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Wave propagation over a beach within a nonlinear theory

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Abstract: Wave propagation over a beach is considered within a nonlinear theory in shallow water. Lagrangian coordinates are used to describe the problem. The solution is expanded in double series involving a small parameter and local oscillations. Two cases are treated: The beach with appreciable inclination on the horizontal (cliff) and the beach of small inclination. We show that finite solutions are obtained, in contrast to the linear theory which involves a logarithmic singularity at the shoreline. For the cliff, it is shown that local oscillations do not appear in the first two orders of approximation, and the incident wave is totally reflected without loss of energy at this order of approximation. The case of an incident wave on the beach is considered. The deformation of this wave is investigated and explicit formulae are obtained for the reflected wave and for the local oscillations, to shed light on the energy transfer due to interaction with the beach.

Keywords: Shallow water theory; Lagrange's coordinates; wave propagation over a beach; local perturbations.

1 Introduction

The propagation of wave motion on beaches has been a subject of permanent interest since the end of the nineteenth century. An interdisciplinary overview on the subject may be found in [1]. Although the real problem in three-dimensional, it is well-known that under certain conditions the two-dimensional model yields satisfactory results.

Early work on the subject was restricted to beaches with uniform slope within the linearized theory (c.f. Lewy [2], Friedrichs [3], John [4], Isaacson [5], Roseau [6]). The problem formulation and solution for waves on beaches may be found in Stoker [7, Ch. 5], with a discussion on the validity of the solutions under different theories. Carrier and Greenspan [8] were the first to use a nonlinear model to investigate the behaviour of a wave as it climbs a sloping beach. Explicit solutions of the equations of the non-linear inviscid shallow-water theory are obtained for several physically interesting wave-forms. In particular it is shown that waves can climb a sloping beach without breaking. Keller [9] investigates the propagation of surface waves in water whose depth varies in a general way.

Wehausen and Laitone [10, p.537] give a detailed description of the problem under the general title of plane wave motion in unbounded regions with fixed boundaries. Lehman and Lewy [11] discuss the uniqueness problem for water waves on sloping beaches and the boundedness of solutions. Peregrine [12] proposed a Boussinesq-type model for long waves in shallow waters of varying depth through nonlinear equations. Shuto [13,14] considers the run-up of long waves on a sloping beach and produces a solution to the three-dimensional problem of long wave propagation for periodic motion using a Lagrangian description. Experimental studies on wave reflection by a sloping beach in a tank and the dependence of the reflection coefficient on wave steepness were carried out by Taira and Nagata [15].

Kakutani [16] studied the effect of an uneven bottom on the long gravity waves by using a nonlinear perturbation method. Germain [17,18] presents a new expansion of the solution describing wave propagation on a beach within shallow water in double series involving a small parameter and local oscillations.

Tuck and Hwang [19] investigate the linear propagation of long waves on a uniformly sloping beach. Near-shore large amplitude waves are also investigated using the nonlinear theory. Suhayda [20] presents measurements associated with standing waves beaches. Green and Naghdi [21] make a derivation of a system of equations for propagation of waves in water of variable depth. The derivation is effected by means of the incompressibility condition, the energy equation, the invariance requirements under superposed rigid-body

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motions, together with a single approximation for the three-dimensional velocity field.

Sachdev and Seshadri [22] propose an approximate analytical solution to the problem of motion of a bore on a sloping beach. Svendsen and Hansen [23] investigate two-dimensional time-periodic water waves on a gently sloping bottom in the long-wave limit. They derive solutions up to the second-order degree of smallness. Goto [24] derives a nonlinear set of equations of long waves in the Lagrangian description by a new perturbation method. Equations are derived by introducing a new method of perturbation in which the finite displacement of water particles from their initial position is allowed. A numerical solution is given to the problem of wave run-up on a uniform slope connected to a channel of constant depth. Comparison is carried out with the analytical solution of the linear theory.

Mahony and Pritchard [25] study wave reflection from beaches and the dependence on friction at the bottom of the reflection coefficient. They note that the absorption of the wave energy is inevitably linked to wave breaking or viscous effects, and that substantial reflection may sometimes be attained in the absence of wave breaking.

Peregrine [26] presents an overview of wave breaking on beaches. Meyer [27,28] investigates the nonlinear equations governing inviscid water waves close to shore over beaches of small slope, with a view to develop a unified theory which describes the whole shoaling process. Yung-Chao Wu [29] studies small-amplitude waves generated by an oscillating, inclined paddle-type wavemaker in a channel of constant depth by a semi-analytical method using collocation. His results are compared to those obtained by integral equation method. Ehrenmark [30] considers the problem of a train of infinitesimal waves propagating over a uniformly sloping beach and discusses solutions having singularities of different orders at the shoreline.

Miles [31] studies wave reflection from a gently sloping beach within the linear theory. In agreement with the results of [25], it is shown that the absence of viscosity implies perfect reflection. Mandal and Kundu [32] re-investigate the two-dimensional problem of incoming wave against a cliff by Fourier transform in the linear theory, no reflection being assumed. They present a simplified solution which includes a logarithmic singularity at the shoreline, earlier obtained by Stoker. Abou-Dina and Helal [33,34] use asymptotic double expansions proposed in [17,18] to study the effect of obstacles on wave propagation in shallow water. The effect of surface tension is considered.

Chakrabarti [35] studies the propagation of waves against a cliff under the assumptions of linearized theory. His solution exhibits a source/sink type behavior of the velocity potential at the shore-line. Gupta [36] proposes an analytic solution describing the motion of a bore over a uniformly sloping beach for the supercritical case. McIver [37] provides an example of non-uniqueness in the twodimensional linear water wave problem. Ehrenmark [38] investigates the flow generated by small amplitude non-breaking gravity waves on a perfect fluid in a wedge shaped domain using a second-order perturbation analysis. Streamlines are sketched for some values of the beach slope. Javam *et al.* [39] undertake a numerical study of internal wave reflection from sloping boundaries within a nonlinear theory. Ehrenmark [40] studies wave trapping above a plane beach by partially or totally submerged obstacles within the linear theory. He underlines a case of non-uniqueness for the water wave problem on a beach.

Liu *et al.* [41] obtain analytical solutions for forced long waves on a sloping beach. Comparison is carried out with previous numerical solutions. Ehrenmark [42] uses a transformation technique to calculate potentials expressed in integral form for the wave motion over a uniformly sloping beach. Bukreev [43] presents experimental results concerning the reflection of a nonlinear wave from a vertical wall. The existing literature deals mainly with uniformly sloping beaches with extension to deep water, or with vertical barriers and cliffs in water of finite or infinite depth. Martin and Taskinen [44] consider the linear water-wave problem in a bounded water-basin with a shallow beach. Simarro *et al.* [45] propose a fully nonlinear model to study wave propagation in deep or in shallow waters.

Xua and Dias [46] give a numerical evaluation and perform comparison of four old solutions of standing waves over a beach of uniform slope. Gallerano *et al.* [47, 48] use a numerical model to solve the three-dimensional Navier-Stokes equations by a non-hydrostatic shock-capturing numerical scheme which is able to simulate the wave propagation from deep water to the shoreline, including the surf zone and swash zone. Durán *et al.* [49] propose a modification of the governing equations, which is asymptotically similar to the initial model for weakly nonlinear waves, while preserving an additional symmetry of the complete water wave problem. This improved system is shown to have well-conditioned dispersive terms in the swash zone, hence allowing for efficient and stable run-up computations. Zhou and Wang [50] consider the wave-breaking phenomenon under a weakly dissipative shallow water equation. Other recent work on wave propagation in shallow water may be found in [51,52,53].

