3-year historical earnings per share growth, 3-year historical sales growth, 2-year forward earnings per share growth, 2-year forward sales growth, and return on equity times (1-payout ratio). The FTSE/JSE growth index evaluates the denationalisation of the South African financial market [1]. There are 60 companies from different Industry Classification Benchmark (ICB) sectors that constitute the FTSE/JSE growth index with total full gross market capital before the invest-ability weight of ZAR2 392 995 million. The top ten companies with a total net market capital of ZAR2 392 995 million and their respective ICB sector are shown in Table 1.

The South African market, just like other emerging markets, is highly volatile and unpredictable, making it a risky market. As well-known in the finance literature, the majority of the financial returns exhibit two stylized facts: heavy tails and volatility clustering [2,3,4]. In this paper, we consider the two stylized facts but focus on the returns of the daily FTSE/JSE growth index. We introduce two types of heavy-tailed distributions i.e. the generalized hyperbolic skewed Student-*t* and the Pearson type-IV distributions into the generalized autoregressive conditional heteroscedasticity (GARCH) framework as in Bollerslev [5]. We compare the relative performance of the two heavy-tailed distributions against the Student-*t* and the skewed Student-*t* distributions. We are interested in determining if the newly developed generalized hyperbolic skewed Student-*t* or the Pearson type-IV distributions outperform the Student-*t* and the skewed Student-*t* distributions in fitting the returns of the FTSE/JSE growth index. Literature is quite rich in models describing the stylized properties of financial returns. These include the autoregressive conditional heteroscedasticity (ARCH)

* Corresponding author e-mail: chifurira@ukzn.ac.za

Table 1 FTSE/JSE Growth index: Top 10 constituents

Rank	Company Name	ICB Sector
1	BHP Billiton	Mining
2	Anglo American	Mining
3	SAB Miller	Beverages
4	Sasol	Oil & Gas Producers
5	MTN Group	Mobile Telecommunications
6	Compagnie Financiere Richemont AG	Personal goods
7	Naspers	Media
8	Kamba Iron Ore	Mining
9	Gold Fields	Mining
10	Liberty International	Real Estate

Source:FTSE-JSE Growth Index Fact Sheet (2018)

model [6], GARCH [5] and typical variants of the GARCH-type model such as the exponential GARCH [7], integrated GARCH [8], GARCH-in-mean [9], long memory GARCH [10] and stable mixture GARCH [11]), to name a few. In order to obtain good estimates for risk management, the challenge is to choose the appropriate GARCH-type model which adequately captures volatility clustering and, at the same time, capturing the heavy-tailed-ness characteristic (leptokurtosis) of financial returns. For instance, Bollerslev [5] considered the GARCH combined with the Student-t distribution so that the Student-t distribution could capture conditional heavy tails of a variety of foreign exchange and stock price indices returns. Paolella [12] and Tavares et al. [13] used the stable-APARCH model to capture both the asymmetric effect and heavy tails of S&P 500 and FTSE returns. Su and Hung [14] showed that the GARCH model with generalized error, normal and skewed normal distributions provide accurate value-at-risk (VaR) estimates of a range of stock indices across international stock markets during the period of the United States of America subprime mortgage crisis. Sin et al. [15] used the TARCH model combined with the generalized error distribution to model crude oil index returns. In literature, there is no agreement on the type of GARCH model and heavy-tailed distribution to be used in order to capture both volatility clustering and heavy tails of financial returns.

In this paper, we follow the model framework in Guo [16] and are particularly interested in the relative performance of the GARCH-type model combined with generalized hyperbolic skewed Student-t and Pearson type-IV distributions in estimating VaR for returns of the FTSE/JSE growth index. We are not aware of any literature relating to an application of the GARCH-type model combined the generalized hyperbolic skewed Student-t and Pearson type-IV distributions to the returns of the FTSE/JSE growth index. Actually, to the best of our knowledge, there is limited research on combining dynamic volatility models with heavy-tailed distributions in modeling South African financial data, more specifically, returns of the FTSE/JSE growth index.

The rest of the paper is organized as follows: the data used in this study is described in Section 2. In section 3, we provide background theory on GARCH-type models, the generalized hyperbolic skewed Student-t distribution and Pearson type-IV distribution, VaR and backtesting. Section 4 presents the empirical results and discussions. Finally, Section 5 concludes this work.

2 Data

In this paper, the data examined consist of the daily closing price of the FTSE/JSE growth index (J280) for the period 4 July 2007 to 19 April 2018 obtained from IRESS. We divide the data into the in-sample dataset (4 July 2007 to 31 December 2014), which gives 1874 observations and an out-of-sample dataset (2 January 2015 to 31 December 2019), which gives 1311 observations. The in-sample data is used for the model estimation and forecasting risk, while the out-sample data is used for testing the value-at-risk (VaR) forecast. Investors are interested in the return of their investment. We, therefore, obtain the daily log returns (r_t) of South Africa's (SAs) growth index. The log-returns are given by

$$r_t = ln\left(\frac{P_t}{P_{t-1}}\right), \quad (1)$$

where r_t is the natural logarithmic return of the daily price of South Africas growth index at time t, P_t is the daily closing price of South Africas growth index at time t and P_t is the daily closing price of South Africas growth index at time $t-1$.