The purpose of the present work is to investigate the propagation of waves on a beach consisting of a horizontal bed and a uniformly sloping bottom. The work is performed along the guidelines formulated by Germain (c.f. [17,18]) which relies on the expansion of the solution in asymptotic double series combining a small parameter and local oscillations located at the point of separation of the horizontal and the variable parts of the flow bed.

The work is arranged as follows: We first exposed the literature on wave propagation over a beach in section 1. Section 2 expands the details about the problem and material. Subsection 2.1 deals with the problem formulation and the basic equations, while, subsection 2.2 is devoted to the method of solution of the problem under investigation. In subsection 2.3 we address three cases: Subsection 2.3.1 treats the case of a beach of appreciable inclination on the horizontal, and the case of a cliff; subsection 2.3.2 considers the case of a sloping beach of small inclination on the horizontal; subsection 2.3.3 uses the results of the previous two subsections to treat the case of an incident wave over a beach consisting of a horizontal bed and a uniformly sloping bottom. Explicit formulae are obtained for the reflected wave and for the local oscillations. In section 3, the results are presented through a numerical application. A brief discussion of tsunami modeling in section 4. Finally, section 5 collects the main results and conclusions of the present work.

2 Materials and Methods

2.1 Problem formulation and basic equations

In what follows, we shall investigate the problem of wave propagation over a beach of a heavy ideal fluid of constant mass density within the shallow theory. The wave progresses in a fluid of finite constant depth before reaching the uniformly inclined beach. A reflected wave will be generated. The phenomenon of wave breaking is not taken in consideration. Impermeability and isobarity conditions are satisfied on the a priori unknown flow free surface. Impermeability holds on the flow bed as well. Radiation conditions will be defined later on. The geometry of the problem is illustrated on Fig. 1. It is worth here noting the difference with previous work in which the inclined beach extends indefinitely.

The conditions for shallow water are usually formulated in terms of two ratios (c.f. [49,54]): The ratio of characteristic wave amplitude to characteristic depth, and the ratio of characteristic depth to characteristic wave length. Both of these ratios must be sufficiently small. The ratio of these two ratios may lead to different shallow water theories. Clearly, these ratios may vary by different orders of magnitude, depending on the problem under consideration. Characteristic depths may vary from a few meters on beaches and rivers, to a few kilometers in deep waters. Similarly, wavelengh may vary from a few meters to tens of kilometers, while wave amplitude can vary from a few centimeters to a few meters.

A material description will be used to solve the problem. While an overwhelming proportion of the research carried out in this field use an Eulerian description, there is very few work done in the Lagrangian framework. Among those we cite Shuto [13, 14] who treats a three-dimensional problem of wave propagation over a beach, and Goto [24] who describes the propagation of long waves over a beach. Advantages and merits of the Lagrangian description may be found in [13].

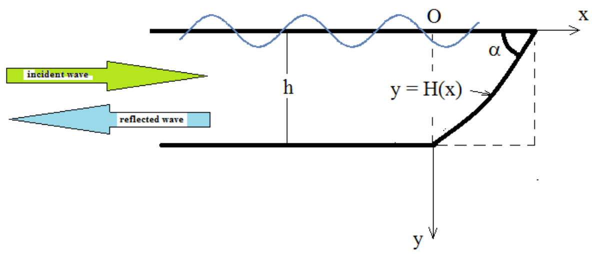


Fig. 1: Geometry of the problem.

The considered problem is two-dimensional. It is believed that such a formulation is useful in many situations of practical interest, especially under laboratory conditions.

The present formulation involves a small parameter representing at the same time the condition of shallow water and the wave steepness:

$$\nu = \frac{\text{water depth above the horizontal bottom}}{\text{horizontal flow extent}} = \frac{\text{wave amplitude}}{\text{wavelength}}.$$

2.2 Basic equations in two dimensions

Let Oxy be a system of orthogonal Cartesian coordinates with x-axis horizontal and taken along the mean position of the free surface, the y-axis is pointing downward passing by the foot of the beach. Let h be the depth of the fluid above the horizontal part. The beach is given by the function

$$y = H(x) \implies x = f(y), \tag{1}$$

while the flat bottom lies in the region with negative abcissae. The equation of the flow bed is:

$$y = G(x) = \begin{cases} h, & -\infty \le x \le 0 \\ H(x), & 0 \le x \le f(0). \end{cases} \tag{2}$$

A fluid particle at the location M with coordinates (x,y) at time moment t occupied the location M_0 with coordinates (X,Y) at time $t = t_0$. At this initial time, the free surface had equation $Y = \eta(X)$.

If $P(X,Y,t)$ is the pressure at the particle at location M at time t, then the unknowns of the problem are:

$$x(X,Y,t), \quad y(X,Y,t), \quad P(X,Y,t).$$

For the determination of these three unknown functions, the following field equations in the bulk and boundary conditions are available:

1. The kinematical condition, or equation of continuity, for the ideal fluid of constant density:

$$\frac{\partial x}{\partial X}\frac{\partial y}{\partial Y} - \frac{\partial x}{\partial Y}\frac{\partial y}{\partial X} = 1. \tag{3}$$

2.The dynamical condition, or condition of circulation preserving motion for the considered fluid:

$$\left(\frac{\partial x}{\partial X}\frac{\partial^2 x}{\partial Y\partial t}-\frac{\partial x}{\partial Y}\frac{\partial^2 x}{\partial X\partial t}\right)+\left(\frac{\partial y}{\partial X}\frac{\partial^2 y}{\partial Y\partial t}-\frac{\partial y}{\partial Y}\frac{\partial^2 y}{\partial X\partial t}\right) = F(X,Y), \quad (4)$$

where F is the double initial vorticity. In what follows, we make the hypothesis that the vorticity was zero initially. The method of solution developed hereafter, however, remains valid in the case $F\neq 0$.

3.The equations of motion involving the unknowns x, y and the pressure gradient. These are used to find the pressure inside the fluid, once the other unknowns x and y have been determined.

$$\frac{\partial^2 x}{\partial t^2}=-\frac{1}{\rho}\left(\frac{\partial p}{\partial X}\frac{\partial y}{\partial Y}-\frac{\partial p}{\partial Y}\frac{\partial y}{\partial X}\right), \quad (5)$$

$$\frac{\partial^2 y}{\partial t^2}=-\frac{1}{\rho}\left(\frac{\partial p}{\partial Y}\frac{\partial x}{\partial X}-\frac{\partial p}{\partial X}\frac{\partial x}{\partial Y}\right)+g. \quad (6)$$

4.The condition on the free surface expressing the isobarity of this surface:

$$\frac{\partial p}{\partial X}=0 \quad \text{at} \quad Y=0 \quad (7)$$

This condition may be formulated in an equivalent form using the equations of motion (5) and (6) to yield:

$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial X}+\frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial X}=g\frac{\partial y}{\partial X} \quad \text{at} \quad Y=0. \quad (8)$$

5.The condition of impermeability of the free surface. This is automatically satisfied.

6.The condition of impermeability of the flow bed:

$$\begin{cases} y(X,h,t)=0 \\ \quad \text{on the horizontal part of the flow bed} \\ x(f(Y),Y,t)+X=f(Y+y(f(Y),Y,t)) \\ \quad \text{on the inclined part of the flow bed.} \end{cases} \quad (9)$$

In addition to these relations, one still needs to impose some limitations on the extent of the flow domain and some radiation conditions, to be explicitly stated later on.

The basic equations are now cast into a more convenient form for later work by considering the displacements $(x-X, y-Y)$ instead of the locations (x,y). Without change in notations, the new kinematical and dynamical conditions now read:

$$\frac{\partial x}{\partial X}+\frac{\partial y}{\partial Y}+\frac{\partial x}{\partial X}\frac{\partial y}{\partial Y}-\frac{\partial x}{\partial Y}\frac{\partial y}{\partial X}=0, \quad (10)$$

$$\left(1+\frac{\partial x}{\partial X}\right)\frac{\partial^2 x}{\partial Y\partial t}-\frac{\partial x}{\partial Y}\frac{\partial^2 x}{\partial X\partial t}+\frac{\partial y}{\partial X}\frac{\partial^2 y}{\partial Y\partial t}-\left(1+\frac{\partial y}{\partial Y}\right)\frac{\partial^2 y}{\partial X\partial t}=0, \quad (11)$$

the equations of motion become:

$$\frac{\partial^2 x}{\partial t^2}=-\frac{1}{\rho}\left[\frac{\partial p}{\partial X}\left(1+\frac{\partial y}{\partial Y}\right)-\frac{\partial p}{\partial Y}\frac{\partial y}{\partial X}\right], \quad (12)$$

$$\frac{\partial^2 y}{\partial t^2}=-\frac{1}{\rho}\left[\frac{\partial p}{\partial Y}\left(1+\frac{\partial x}{\partial X}\right)-\frac{\partial p}{\partial X}\frac{\partial x}{\partial Y}\right]-g, \quad (13)$$

while the isobarity at the free surface yields:

$$\frac{\partial^2 x}{\partial t^2}\left(1+\frac{\partial x}{\partial X}\right)+\frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial X}=g\frac{\partial y}{\partial X} \quad \text{at} \quad Y=0. \quad (14)$$

We shall assume in what follows that all the functions appearing in the problem formulation, together with their various derivatives, are bounded in the domain occupied by the fluid, unless otherwise specified.

2.3 Method of solution

The method of solution to be used in the sequel was proven to be a powerful tool for solving nonlinear problems of shallow water theory. It was introduced by Germain [17,18] as an extention of the classical theory in order to solve problems of wave generation in a semi-infinite channel of constant depth. Few researches have applied this same method [55].

The method relies on the introduction of a distortion ε in the independent variables:

$$\xi=\varepsilon X, \qquad \zeta=Y, \qquad \tau=\varepsilon\sqrt{gh}t. \quad (15)$$

The small parameter ε is related to the physical parameters of the motion, it will not be precised for the time being. In terms of the new variables, the basic equations take the form:

The kinematical condition:

$$\frac{\partial y}{\partial \zeta}+\varepsilon\left(\frac{\partial x}{\partial \xi}+\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \zeta}-\frac{\partial x}{\partial \zeta}\frac{\partial y}{\partial \xi}\right)=0, \quad (16)$$

The dynamical condition:

$$\frac{\partial^2 x}{\partial \zeta\partial \tau}+\varepsilon\left(\frac{\partial x}{\partial \xi}\frac{\partial^2 x}{\partial \zeta\partial \tau}-\frac{\partial x}{\partial \zeta}\frac{\partial^2 x}{\partial \xi\partial \tau}+\frac{\partial y}{\partial \xi}\frac{\partial^2 y}{\partial \zeta\partial \tau}-\left(1+\frac{\partial y}{\partial \zeta}\right)\frac{\partial^2 y}{\partial \xi\partial \tau}\right)=0, \quad (17)$$

The free surface condition:

$$\frac{\partial y}{\partial \xi}=\varepsilon h\frac{\partial^2 x}{\partial \tau^2}+\varepsilon^2 h\left(\frac{\partial x}{\partial \xi}\frac{\partial^2 x}{\partial \tau^2}+\frac{\partial y}{\partial \xi}\frac{\partial^2 y}{\partial \tau^2}\right) \quad \text{at} \quad \zeta=0. \quad (18)$$

The flow bed condition:

$$\begin{cases} y(\xi,h,\tau)=0 \\ \qquad \text{on the horizontal part of the bottom} \\ \varepsilon x(\varepsilon f(\zeta),\zeta,\tau)+\xi=\varepsilon f(\zeta+y(\varepsilon f(\zeta),\zeta,\tau)) \\ \quad \text{on the inclined part of the bottom } \xi=\varepsilon f(\zeta) \end{cases} \tag{19}$$

The condition on the bottom for $\xi \geq 0$ is re-formulated in a more convenient way for later use as:

$$\begin{aligned} &\left[x - yf'(\zeta) - \frac{1}{2!}y^2 f''(\zeta) - \cdots\right] \\ &\quad + \varepsilon f(\zeta)\left[\frac{\partial}{\partial\xi}\{x - yf'(\zeta) - \frac{1}{2!}y^2 f''(\zeta) - \cdots\}\right] \\ &+ \frac{1}{2!}\varepsilon^2 f(\zeta)^2\left[\frac{\partial^2}{\partial\xi^2}\{x - yf'(\zeta) - \frac{1}{2!}y^2 f''(\zeta) - \cdots\}\right] \\ &\qquad = 0 \quad \text{at} \quad \xi = 0. \end{aligned}$$

so that the application of this boundary condition at any approximation order takes place on the vertical flow cross-section $\xi = 0$.

In what follows, we shall consider two cases, for which the beach inclination is either of order zero, or of order 1 in the small parameter ε.

Unlike the classical theory of shallow water for which the solutions are expanded in Poincaré small parameter, the present method allows for an extension of this classical theory by introducing expansions that include local perturbations in the form initially proposed by Germain and later on generalized by Badawi (c.f. Germain [17,18] and Badawi [56]):

$$x^{\pm} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \varepsilon^n x^{\pm}_{m,n}(\xi,\zeta,\tau)\, e^{m\lambda^{\pm}(\xi)/\varepsilon} \tag{20}$$

$$y^{\pm} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \varepsilon^n y^{\pm}_{m,n}(\xi,\zeta,\tau)\, e^{m\lambda^{\pm}(\xi)/\varepsilon}, \tag{21}$$

where $\pm$ denote the regions $\xi > 0$ and $\xi < 0$ respectively, $\lambda^{\pm}(\xi)$ are functions to be determined in the process of the solution. Without loss of generality, one may take $\lambda^{\pm}(0) = 0$. Clearly, this function must satisfy:

$$\begin{cases} \lambda^{+'}(\xi) < 0 & \xi > 0, \\ \lambda^{-'}(\xi) > 0 & \xi < 0. \end{cases} \tag{22}$$

The double summation over m,n does not include the term with $m = n = 0$ as the corresponding solution is verified to vanish identically for the considered application. However, there are cases when this term is essential and must be included in the solution, for example problems involving uniform flow (c.f. Ghaleb and Hefni [55]). As noted by Germain [17,18], the series in the expressions for the solution have an asymptotic nature.

In what follows, all considerations will be restricted to beaches with uniform slope.

2.3.1 Beach of steep inclination

It is required to solve equations (16), (17), (18), (20) under the condition of initial rest of the fluid and appropriate disturbance reaching the beach near the time $t = 0$.

Next, we write down the basic equations and boundary conditions in the first few orders of approximation (m,n) and provide the corresponding solutions.