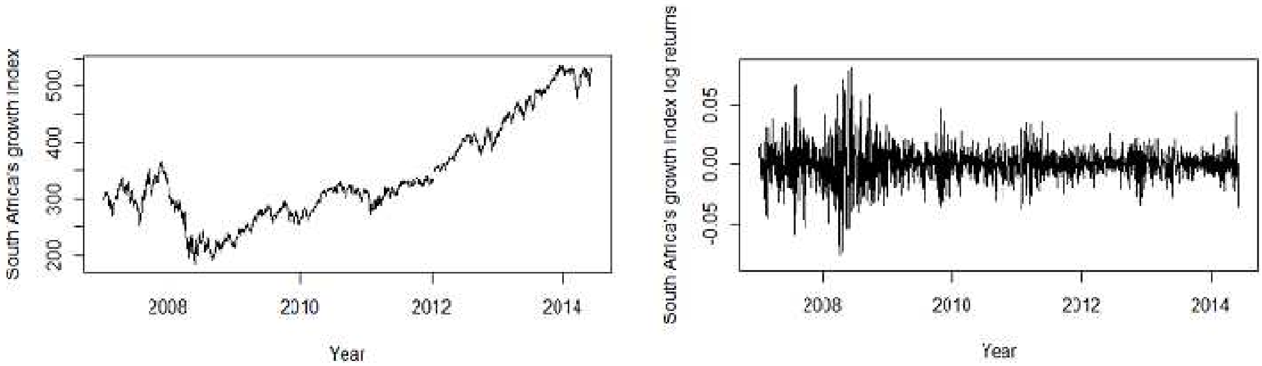

Fig. 1 Time series plot of (Left Panel) daily FTSE/JSE Growth Index (Right Panel) daily returns of FTSE/JSE Growth Index from 4 July 2007-31 December 2014 (in-sample data set)

Figure 1(Left Panel) shows the times series plot of SA's growth index, and Figure 1(Right Panel) shows the log-returns plots of the in-sample data.

The time series plot shows that the daily growth index has a trend. This is confirmed by Mann-Kendall tau test statistic (0.6540) with a p-value < 0.001 thus, rejecting the null hypothesis of no trend at 5% significance level. Therefore, it seems to be non-stationary in the mean and variance. From Figure 1(Right Panel), it seems FTSE/JSE Growth Index returns are stationary in the mean but with a non-constant variance indicating volatility clustering. The augmented-Dickey-Fuller (ADF) test is used to test for stationarity in the mean and variance formally. The null hypothesis is that the log-return series is non-stationary. The ADF statistic is -12.67 with a p-value < 0.001 thus, rejecting the null hypothesis at 5% significance level, indicating that the growth index log-returns are stationary. Table 2 presents the descriptive statistics of the daily returns of the FTSE/JSE growth index.

Table 2 Descriptive statistics for daily returns of FTSE/JSE growth index (J280)

No. of obs	Mean	Std. dev.	Min	Max	Skewness	Excess Kurtosis	p-value of Ljung-Box statistic	p-value of ARCH LM statistic
1873	0.0003	0.0149	-0.0832	0.0819	-0.0394	3.9620	0.0017	< 0.0001

The table reports summary statistics for the daily log returns (r_t) of FTSE/JSE growth index. The Ljung-Box Q(15) statistic test for serial correlation up to 15 lags for r_t.

The mean of the log-returns is close to zero and a significant moment of excess kurtosis (3.9620) illustrates the non-normality (asymmetric property of the log-returns) of returns from the FTSE/JSE growth index. Since the p-value (0.0017) for the Ljung-Box Q statistic is less than 0.05, we reject the null hypothesis of no presence of serial correlation in the log returns. The p-value (< 0.0001) for the ARCH Lagrange Multiple (ARCH LM) statistic is less than 0.05. Thus, we reject the null hypothesis of the absence of potential time-varying volatility (no arch effect) up to lag 15. These findings led to the adoption of GARCH-type models, as discussed in Section 3.

3 Methodology

In this section, we present background theory on GARCH-type models combined with generalized hyperbolic skewed Student-t and Pearson type-IV distributions. We also discuss VaR and backtesting procedures.

3.1 GARCH-type models

A common finding in financial time series modeling is that the financial returns are correlated with their own lagged values. Autoregressive moving average (ARMA) models are widely used to predict financial returns time series, and the models' properties are well documented in the literature. An ARMA(p,q) specification is of the form

$$r_t = \mu + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t, \tag{2}$$

where μ is a constant term, ϕ_i is the i^{th} autoregressive (AR) coefficient, θ_j is the j^{th} moving average (MA) coefficient and ε_t the error term (innovation) at time t. p and q are the orders of AR and MA terms, respectively.