1. **Approximation** $(0,1)$
The condition at the free surface is identically satisfied. The equations in the mass yield:

$$\frac{\partial y_{0,1}}{\partial\zeta} = 0, \tag{23}$$

$$\frac{\partial^2 x_{0,1}}{\partial\zeta\partial\tau} = 0, \tag{24}$$

and the condition on the bottom is:

$$y_{0,1}(\xi,h,\tau) = 0. \tag{25}$$

One gets:

$$x_{0,1}(\xi,\zeta,\tau) = x_{0,1}(\xi,\tau), \qquad y_{0,1}(\xi,\zeta,\tau) = 0, \tag{26}$$

where a function of (ξ,ζ) has been omitted in the expression for $x_{0,1}$ without loss of generality, as it can be added as a term of order ε to the above transformation of coordinates.

2. **Approximation** $(0,2)$
The equations in the bulk:

$$\frac{\partial y_{0,2}}{\partial\zeta} + \frac{\partial x_{0,1}}{\partial\xi} = 0, \tag{27}$$

$$\frac{\partial^2 x_{0,2}}{\partial\zeta\partial\tau} = 0, \tag{28}$$

equation (27) gives $x_{0,2} = x_{0,2}(\xi,\tau)$. Condition at the bottom:

$$y_{0,2}(\xi,h,\tau) = 0. \tag{29}$$

Condition at the free surface:

$$h\frac{\partial^2 x_{0,1}}{\partial\tau^2} = \frac{\partial y_{0,2}}{\partial\xi}. \tag{30}$$

Integrate (27) and use the second condition at the bottom to get:

$$y_{0,2}(\xi,\zeta,\tau) = -(\zeta-h)\frac{\partial x_{0,1}}{\partial\xi}. \tag{31}$$

This same expression is taken for all values of ξ. Combining this result with the condition at the free surface, one is finally left with:

$$\frac{\partial^2 x_{0,1}}{\partial\xi^2} - \frac{\partial^2 x_{0,1}}{\partial\tau^2} = 0. \tag{32}$$

Integrating:

$$x_{0,1}(\xi,\tau) = U_i(\xi-\tau) + U_r(\xi+\tau), \tag{33}$$

the indices (i) and (r) referring to "incident" and "reflected" waves respectively. One gets:

$$y_{0,2}(\xi,\zeta,\tau) = -(\zeta-h)\left[U_i'(\xi-\tau)+U_r'(\xi+\tau)\right], \tag{34}$$

where the dash refers to the derivative of the function with respect to its argument.
Again, equation (28) gives $x_{0,2} = x_{0,2}(\xi,\tau)$ and the condition on the beach yields:

$$\begin{aligned} x_{0,2} &= x_{0,2}(\xi,\tau) \\ &= C(\tau) + f_1 h \xi \left[U_i''(\xi-\tau)+U_r''(\xi+\tau)\right], \end{aligned} \tag{35}$$

where $C(\tau)$ is an arbitrary constant.

3. **Approximation** $(1,1)$
Using the rest initial condition, it is easy to verify that:

$$\frac{\partial y_{1,1}}{\partial \zeta} + \lambda'(\xi) x_{1,1} = 0, \tag{36}$$

$$\frac{\partial x_{1,1}}{\partial \zeta} - \lambda'(\xi) y_{1,1} = 0, \tag{37}$$

the solution of which yields:

$$\begin{aligned} x_{1,1} &= -K_{1,1}(\tau)\cos\left[|\lambda'(\xi)|\zeta\right], \\ y_{1,1} &= K_{1,1}(\tau)\sin\left[|\lambda'(\xi)|\zeta\right], \end{aligned} \tag{38}$$

where $K_{1,1}(\tau)$ is a bounded function of time and, moreover, tends to zero as $t \to 0$. This function is the only arbitrary function at this order of approximation. Note that this solution satisfies the free surface condition. It remains now to satisfy the condition at the bottom. When $\xi < 0$ it is sufficient to choose $\lambda^-(\xi) = \frac{\pi\xi}{h}$.

4. **Approximation** $(m,1), m > 1$
It is worth noting that the solution at the approximation $(m,1)$, $m > 1$ follows exactly the same pattern as for the order $(1,1)$. For this reason we shall not work it out but quote the results directly. Finally, the first-order approximation in ε in the region $\xi < 0$ can be written as:

$$x = x_{0,1}(\xi,\tau) - \sum_{m=1}^{\infty} K_{m,1}(\tau)\, e^{m\pi\xi/\varepsilon h} \cos\left(m\frac{\pi}{h}\zeta\right) \tag{39}$$

$$y = \sum_{m=1}^{\infty} K_{m,1}(\tau)\, e^{m\pi\xi/\varepsilon h} \sin\left(m\frac{\pi}{h}\zeta\right). \tag{40}$$

The function

$$F(\mu + i\zeta) = x + iy, \qquad \mu = \frac{\xi}{\varepsilon}$$

can now be viewed as being holomorphic in the semi-infinite strip $(-\infty < \mu < 0, 0 < \zeta < 1)$, its imaginary part assuming zero values at the upper and the lower edges. At the end $\mu = 0$, the real and imaginary parts of this function are related by eq.(26). By Schwarz Reflection Principle, this function may be extended to its complex conjugate in the strip $(-\infty < \mu < 0, -1 < \zeta < 0)$.
The boundary condition at $\xi = 0$ is written as:

$$\begin{cases} x(0,\tau) = -y(0,\tau) f_1, & -h \le \zeta < 0 \\ x(0,\tau) = y(0,\tau) f_1, & 0 < \zeta \le h \end{cases} \tag{41}$$

or

$$\begin{cases} \sum_{m=0}^{\infty} K_{m,1}(\tau)\cos\left(m\frac{\pi}{h}\zeta\right) - \sum_{m=1}^{\infty} f_1 K_{m,1}(\tau)\sin\left(m\frac{\pi}{h}\zeta\right) = U_i(-\tau), & -h \le \zeta < 0 \\ \sum_{m=0}^{\infty} K_{m,1}(\tau)\cos\left(m\frac{\pi}{h}\zeta\right) + \sum_{m=1}^{\infty} f_1 K_{m,1}(\tau)\sin\left(m\frac{\pi}{h}\zeta\right) = U_i(-\tau), & 0 < \zeta \le h \end{cases} \tag{42}$$

with $K_{0,1}(\tau) = -U_r(\tau)$ and f_1 is the constant value of the derivative of function $f(\zeta)$. These two dual series equations have an obvious solution:

$$K_{0,1}(\tau) = U_i(-\tau), \quad K_{m,1}(\tau) = 0, \quad m = 1,2,\cdots. \tag{43}$$

Thus there are no local oscillations at this order of approximation, a result similar to those obtained in [33,34].
Before proceeding further, let us take a look at the functions $U_i(-\tau)$ and $U_r(\tau)$ in equations (42). Consider the space of the coordinates (ξ,τ). Fig. 2 shows two pairs of characteristic straight lines for the propagating incident and reflected waves emanating from a general point (ξ,τ) and from the point $(0,\tau)$.

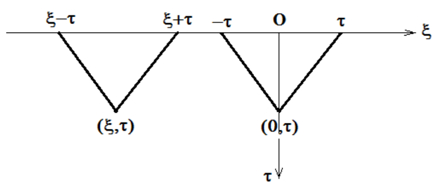


Fig. 2: Characteristics for the incident and the reflected waves.

One easily sees that

$$U_i(-\tau) = U_i(0,\tau), \qquad U_r(\tau) = U_r(0,\tau).$$

Finally, one has

$$U_r(0,\tau) = -U_i(0,\tau)$$

so that the incident wave is completely reflected at this order of approximation.