The GARCH-type models were first described and used by Engle [6]) and Bollerslev [5] and have been used extensively in financial time series analysis that displays time-varying volatility. Engle [6] found that large returns often follow large returns, or small returns often follow small returns, i.e. large returns or small returns often appear in clusters, which is called conditional heteroscedasticity or the ARCH effect [17]. The ARCH effect explains the variation of the returns as not a constant but rather dependent on the value of the previous volatility term. Assuming that the innovation ε_t follows a GARCH-type process, the specification of the ARMA(p,q)-GARCH(m,s) model is of the form

$$\begin{aligned} r_t &= \mu + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t, \\ a_t &= \sigma_t \varepsilon_t, \\ \sigma_t^2 &= \alpha_0 + \sum_{i=1}^{m} \alpha_i a_{t-i}^2 + \sum_{j=1}^{s} \beta_j \sigma_{t-j}^2 \end{aligned} \tag{3}$$

where α_0 and β_j are constant coefficients, $m$and s are orders and ε_t is a white noise sequence with mean 0 and variance 1. When m and s are both equal to 1, the GARCH(1,1) model is also called the standard GARCH model, denoted as the sGARCH model in this study. There are some variants of the GARCH-type models which capture the nonlinear property of financial returns. We briefly describe the exponential GARCH (EGARCH) model used in this study. Nelson [7] proposed the EGARCH model to overcome some weaknesses of the GARCH model, such as it responds equally to positive and negative shocks, and the tail behavior of the GARCH model remains too short even with heavy-tailed distributions governing the innovations. Formally, an ARMA(p,q)-EGARCH(m,s) model is defined as

$$\begin{aligned} r_t &= \mu + \sum_{i=1}^{p} \phi_i r_{t-i} + \sum_{j=1}^{q} \theta_j \varepsilon_{t-j} + \varepsilon_t, \\ ln(\sigma_t^2) &= \alpha_0 + \sum_{i=1}^{m} \alpha_i ln(\alpha_{t-i}^2) + \sum_{j=1}^{s} \beta_j g(z_{t-j}) \\ g(z_{t-j}) &= \theta z_{t-j} + \gamma(|z_{t-j}|) - \sqrt{\frac{2}{\pi}} \\ a_t &= \sigma_t \varepsilon_t \end{aligned} \tag{4}$$

where γ is usually set to 1. The GARCH model in (3) imposes the nonnegative constraints on the parameters α_i and β_j; however in the EGARCH model, these restrictions are removed [18]. In this study, an ARMA(p,q)-EGARCH(m,s) model that adds a heteroscedasticity term into the mean equation to show the impact of the volatility of the FTSE/JSE growth index returns is used.

3.2 Distributions

An ARMA-GARCH-type model with Gaussian innovations usually fails to capture the heavy tailed-ness of financial returns. In practice, the Student-t and the skewed Student-t distributions have been used to govern the innovations of ARMA-GARCH-type models to capture the non-normality stylized property of financial returns. In this study, we explore the performance of the generalized hyperbolic skewed Student-t and Pearson's type-IV distributions.

The generalized hyperbolic skewed Student-*t* distribution

The generalized hyperbolic skewed Student-*t* distribution is a special case of the widely used generalized hyperbolic distribution (GHD). Barndorff-Nielsen and Halgreen [19] introduced the generalized hyperbolic distributions and at first applied them to model grain size distributions of wind-blown sands. An important aspect is that GHDs embrace many special cases, respectively limiting distributions of hyperbolic, normal inverse Gaussian (NIG), Student-*t*, variance-gamma, and normal distributions. All of them have been used to model financial returns. The probability density function of the GHD is given by

$$f_{GHD}(x) = \frac{(\alpha^2-\beta^2)^{\frac{\lambda}{2}} K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2})\exp(\beta(x-\mu))}{\sqrt{2\pi}\alpha^{\lambda-\frac{1}{2}}\delta^{\lambda}K_{\lambda}(\delta\sqrt{(\alpha^2-\beta^2)})(\sqrt{\delta^2+(x-\mu)^2})^{\frac{1}{2}-\lambda}} \tag{5}$$

where $K_\lambda(.)$ is the modified Bessel function of the third kind with index λ and

$\delta \geq 0, |\beta| < \alpha$ if $\lambda > 0$

$\delta > 0, |\beta| < \alpha$ if $\lambda = 0$

$\delta > 0, |\beta| \leq \alpha$ if $\lambda < 0$

λ is the parameter influencing the kurtosis, α is the parameter that determines the shape, β determines the skewness of the distribution and satisfying $0 \leq |\beta| < \alpha$. μ is the location parameter, and δ is the scale parameter. Letting $\lambda = -\frac{v}{2}$ and $\alpha \to |\beta|$, we have the generalized hyperbolic skewed Student-*t* distribution (GH*t*D) as

$$f_{GHtD}(x) = \frac{2^{\frac{1-v}{2}}|\beta|^{\frac{v+1}{2}}K_{\frac{(v+1)}{2}}(\sqrt{\beta^2(\delta^2+(x-\mu)^2)})\exp(\beta(x-\mu))}{\Gamma\left(\frac{v}{2}\right)\sqrt{\pi}(\sqrt{\delta^2+(x-\mu)^2})^{\frac{v+1}{2}}} \tag{6}$$

where $\delta > 0$ and $K_v(.) \sim \sqrt{\frac{\pi}{2x}}\exp(-x)$ for $x \to \pm\infty$ is also the modified Bessel function. The distribution has the important property that one tail has a polynomial and the other an exponential behavior [19]. Another property of the GH*t*D is that the distribution is almost tractable and the maximum likelihood estimation of its parameters is quite straight forward using the EM-algorithm, making it very useful for financial application [20,21]. The GH*t*D was briefly mentioned by Prause [22], Barndorff-Nielsen and Shepard [23], Mencia and Sentana [24] and Stefano and McNeil [25]. Aas and Haff [20] showed that the GH*t*D fitted well to four different kinds of market variables; the total index for Norwegian stocks, the SSBWG hedged bond index for international bonds, the Norwegian Krones/EUR exchange rate, and the EURIBOR 5-year interest rate.

The Pearson type-IV distribution

The generalized family of frequency curves, now known as the Pearsonian system of curves, was first developed by Karl Pearson [26]. The Pearsons family includes members such as the normal, Student-*t*, F, gamma, beta, inverse Gaussian, Pareto, and Pearson type-IV distributions. The probability density function (pdf) of Pearson type-IV distribution (PIVD) is given by

$$f_{PIVD}(x) = k\left[1+\left(\frac{x-\lambda}{a}\right)^2\right]^{-m}\exp\left[-v\tan^{-1}\left(\frac{x-\lambda}{a}\right)\right] \tag{7}$$

where $m > \frac{1}{2}$, $v, a > 0$, λ are real-valued parameters, and $-\infty < x < \infty$. $k = \frac{2^{2m-2}|\Gamma(m-\frac{iv}{2})|^2}{\pi a\Gamma(2m-1)}$ is a normalization constant that depends on m, v and a. The pdf of the PIVD is invariant under simultaneous change (a to $-a$, v to v). We specify $a > 0$ so that the curve is always bell-shaped. λ and a are the location and scale parameters, respectively, and v is the skewness parameter. If $v > 0$, then the distribution is positive, while if $v < 0$, the distribution is negative. Parameter m controls the tail thickness and can thus be regarded as a kurtosis parameter. If m is decreased, the kurtosis is increased, and for smaller values of m, the tails of PIVD are much heavier than those of a Gaussian distribution. The PIVD is essentially an asymmetric version of the Student-*t* distribution i.e. when $v = 0$. In literature, the PIVD has been used to model returns from financial stock indices. Zhu and Li [27] showed that the PIVD fits well to the daily DJIA, FTSE, HSI, and NASDAQ indexes from 3 January 2000, to 27 December 2007. Stavroyiannis et al. [3] compared the relative performance of the GARCH(1,1)-PIVD model against the skewed Student-*t* distribution in estimating VaR of daily returns of DJIA, NASDAQ Composite, FTSE100, CAC40, DAX, and S&P500. Bhattacharyya et al. [28] used a combination of PIVD and GARCH(1,1) model to estimate VaR for stock indices of 14 countries.

3.3 Value-at-Risk and Backtesting

Value-at-Risk (VaR) is a risk management tool that has become a benchmark for measuring market risks. This risk measure is used to evaluate the maximum possible loss for a portfolio over a given time period [29]. According to Brooks and Persand [30], there are two main approaches to calculating VaR for financial data, namely the parametric method and the non-parametric method. In this paper, we estimate VaR using the proposed distributions (the parametric method) and compare them with the historical VaR values derived through a non-parametric method.

For a random variable X with distribution function F over a specified time period, the VaR (for a given probability p) can be defined as the p-th quantile of F, i.e.,

$$\mathrm{VaR}_p = F^{-1}(1-p),$$

where F^{-1} is the quantile function [31].

The strength of a model is its ability to forecast accurate VaR estimates for adequate capitalization. In this work, we test VaR model identification and effectiveness by utilizing the widely accepted Kupiec likelihood ratio (LR) unconditional coverage test [32]. The Kupiec test utilizes the fact that a good model should have its proportion of violations of VaR estimates close to the corresponding tail probability, α. The method consists of calculating x^α, the number of times the observed returns fall below (for long positions) or above (for short positions) the VaR estimate at level α, i.e., $r_t < \mathrm{VaR}^\alpha$ or $r_t > \mathrm{VaR}^\alpha$, and then comparing the corresponding failure rates to α [33, 34]. The null hypothesis is that the expected proportion of violations is equal to α. Under this null hypothesis, where N is the sample size, the Kupiec statistic, given by

$$LR_{UC} = 2\ln\left(\left(\frac{x^\alpha}{N}\right)^{x^\alpha}\left(1-\frac{x^\alpha}{N}\right)^{N-x^\alpha}\right) - 2\ln\left(\alpha^{x^\alpha}(1-\alpha)^{N-x^\alpha}\right), \tag{8}$$

is asymptotically distributed according to a chi-square distribution with one degree of freedom.

3.4 Approach

In this paper, the following steps are used for calculating VaR and then backtesting using the Kupiec test. An ARMA(1,1)-EGARCH(1,1) model is fitted to the return using the pseudo maximum likelihood procedure and using a normal distribution governing the innovations.

(a)The standardized residuals are extracted from the model.
(b)The GHtD (Model 1) and PIVD (Model 2) are fitted to the standardized residuals using the maximum likelihood estimation.
(c)The VaR is calculated for the two models.

4 Empirical Results

This section presents the empirical evidence from the returns of the FTSE/JSE growth index dataset. In this section, we report the parameter estimates for all the models proposed in Section 3 and the accuracy of VaR forecast.