5.**Approximation** $(m,2), m>0$
The kinematical and the kinetic conditions give, respectively:

$$\frac{\partial y_{m,2}}{\partial \zeta}+\frac{m\pi}{h}x_{m,2}=-\frac{\partial K_{m,1}}{\partial \xi}\cos\frac{m\pi}{h}(\zeta-h)+\sum_{i=1}^{m-1}R_i\cos(2i-m)\frac{\pi}{h}(\zeta-h) \quad (44)$$

$$\frac{\partial^2 x_{m,2}}{\partial \zeta\partial\tau}-\frac{m\pi}{h}y_{m,2}=-\frac{\partial^2 K_{m,1}}{\partial \xi\partial\tau}\sin\frac{m\pi}{h}(\zeta-h)-\sum_{i=1}^{m-1}S_i\sin(2i-m)\frac{\pi}{h}(\zeta-h), \quad (45)$$

where

$$R_i=i(m-i)\frac{\pi^2}{h^2}K_{i,1}(\xi,\tau)K_{m-i,1}(\xi,\tau),$$
$$S_i=i(m-i)\frac{\pi^2}{h^2}\left[K_{i,1}(\xi,\tau)\frac{\partial K_{m-i,1}}{\partial\tau}-K_{m-i,1}(\xi,\tau)\frac{\partial K_{i,1}}{\partial\tau}\right], \quad i=1,2,\cdots,m-1. \quad (46)$$

Coefficients on the r.h.s. of (44) and (45) all vanish, hence these equations simplify to:

$$\frac{\partial y_{m,2}}{\partial \zeta}+\frac{m\pi}{h}x_{m,2}=0 \quad (47)$$
$$\frac{\partial^2 x_{m,2}}{\partial \zeta\partial\tau}-\frac{m\pi}{h}y_{m,2}=0, \quad (48)$$

with general solution satisfying the condition at the bottom:

$$x_{m,2}(\xi,\zeta,\tau)=-K_{m,2}(\tau)\cos\frac{m\pi}{h}\zeta,$$
$$y_{m,2}(\xi,\zeta,\tau)=K_{m,2}(\tau)\sin\frac{m\pi}{h}\zeta. \quad (49)$$

The second-order approximation in ε in the region $\xi<0$ is:

$$x=C(\tau)+f_1h\xi\left[U_i''(\xi-\tau)+U_r''(\xi+\tau)\right]-\sum_{m=1}^{\infty}K_{m,2}(\tau)e^{m\pi\xi/\varepsilon h}\cos m\frac{\pi}{h}\zeta,$$
$$y=-(\zeta-h)\left[U_i'(\xi-\tau)+U_r'(\xi+\tau)\right]+\sum_{m=1}^{\infty}K_{m,2}(\tau)e^{m\pi\xi/\varepsilon h}\sin m\frac{\pi}{h}\zeta. \quad (50)$$

The boundary condition at $\xi=0$ gives:

$$\sum_{m=1}^{\infty}mK_{m,2}(\tau)\cos m\frac{\pi}{h}\zeta+f_1\sum_{m=1}^{\infty}mK_{m,2}(\tau)\sin m\frac{\pi}{h}\zeta=0, \quad 0<\zeta\le h. \quad (51)$$

As before, we make an extension of the analytic function $x+iy$ to the interval $-h\le\zeta<0$ using Schwarz Reflection Principle to get:

$$\sum_{m=1}^{\infty}mK_{m,2}(\tau)\cos m\frac{\pi}{h}\zeta-f_1\sum_{m=1}^{\infty}mK_{m,2}(\tau)\sin m\frac{\pi}{h}\zeta=0, \quad -h\le\zeta<0. \quad (52)$$

Now let

$$\sum_{m=1}^{\infty}mK_{m,2}(\tau)\cos m\frac{\pi}{h}\zeta+f_1\sum_{m=1}^{\infty}K_{m,2}(\tau)\sin m\frac{\pi}{h}\zeta=\begin{cases}Q(\zeta,\tau), & -h\le\zeta<0\\ 0, & 0<\zeta\le h,\end{cases} \quad (53)$$

where $Q(\zeta,\tau)$ is a function to be determined. Expressions for the coefficients in this last relation can be written down at once as:

$$K_{m,2}(\tau)=\frac{1}{hm}\int_{-h}^{0}Q(\zeta',\tau)\cos m\frac{\pi}{h}\zeta'\,d\zeta', \quad m=1,2,\cdots. \quad (54)$$

Substitution of expressions (43) into the first of eqs.(52) yields the following integral equation for the function $Q(\zeta)$:

$$\frac{1}{h}\int_{-h}^{0}K(\zeta',\zeta)Q(\zeta',\tau)\,d\zeta'=0, \quad -h\le\zeta<0 \quad (55)$$

with kernel

$$\begin{aligned}K(\zeta',\zeta)&=\sum_{m=1}^{\infty}\cos m\frac{\pi}{h}\zeta'\cos m\frac{\pi}{h}\zeta-\sum_{m=1}^{\infty}f_1\cos m\frac{\pi}{h}\zeta'\sin m\frac{\pi}{h}\zeta\\&=\frac{1}{2}\sum_{m=1}^{\infty}\left[\cos m\frac{\pi}{h}(\zeta'+\zeta)+\cos m\frac{\pi}{h}(\zeta'-\zeta)\right]-\frac{1}{2}f_1\sum_{m=1}^{\infty}\left[\sin m\frac{\pi}{h}(\zeta'+\zeta)-\sin m\frac{\pi}{h}(\zeta'-\zeta)\right]\\&=-\frac{1}{2}+\frac{1}{2}\pi\sum_{k=-\infty}^{\infty}\delta\left(\frac{\pi}{h}(\zeta'+\zeta)-2\pi k\right)+\frac{1}{2}\pi\sum_{k=-\infty}^{\infty}\delta\left(\frac{\pi}{h}(\zeta'-\zeta)-2\pi k\right)\\&\quad-\frac{1}{4}f_1\left[\cot\frac{\pi}{2h}(\zeta'+\zeta)-\cot\frac{\pi}{2h}(\zeta'-\zeta)\right]\end{aligned} \quad (56)$$

The only term that gives non-trivial contribution is for $k=0$ in the second summation. Thus:

$$K(\zeta',\zeta) = -\frac{1}{2}+\frac{1}{2}\pi\,\delta\left(\frac{\pi}{h}(\zeta'-\zeta)\right) - \frac{1}{4}f_1\left[\cot\frac{\pi}{2h}(\zeta'+\zeta)-\cot\frac{\pi}{2h}(\zeta'-\zeta)\right] \quad (57)$$

or

$$K(\zeta',\zeta) = -\frac{1}{2}+\frac{1}{2}\,\delta\left(\frac{\zeta'-\zeta}{h}\right) - \frac{1}{4}f_1\left[\cot\frac{\pi}{2h}(\zeta'+\zeta)-\cot\frac{\pi}{2h}(\zeta'-\zeta)\right] \quad (58)$$

One is finally left with the following Fredholm integral equation of the second kind for the function $Q(\zeta,\tau)$:

$$Q(\zeta,\tau)-\frac{1}{2}\frac{1}{h}\int_{-h}^{0}\left[2+f_1\cot\frac{\pi}{2h}(\zeta'+\zeta)-\cot\frac{\pi}{2h}(\zeta'-\zeta)\right]Q(\zeta',\tau)\,d\zeta' = 0, \quad -h\le\zeta<0 \quad (59)$$

and upon simplification:

$$Q(\zeta,\tau)-\frac{1}{2}\frac{1}{h}\int_{-h}^{0}\left[2-f_1\frac{\sin\frac{\pi\zeta}{h}}{\sin\frac{\pi}{2h}(\zeta'+\zeta)\sin\frac{\pi}{2h}(\zeta'-\zeta)}\right]Q(\zeta',\tau)\,d\zeta' = 0, \quad -h\le\zeta<0 \quad (60)$$

This last equation clearly shows that $Q(\zeta,\tau)$ does not depend on time, and is in fact identically equal to zero as no source terms are present in this equation. Thus, there are no local oscillations at this order of approximation.