4.1 GARCH-type model fitting

In the first step, we fit the GARCH type models to the returns and check its adequacy as the returns have a significant moment of excess kurtosis. The GARCH(1,1) and the EGARCH(1,1) models are fitted to the FTSE/JSE growth index log-returns using the MLE method. Table 3 shows the maximum likelihood parameter estimates and the standard errors in brackets for the GARCH-type models with normal distribution innovations. The Akaike information criterion (AIC) and Bayesian information criterion (BIC) model selection criteria are also reported in Table 3.

From Table 3, it is observed that the ML parameters estimates for the GARCH-type models fitted to the returns of the FTSE/JSE growth index are significant at a 5% level of significance. The ARMA(1,1)-EGARCH (1,1) model has the lowest AIC and BIC values and is thus selected as the best GARCH-type model. The extracted standardized residuals of the ARMA(1,1)-EGARCH(1,1) model has no serial correlation since the p-value of the Ljung-Box statistic $= 0.3779 > 0.05$. In addition, the model has captured the volatility clustering with a p-value of the ARCH-LM statistic $= 0.1488 > 0.05$. Table 4 shows the descriptive statistics of the extracted standardized residuals.

Table 3 ML Parameter estimates of GARCH-type models

Parameter estimate	ARMA(1,1)- sGARCH	ARMA(1,1)- EGARCH (1,1)
$\hat{\mu}$	$0.0014(0.0038)**$	–
$\hat{\phi}_1$	$-0.9749(0.0001)^{***}$	$-0.6608(0.0415)^{***}$
$\hat{\theta}_1$	$0.9849(0.0001)^{***}$	$0.6960(0.0397)^{***}$
$\hat{\alpha}_0$	$0.0001(0.0054)^{**}$	$-0.0939(0.0014)^{***}$
$\hat{\alpha}_1$	$0.0727(0.0001)^{***}$	$0.1077(0.0094)^{***}$
$\hat{\beta}_1$	$0.9209(0.0001)^{***}$	$0.9891(0.0000)^{***}$
$\hat{\gamma}$	–	$0.0850(0.0058)^{***}$
AIC	-5.9725	-6.0047
BIC	-5.9548	-5.9870

Note: $*, **, ***$ indicates p-value that is significant at 10%, 5%, and 1% level of significance respectively.

Table 4 Descriptive statistics of standardized residuals of the ARMA(1,1)-EGARCH(1,1) model

No. of obs	Mean	Std. dev.	Min	Max	Skewness	Excess Kurtosis	p-value of Ljung-Box statistic	p-value of ARCH LM statistic
1873	0.0211	0.9996	-3.5200	4.6038	-0.1521	0.3234	0.3779	< 0.1488

The table reports summary statistics for the standardized residuals

From Table 4, it is observed that the moment of excess kurtosis (0.3234) of the standardized residuals from the fitted the ARMA(1,1)-EGARCH(1,1) model with a normal distribution governing the innovation is significantly different from zero. This indicates that there is still relatively more value in the tail; therefore, the standardized residuals seem to have a tail heavier than that of the normal distribution. To check for the non-normality of the standardized residuals, the Q-Q plot and Jarque-Bera test are employed. Figure 2 shows the Q-Q plot of the standardized residuals.

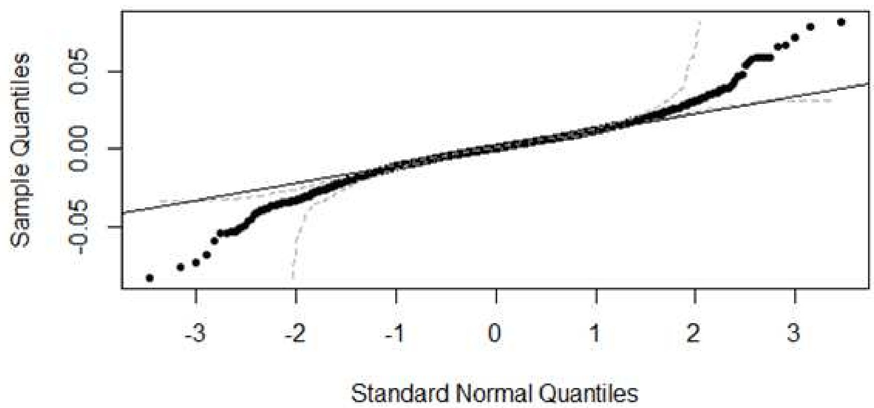

Fig. 2 Q-Q plot of standardized residuals of the ARMA(1,1)-EGARCH(1,1) model with normal distribution governing the innovations

From Figure 2, the Q-Q plot suggests that the standardized residuals seem to diverge from the normal distribution at the tails. This is confirmed by the Jarque-Bera test statistic (13.5808) with a p-value$(0.0011) > 0.05$. This confirms that the standardized residuals of the ARMA(1,1)-EGARCH(1,1) model have a much heavier tail than that of the normal distribution. This suggests that the standardized residuals are realized from a heavy-tailed distribution. Thus, justifying the use of heavy-tailed distributions to model the extracted standardized residuals from the ARMA(1,1)-EGARCH(1,1) model.

In this study, we fit the generalized hyperbolic skewed Student-*t* (GH*t*) and the Pearson type-IV (PIV) distributions to the standardized residuals from the ARMA(1,1)-EGARCH(1,1) model. We also fit the Student-*t* and the skewed Student-*t* distributions. In literature, the Student-*t* distribution (S*t*D) and the skewed Student-*t* distribution (SS*t*D) are most commonly used to estimate market risk. Fitting a statistical distribution usually assumes that the data are independent and identically distributed, i.e. randomness, with no serial correlation and no heteroscedasticity. We tested for randomness using the Bartels rank test. The null hypothesis is independent and identically distributed (i.i.d.). The p-value of the Bartels rank test statistic is $0.7136 > 0.05$, indicating that the standardized residuals are i.i.d. From Table 3, we noted that the standardized residuals are not serially correlated and have no heteroscedasticity. The parameters are estimated using the method of maximum likelihood. Table 5 reports the ML parameter estimates with standard errors in brackets of the fitted distributions.

Table 5 ML Parameter estimates of GARCH-type models

Model	$\hat{\mu}$	$\hat{\delta}$	$\hat{\sigma}$	$\hat{\eta}$	AD-statistic
S*t*D	0.0268	–	0.9919	24.7950	0.9986(0.3580)
SS*t*D	0.0207	0.8999	0.9996	23.6687	0.1158(0.9999)
GH*t*D	0.6121	4.5930	−0.6127	23.83668	0.2083(0.9880)
PIVD	11.5479	4.3386	0.92389	4.3910	0.1894(0.9930)

Note: p-value of the AD statistics is given in parenthesis

From Table 5, it is evident that the distributions fit the extracted standardized residuals well with the p-values of the AD statistic> 0.05. We then calculate VaR estimates for each model. The VaR estimates are calculated for both the long and the short positions. The VaR for the short position is associated with the right quantiles of the distribution at a given probability level. The VaR for the long position is associated with the left quantiles of the distribution at a given probability level. Table 6 presents the VaR estimates for the ARMA(1,1)-EGARCH(1,1)-S*t*D, ARMA(1,1)-EGARCH(1,1)-SS*t*D, ARMA(1,1)-EGARCH(1,1)- GH*t*D and ARMA(1,1)-EGARCH(1,1)- PIVD models at different levels of significance for both the long and short position.

From Table 6, we note that the ARMA(1,1)-EGARCH(1,1) with the S*t*D governing the innovations produced high VaR estimates at the short position, and low VaR estimates at the long position. This suggests that the ARMA(1,1)-EGARCH(1,1)-S*t*D model is inadequate to fully capture the 'stylized facts' exhibited by the FTSE/JSE SA's growth index returns. This phenomenon is well known in the literature.

In order to check model adequacy in estimating the VaR estimate, the VaR estimates are backtested using the Kupiec likelihood ratio test. Table 7 shows the p-value of the Kupiec likelihood ratio test statistic at different levels for the in-sample data.

From Table 7, the VaR estimates from the ARMA(1,1)-EGARCH(1,1)-S*t*D model produced the lowest p-value for the Kupiec likelihood ratio test statistic at all VaR levels. The ARMA(1,1)-EGARCH(1,1)- S*t*D model produced a p-value < 0.05 for the Kupiec likelihood ratio test statistic at a 95% VaR level. This indicates that the ARMA(1,1)-EGARCH(1,1)-S*t*D model performs weakly in estimating VaR for the FTSE/JSE growth index returns. The best model for VaR estimation for the FTSE/JSE growth index returns differ for different VaR levels. We observe that for the long position, at a 2.5% VaR level, the ARMA(1,1)-EGARCH(1,1)-PIVD model produced the highest and significant p-value. This indicates that at this level, the best VaR model is the ARMA(1,1)-EGARCH(1,1)-PIVD model. While at a 5% VaR level, the ARMA(1,1)-EGARCH(1,1)-SS*t*D model produced the highest p-value. At a 10% VaR level, the ARMA(1,1)-EGARCH(1,1)-GH*t*D and ARMA(1,1)-EGARCH(1,1)-PIVD models produced the highest p-values. For the short position, we observe the ARMA(1,1)-EGARCH(1,1)- SS*t*D model outperforming the other models under investigation at all VaR levels. We also

Table 6 VaR estimates for the ARMA(1,1)-EGARCH(1,1) model combined with different distributions

Distribution governing the innovations	VaR estimates					
	Long position			Short position		
	2.5%	5%	10%	97.5%	95%	90%
S*t*D	-1.9471	-1.6101	-1.2346	2.0007	1.6638	1.2882
SS*t*D	-2.0423	-1.6742	-1.2669	1.9022	1.5960	1.2529
GH*t*D	-2.0302	-1.6589	-1.2535	1.9204	1.6130	1.2647
PIVD	-2.0358	-1.6623	-1.2547	1.9160	1.6092	1.2622

Table 7 : In-sample dataset: VaR backtesting of returns of FTSE/JSE growth index

Distribution governing the innovations	**In-Sample dataset**					
	the *p*-value of Kupiec likelihood ratio test statistic					
	Long position			**Short position**		
	2.5%	5%	10%	97.5%	95%	90%
S*t*D	0.1451	0.2806	0.8356	0.4677	0.0088	0.1765
SS*t*D	0.6420	**0.7239**	0.8591	**0.7857**	**0.8607**	**0.9570**
GH*t*D	0.5423	0.4414	**0.9816**	0.6728	0.4111	0.9202
PIVD	**0.6421**	0.5052	**0.9816**	0.6728	0.5453	0.9201