2.3.2 Beach of small inclination

Let the equation of the beach be:

$$y = h+\bar{\mu}x, \quad (61)$$

where h and $|\bar{\mu}|\ll 1$ are constants.

The fluid initially at rest occupies the region:

$$0\le X\le -\frac{h}{\bar{\mu}}, \qquad 0\le Y\le h+\bar{\mu}X.$$

Using the distortion $(X,Y)\to(\xi,\zeta)$ previously introduced in (15), and assuming, moreover, that $\bar{\mu}$ is a small parameter of order ε:

$$\bar{\mu}=\varepsilon\mu, \qquad \mu=O(1),$$

the basic equations describing the problem (16) - (18) remain the same, only the condition on the flow bed is changed to become:

$$y=\mu\varepsilon x \quad \text{on the beach with equation } \zeta=h+\mu\xi. \quad (62)$$

We shall require that the velocity field be finite. The form of the solution is as before. The solutions to the different orders of approximation will be exposed without details.

1. **Approximation** $(0,1)$
   It is easy to show that

$$x_{0,1}(\xi,\zeta,\tau)=x_{0,1}(\xi,\tau), \qquad y_{0,1}(\xi,\zeta,\tau)=0. \quad (63)$$

2. **Approximation** $(m,1), m>0$
   The solutions satisfying the equations in the bulk and the bottom boundary condition are:

$$x_{m,1}(\xi,\zeta,\tau)=L_{m,1}(\xi,\tau)\cos\left[m\lambda'(\xi)(\zeta-h-\mu\xi)\right], \quad (64)$$
$$y_{m,1}(\xi,\zeta,\tau)=L_{m,1}(\xi,\tau)\sin\left[m\lambda'(\xi)(\zeta-h-\mu\xi)\right]. \quad (65)$$

Function $L_{m,1}(\xi,\tau)$ may be interpreted as the horizontal component of displacement on the bottom at this order of approximation. The condition on the free surface yields:

$$\lambda'(\xi)=\frac{\pi}{h+\mu\xi}. \quad (66)$$

This, together with the condition $\lambda(0)=0$, gives the following expression for λ:

$$\lambda(\xi)=\frac{\pi}{\mu}\ln\frac{h+\mu\xi}{h}=\ln\left(\frac{h+\mu\xi}{h}\right)^{\frac{\pi}{\mu}}. \quad (67)$$

Therefore:

$$x_{m,1}(\xi,\zeta,\tau)=(-1)^m L_{m,1}(\xi,\tau)\cos\frac{m\pi\zeta}{h+\mu\xi}, \quad (68)$$
$$y_{m,1}(\xi,\zeta,\tau)=(-1)^m L_{m,1}(\xi,\tau)\sin\frac{m\pi\zeta}{h+\mu\xi}. \quad (69)$$

Summing up, the solution at order 1 may be written as:

$$x_1(\xi,\zeta,\tau)=x_{0,1}(\xi,\tau)+\sum_{k=1}^{\infty}(-1)^m L_{m,1}(\xi,\tau)\cos\frac{m\pi\zeta}{h+\mu\xi}\left(\frac{h+\mu\xi}{h}\right)^{-m\pi/\mu\varepsilon}, \quad (70)$$

$$y_1(\xi,\zeta,\tau)=\sum_{k=1}^{\infty}(-1)^m L_{m,1}(\xi,\tau)\sin\frac{m\pi\zeta}{h+\mu\xi}\left(\frac{h+\mu\xi}{h}\right)^{-m\pi/\mu\varepsilon}. \quad (71)$$

3. **Approximation** $(0,2)$
It is straightforward to find out that the solution $x_{0,1}$ at this order of approximation satisfies the wave equation:

$$\frac{\partial^2 x_{0,1}}{\partial \gamma^2} - \frac{\partial^2 x_{0,1}}{\partial \tau^2} + \frac{3}{\gamma}\frac{\partial x_{0,1}}{\partial \gamma} = 0, \tag{72}$$

where the following transformation of independent variables $\xi \to \gamma$ was introduced:

$$\gamma = -\frac{2h}{\mu}\sqrt{1+\frac{\mu\xi}{h}}. \tag{73}$$

Again,

$$y_{0,2}(\xi,\zeta,\tau) = \mu x_{0,1}(\xi,\tau) + (h+\mu\xi-\zeta)\frac{\partial x_{0,1}}{\partial \xi}. \tag{74}$$

The equation for v becomes:

$$\frac{\partial^2 v}{\partial \gamma^2} - \frac{\partial^2 v}{\partial \tau^2} - \frac{3}{4}\frac{1}{\gamma^2} v = 0. \tag{75}$$

Use separation of variables to finally get:

$$x_{0,1} = \frac{1}{\gamma} J_1(\beta\gamma)\,(A\cos\beta\tau + B\sin\beta\tau). \tag{76}$$

4. **Approximation** $(m,2), m > 0$ Straightforward calculations lead to a particular solution of the form:

$$\begin{aligned} x_{m,2} &= L_{m,2}(\xi,\tau)\cos m\lambda'(\xi)\zeta \\ &- \frac{\mu\lambda'(\xi)^2}{\pi^2}\zeta L_{m,1}(\xi,\tau)\sin m\lambda'(\xi)\zeta \\ &- \frac{m\mu\lambda'(\xi)^3}{2\pi^2}\zeta^2 L_{m,1}(\xi,\tau)\cos m\lambda'(\xi)\zeta, \\ y_{m,2} &= L_{m,2}(\xi,\tau)\sin m\lambda'(\xi)\zeta \\ &+ \frac{\mu\lambda'(\xi)^2}{\pi^2}\zeta L_{m,1}(\xi,\tau)\cos m\lambda'(\xi)\zeta \\ &- \frac{m\mu\lambda'(\xi)^3}{2\pi^2}\zeta^2 L_{m,1}(\xi,\tau)\sin m\lambda'(\xi)\zeta, \end{aligned} \tag{77}$$

so that the solution to order 2 is:

$$\begin{aligned} x_2(\xi,\zeta,\tau) &= x_{0,2}(\xi,\tau) + \sum_{m=1}^{\infty} x_{m,2}(\xi,\tau)\left(\frac{h+\mu\xi}{h}\right)^{m\pi/\mu\varepsilon}, \\ y_2(\xi,\zeta,\tau) &= y_{0,2}(\xi,\zeta,\tau) + \sum_{m=1}^{\infty} y_{m,2}(\xi,\tau)\left(\frac{h+\mu\xi}{h}\right)^{m\pi/\mu\varepsilon}. \end{aligned} \tag{78}$$