Note: Values in **Bold** are the highest *p*-values at a given probability level

Table 8 : Out-of-sample dataset: VaR backtesting of returns of FTSE/JSE growth index

Distribution governing the innovations	**Out-of-Sample dataset**					
	the *p*-value of Kupiec likelihood ratio test statistic					
	Long position			**Short position**		
	2.5%	5%	10%	97.5%	95%	90%
S*t*D	0.0521	0.2223	0.2222	0.0191	0.0260	0.0877
SS*t*D	**0.8974**	**0.9808**	0.9721	**0.9247**	0.6520	**0.9353**
GH*t*D	**0.8974**	0.8922	0.9353	0.8974	**0.9808**	0.7884
PIVD	**0.8974**	**0.9808**	**0.9722**	**0.9247**	0.8922	0.8797

Note: Values in **Bold** are the highest *p*-values at a given probability level

checked the performance of the models in the out-of-sample dataset. Table 8 shows the *p*-value of the Kupiec likelihood ratio test statistic at different levels for the out-of-sample data.

From Table 8, it is observed that for the long position, all the models under investigation adequately estimate the VaR. At a 2.5% VaR level, the ARMA(1,1)-EGARCH(1,1)-SS*t*D, ARMA(1,1)-EGARCH(1,1)-GH*t*D and ARMA(1,1)-EGARCH(1,1)-PIVD models produced the highest *p*-values. While at a 5% VaR level, the ARMA(1,1)-EGARCH(1,1)-SS*t*D and ARMA(1,1)-EGARCH(1,1)-PIVD models produced the highest *p*-values. At 10% VaR level, the ARMA(1,1)-EGARCH(1,1)-PIVD model outperforms the other models. For the short position; at a 97.5% VaR level, the ARMA(1,1)-EGARCH(1,1)-SS*t*D and ARMA(1,1)-EGARCH(1,1)-PIVD models are the best VaR estimating models and at a 95% VaR level, the ARMA(1,1)-EGARCH(1,1)-GH*t*D model outperforms the other models in estimating VaR. At a 90% VaR level, the best performing model is the ARMA(1,1)-EGARCH(1,1)- SS*t*D model.

5 Conclusion

In this paper, we examined the suitability of using the ARMA(1,1)-EGARCH(1,1) framework combined with heavy-tailed distributions for modeling VaR for the FTSE/JSE Growth Index log returns. The ARMA(1,1)-EGARCH(1,1) framework was used to capture volatility and asymmetric characteristics exhibited by financial returns, while the heavy-tailed distributions are used to capture the heavy-tailed-ness of the actual return distributions. The GH*t* and PIV distributions are applied to the independent and identically standardized residuals from the ARMA(1,1)-EGARCH(1,1) model with normal innovations, and the VaR is calculated at different levels. The VaR models are compared to the ARMA(1,1)-EGARCH(1,1) models with the S*t*D and SS*t*D governing the innovations. Adequacy of the resulting VaR estimates was tested using the Kupiec likelihood ratio test. Backtesting using the Kupiec likelihood ratio test has shown that the ARMA(1,1)-EGARCH(1,1) with the PIVD governing the innovations is the most robust model for both the long and short positions. The ARMA(1,1)-EGARCH(1,1) with a GH*t*D governing the innovation is the most robust model at a 95% VaR level. Thus, for the long position, the backtesting procedure emphasized the superiority of PIVD model over the Student-*t* distribution and the GH*t*D models. While, for the short position, the backtesting procedure emphasized the superiority of the GH*t*D model over the other competing models at a 95% VaR level, thus providing an excellent candidate as an alternative distributional scheme for estimating VaR. For future studies, we recommend developing a VaR estimating framework using a GH*t*-normal- PIV distribution mixture model.