2.3.3 Wave propagation over a beach

We begin by writing down the expressions for the solution to the first two orders of approximation in the small parameter ε in the flow regions:

$$x_1^-(\xi,\zeta,\tau) = U_i(\xi-\tau) + U_r(\xi+\tau) + \sum_{m=1}^{\infty} K_{m,1}^-(\tau)\cos m\frac{\pi}{h}(\zeta-h)\,e^{m\pi\xi/\varepsilon h}, \tag{79}$$

$$y_1^-(\xi,\zeta,\tau) = -\sum_{m=1}^{\infty} K_{m,1}^-(\tau)\sin m\frac{\pi}{h}(\zeta-h)\,e^{m\pi\xi/\varepsilon h} \tag{80}$$

$$x_2^-(\xi,\zeta,\tau) = \sum_{m=1}^{\infty} K_{m,2}^-(\tau)\cos m\frac{\pi}{h}(\zeta-h)\,e^{m\pi\xi/\varepsilon h}, \tag{81}$$

$$y_2^-(\xi,\zeta,\tau) = -(\zeta-h)\left[U_i'(\xi-\tau)+U_r'(\xi+\tau)\right] - \sum_{m=1}^{\infty} K_{m,2}^-(\tau)\sin m\frac{\pi}{h}(\zeta-h)\,e^{m\pi\xi/\varepsilon h}. \tag{82}$$

and

$$x_1^+(\xi,\zeta,\tau) = F(\xi,\tau) + \sum_{k=1}^{\infty}\frac{K_{m,1}^+(\tau)}{h+\mu\xi}\cos\frac{m\pi(\zeta-h-\mu\xi)}{h+\mu\xi}\left(\frac{h+\mu\xi}{h}\right)^{-m\pi/\mu\varepsilon}, \tag{83}$$

$$y_1^+(\xi,\zeta,\tau) = \sum_{k=1}^{\infty}\frac{K_{m,1}^+(\tau)}{h+\mu\xi}\sin\frac{m\pi(\zeta-h-\mu\xi)}{h+\mu\xi}\left(\frac{h+\mu\xi}{h}\right)^{-m\pi/\mu\varepsilon}, \tag{84}$$

$$x_2^+(\xi,\zeta,\tau) = \sum_{k=1}^{\infty}\frac{K_{m,2}^+(\tau)}{h+\mu\xi}\cos\frac{m\pi(\zeta-h-\mu\xi)}{h+\mu\xi}\left(\frac{h+\mu\xi}{h}\right)^{-m\pi/\mu\varepsilon}, \tag{85}$$

$$y_2^+(\xi,\zeta,\tau) = \mu F + (h+\mu\xi-\zeta)\frac{\partial F}{\partial \xi} + \sum_{k=1}^{\infty}\frac{K_{m,2}^+(\tau)}{h+\mu\xi}\sin\frac{m\pi(\zeta-h-\mu\xi)}{h+\mu\xi}\left(\frac{h+\mu\xi}{h}\right)^{-m\pi/\mu\varepsilon}, \tag{86}$$

where

$$F(\xi,\tau) = \int_0^{\infty}\left[A(\sigma)\cos(\sigma\tau)+B(\sigma)\sin(\sigma\tau)\right]\frac{J_1\left(-\frac{2h}{\mu}\sqrt{1+\frac{\mu\xi}{h}}\,\sigma\right)}{\frac{-2h}{\mu}\sqrt{1+\frac{\mu\xi}{h}}}\,d\sigma. \tag{87}$$

One has

$$\frac{\partial F}{\partial \xi} = \int_0^\infty [A(\sigma)\cos(\sigma\tau)+B(\sigma)\sin(\sigma\tau)] \left[\frac{\mu\sigma}{2h}\frac{J_0\left(-\frac{2h}{\mu}\sqrt{1+\frac{\mu\xi}{h}}\,\sigma\right)}{1+\frac{\mu\xi}{h}}+\frac{\mu^2}{2h^2}\frac{J_1\left(-\frac{2h}{\mu}\sqrt{1+\frac{\mu\xi}{h}}\,\sigma\right)}{\left(1+\frac{\mu\xi}{h}\right)^{3/2}}\right]d\sigma. \tag{88}$$

The function U_i is given, while the unknowns to be determined in the above equations from the continuity of the solution at $\xi=0$ are $U_r, K^{\pm}_{m,1}, K^{\pm}_{m,2}, A, B$.

It is easy to show that continuity of the solution at $\xi=0$ at the first order of approximation yields the results:

$$K^-_{m,1}=K^+_{m,1}=0, \qquad m=1,2,\cdots \tag{89}$$

and

$$U_r(\tau)=F(0,\tau)-U_i(-\tau). \tag{90}$$

This last relation determines the reflected wave, once function F has been obtained. Differentiation of (90) w.r.to t gives:

$$U_r'(\tau)=\int_0^\infty[-\sigma A(\sigma)\sin(\sigma\tau)+\sigma B(\sigma)\cos(\sigma\tau)]\frac{J_1\left(\frac{-2h\sigma}{\mu}\right)}{\frac{-2h}{\mu}}d\sigma+U_i'(-\tau). \tag{91}$$

Continuity at the second order of approximation gives:

$$K^-_{m,2}(\tau)=\frac{1}{h}K^+_{m,2}(\tau) \tag{92}$$

and

$$-(\zeta-h)\left[U_i'(-\tau)+U_r'(\tau)\right] - \sum_{m=1}^{\infty}K^-_{m,2}(\tau)\sin m\frac{\pi}{h}(\zeta-h)= \mu F(0,\tau)+(h-\zeta)\left.\frac{\partial F}{\partial \xi}\right|_{\xi=0}+\sum_{k=1}^{\infty}\frac{K^+_{m,2}(\tau)}{h}\sin\frac{m\pi(\zeta-h)}{h}.$$

Now use the expansion

$$\zeta=-\sum_{m=1}^{\infty}\frac{2h}{m\pi}\sin m\frac{\pi}{h}(\zeta-h) \tag{93}$$

and the orthogonality property of trigonometric functions to get:

$$\mu F(0,\tau)+h\left.\frac{\partial F}{\partial \xi}\right|_{\xi=0}=G(\tau), \tag{94}$$

$$K^-_{m,2}(\tau)=\frac{K^+_{m,2}(\tau)}{h}=\frac{1}{m\pi}G(\tau)-\frac{h}{m\pi}\left.\frac{\partial F}{\partial \xi}\right|_{\xi=0}, \quad m=1,2,\cdots, \tag{95}$$

where

$$G(\tau)=h\left[U_i'(-\tau)+U_r'(\tau)\right]= h\left[2U_i'(-\tau)+\int_0^\infty[-\sigma A(\sigma)\sin(\sigma\tau)+\sigma B(\sigma)\cos(\sigma\tau)]\frac{J_1\left(-\frac{2h\sigma}{\mu}\right)}{\frac{-2h}{\mu}}d\sigma\right] \tag{96}$$

Now

$$F(0,\tau)=\int_0^\infty[A(\sigma)\cos(\sigma\tau)+B(\sigma)\sin(\sigma\tau)]\frac{J_1\left(-\frac{2h\sigma}{\mu}\right)}{\frac{-2h}{\mu}}d\sigma$$

$$\left.\frac{\partial F}{\partial \xi}\right|_{\xi=0}= \int_0^\infty[A(\sigma)\cos(\sigma\tau)+B(\sigma)\sin(\sigma\tau)] \left[\frac{\mu\sigma}{2h}J_0\left(\frac{-2h\sigma}{\mu}\right)+\frac{\mu^2}{2h^2}J_1\left(\frac{-2h\sigma}{\mu}\right)\right]d\sigma.$$

Extend the definition of function U_i' to all real values of times and let:

$$G_1(\tau)=\frac{1}{2}\left[U_i'(\tau)+U_i'(-\tau)\right], \quad G_2(\tau)=\frac{1}{2}\left[U_i'(\tau)-U_i'(-\tau)\right]. \tag{97}$$