Acknowledgement

The authors acknowledges the financial support by the University Capacity Development Programme (UCDP) National Collaborative Project: Strengthening Academic Staff Development in Mathematical and Statistical Sciences in South Africa.
The authors are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] M.R. Makhwiting, C. Sigauke and M Lesaoana, Modeling tail behavior of returns using the generalized extreme value distribution. *Economics, Management and Financial Markets*, **9(1)**, 41-52, (2014).
[2] A. Maree, P. Card, A. Murphy and P. Kidman, *GARCH models with a new class of heavy-tailed distribution for the silver returns*. Working paper, (2017).
[3] S. Stavroyiannis, I. Makris, V. Nikolaidis and L. Zarangas, Econometric modeling and value-at-risk using the Pearson type-IV distribution. *International Review of Financial Analysis*, **22**, 10-17 , (2012).
[4] B. Mandelbrot, The variation of some other speculative prices. *The Journal of Business*, **40(4)**, 393-413 , (1967).
[5] T. Bollerslev, A conditionally heteroskedastic time series model for speculative prices and rates of return. *The review of economics and statistics*, 542-547,(1987).
[6] R.F. Engle, Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, 987-1007, (1982).
[7] D.B. Nelson, Conditional Heteroskedasticity in Asset Return: A New Approach. *Econometrica*, **59**, 347-370 , (1991).
[8] R.F. Engle and T. Bollerslev, Modeling the persistence of conditional variances. *Econometric Reviews*, **5(1)**, 1-50,(1986).
[9] R.F. Engle, D.M. Lilien and R.P. Robins, Estimating time-varying risk premia in the term structure: The ARCH-M model. *Econometrica: Journal of the Econometric Society*, 391-407 , (1987).
[10] C. Conrad and M. Karanasos, The impulse response function of the long memory GARCH process. *Economics Letters*, **90(1)**, 34-41,(2006).
[11] S.A. Broda, M. Haas, J. Krause, M. S. Paolella and S.C. Steude, Stable mixture GARCH models. *Journal of Econometrics*, **172(2)**, 292-306, (2013).
[12] M.S. Paolella, Stable-GARCH models for financial returns: fast estimation and tests for stability. *Econometrics*, **4(2)**, 1-28, (2016).
[13] A. Tavares, J. Curto and G. Tavares, Modeling heavy tails and asymmetry using ARCH-type models with stable Paretian distributions. *Nonlinear Dynamics*, **51(1)**, 231-243, (2008).
[14] J.B. Su and J.C. Hung, An empirical analysis of jump dynamics, heavy-tails and skewness on value-at-risk estimation. *Economic Modelling*, **28(3)**, 1117-1130, (2011).
[15] K.J. Sin, C.W. Cheong and T.S. Hooi, *Level shift two-components autoregressive conditional heteroscedasticity modeling for WTI crude oil market*, In AIP Conference Proceedings , (2017).
[16] Z.Y. Guo, GARCH models with fat-tailed distributions and the Hong Kong stock market returns. *International Journal of Business and Management*, **12(9)**, 28, (2017).
[17] E. J. Lee, N. Klumpe, J. Vlk and S.H. Lee, Modeling Conditional Dependence of Stock Returns Using a Copula-based GARCH Model. *International Journal of Statistics and Probability*, **6(2)**, 32-41 , (2017).
[18] D.B. Nelson and C.Q. Cao, Inequality constraints in the univariate GARCH model. *Journal of Business and Economic Statistics*, **10(2)**, 229-235, (1992).
[19] O. E. Barndorff-Nielsen and C. Halgreen, , Infinite divisibility of the hyperbolic and generalized inverse Gaussian distributions. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, **38(4)**, 309-311, (1977).
[20] K. Aas and I. H. Haff, The generalized hyperbolic skew Student-t distribution. *Journal of financial econometrics*, **4(2)**, 275-309,(2006).
[21] M.C. Jones and M.J. Faddy, A skew extension of the t-distribution, with applications. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **65(1)**, 159-174 , (2003).
[22] K. Prause, *The generalized hyperbolic model: Estimation, financial derivatives, and risk measures*, PhD thesis, University of Freiburg, Germany, (1999).
[23] O.E. Barndorff-Nielsen, and N. Shephard, Non-Gaussian OrnsteinUhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **63(2)**, 167-241, (2001).
[24] J. Menca and E. Sentana, *Estimation and testing of dynamic models with generalized hyperbolic innovations*, (2004), https://ssrn.com/abstract=790704.

[25] D. Stefano and A. McNeil, The *t* copula and related copulas,(2004), http://www, math. ethz. ch/–mcneil/ftp/tCopula, pdf.
[26] R. Cheng, *Using Pearson type IV and other cinderella distributions in simulation*, Proceedings of the 2011 Winter Simulation Conference, University of Southampton, 457-468 , (2011).
[27] K. Zhu and W.K. Li, A new Pearson-type QMLE for conditionally heteroscedastic models. *Journal of Business and Economic Statistics*, **33(4)**, 552-565, (2015).
[28] M. Bhattacharyya, A. Chaudhary and G. Yadav, Conditional VaR estimation using Pearsons type IV distribution. *European Journal of Operational Research*, **191(2)**, 386-397, (2008).
[29] A. J. McNeil and R. Frey, Estimation of tail-related risk measures for heteroscedastic financial time series: an extreme value approach. *Journal of empirical finance*, **7(3-4)**, 271-300 , (2000).
[30] C. Brooks and G. Persand, The effect of asymmetries on stock index return Value-at-Risk estimates. *The Journal of Risk Finance*, **4(2)**, 29-42, (2003).
[31] R.S. Tsay, *An introduction to the analysis of financial data with R*,John Wiley and Sons, New Jersey, (2014).
[32] P. Kupiec, Techniques for verifying the accuracy of risk measurement models. *Journal of Derivatives*, **2**, 173-184 , (1995).
[33] R. Chifurira and K. Chinhamu, Using the generalized Pareto and Pearson type-iv distributions to measure value-at-risk for the daily South African mining index. *Studies in Economics and Econometrics*, **41(1)**, 33-54, (2017).
[34] K. Chinhamu, C. K. Huang, C.S. Huang and J. Hammujuddy, Empirical Analyses of Extreme Value Models for the South African Mining Index. *South African Journal of Economics*, **83(1)**, 41-55, (2015).