Equation (94) will be satisfied if one chooses:

$$\int_0^\infty\frac{\mu\sigma}{4h}\left[A(\sigma)J_0\left(\frac{-2h\sigma}{\mu}\right)+B(\sigma)J_1\left(\frac{-2h\sigma}{\mu}\right)\right]\cos(\sigma\tau)d\sigma =G_1(\tau), \tag{98}$$

$$\int_0^\infty\frac{\mu\sigma}{4h}\left[B(\sigma)J_0\left(\frac{-2h\sigma}{\mu}\right)-A(\sigma)J_1\left(\frac{-2h\sigma}{\mu}\right)\right]\sin(\sigma\tau)d\sigma =G_2(\tau), \tag{99}$$

from which one obtains by inversion of Fourier sine and cosine transforms:

$$A(\sigma)J_0\left(\frac{-2h\sigma}{\mu}\right)+B(\sigma)J_1\left(\frac{-2h\sigma}{\mu}\right) =\frac{8h}{\pi\mu\sigma}\int_0^\infty G_1(\tau)\cos(\sigma\tau)d\tau, \tag{100}$$

$$B(\sigma)J_0\left(\frac{-2h\sigma}{\mu}\right)-A(\sigma)J_1\left(\frac{-2h\sigma}{\mu}\right) =\frac{8h}{\pi\mu\sigma}\int_0^\infty G_2(\tau)\sin(\sigma\tau)d\tau. \tag{101}$$

The solution of this system of linear algebraic equations yields the values of $A(\sigma), B(\sigma)$:

$$A(\sigma)=\frac{8h}{\pi\mu\sigma}\frac{1}{\Delta(\sigma)}\left[J_1\left(\frac{-2h\sigma}{\mu}\right)\int_0^\infty G_1(\tau)\cos(\sigma\tau)d\tau+ J_0\left(\frac{-2h\sigma}{\mu}\right)\int_0^\infty G_2(\tau)\cos(\sigma\tau)d\tau\right], \tag{102}$$

$$B(\sigma)=\frac{8h}{\pi\mu\sigma}\frac{1}{\Delta(\sigma)}\left[J_0\left(\frac{2h\sigma}{\mu}\right)\int_0^\infty G_1(\tau)\cos(\sigma\tau)d\tau- J_1\left(\frac{2h\sigma}{\mu}\right)\int_0^\infty G_2(\tau)\cos(\sigma\tau)d\tau\right], \tag{103}$$

where

$$\Delta(\sigma) = \left[J_0\left(\frac{-2h\sigma}{\mu}\right)\right]^2 + \left[J_1\left(\frac{-2h\sigma}{\mu}\right)\right]^2. \quad (104)$$

This completes the solution of the problem.

To find the flow free surface, we set $\zeta = 0$ in the above equations to get:

$$x_1^-(\xi,0,\tau) = U_i(\xi-\tau) + U_r(\xi+\tau), \quad (105)$$

$$y_1^-(\xi,0,\tau) = 0 \quad (106)$$

$$x_2^-(\xi,0,\tau) = \sum_{m=1}^{\infty} K_{m,2}^-(\tau)(-1)^m e^{m\pi\xi/\varepsilon h}, \quad (107)$$

$$y_2^-(\xi,0,\tau) = h\left[U_i'(\xi-\tau) + U_r'(\xi+\tau)\right] \quad (108)$$

and

$$x_1^+(\xi,0,\tau) = F(\xi,\tau), \quad (109)$$

$$y_1^+(\xi,0,\tau) = 0, \quad (110)$$

$$x_2^+(\xi,0,\tau) = \sum_{k=1}^{\infty} \frac{K_{m,2}^+(\tau)}{h+\mu\xi}(-1)^m\left(\frac{h+\mu\xi}{h}\right)^{-m\pi/\mu\varepsilon} \quad (1,11)$$

$$y_2^+(\xi,0,\tau) = \mu F + (h+\mu\xi)\frac{\partial F}{\partial \xi}. \quad (112)$$

3 Results

In this section, we show the numerical results of the free surface of the propagating waves over the beach. For the numerical application, we have taken

$$h = 1m, \quad \mu = -10, -11, -12$$

and

$$U_i(\xi,\tau) = 1 - \tanh\frac{3}{4}(\xi-\tau). \quad (113)$$

For this case $G_2(\tau) = 0$.

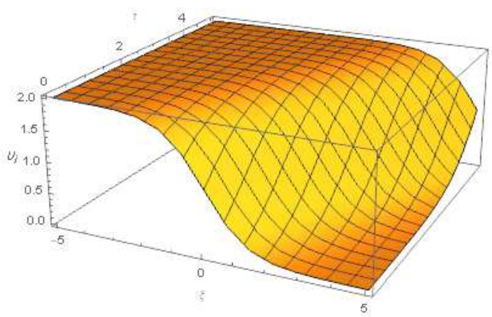


Fig. 3: The incident wave $U_i(\xi,\tau)$.

The incident wave is described in the space by Fig. 3. We observe the location $\xi = 0$ for different times as

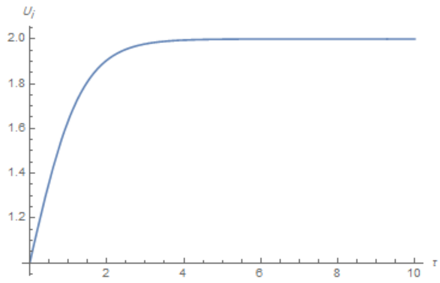


Fig. 4: The incident wave $U_i(0,\tau)$.

illustrated on Fig. 4. Its amplitude increases from zero to a saturation level of value 2.

We have represented on Fig. 5 the reflected wave as observed from the location $\xi = 0$. It is noticed that the amplitude at $\tau = 0$ is less than 2, which means that part of the incident wave energy has been transfered to the fluid to produce the local oscillations. The energy loss of the incident wave is lower as the beach is steeper. Calculations have shown that there is a value for $|\mu| \simeq 8$ below which unphysical results are obtained as the energy of the reflected wave becomes larger than that of the incident wave.

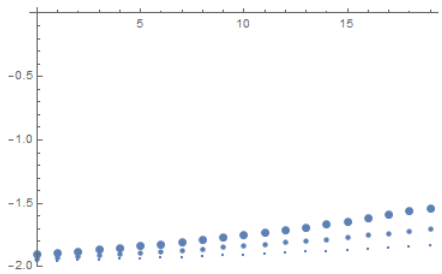


Fig. 5: The reflected wave at $\xi = 0$ for: $\mu = -0.7$ (large dots), $\mu = -0.5$ (medium dots), $\mu = -0.3$ (small dots).

Because the problem reduces at each order of approximation to a set of linear problems, we would like here to examine the shape of the free surface due to the propagating incident wave only. One has:

$$x = \frac{\xi}{\varepsilon} + \varepsilon U_i(\xi-\tau) = \frac{\xi}{\varepsilon} + \varepsilon\left[1 - \tanh\frac{3}{4}(\xi-\tau)\right], \quad (114)$$

$$y = 1 \quad (115)$$

Fig. 6 illustrates the shape of the free surface over the inclined beach at consecutive time moments $\tau = \{-5,-4,-3,-2,-1,0,1,2,3\}$. Fig. 7 shows the shapes of the free surface at three locations $\xi = \{0.3, 0.9, 1.5\}$ on the inclined part of the beach as functions of time.

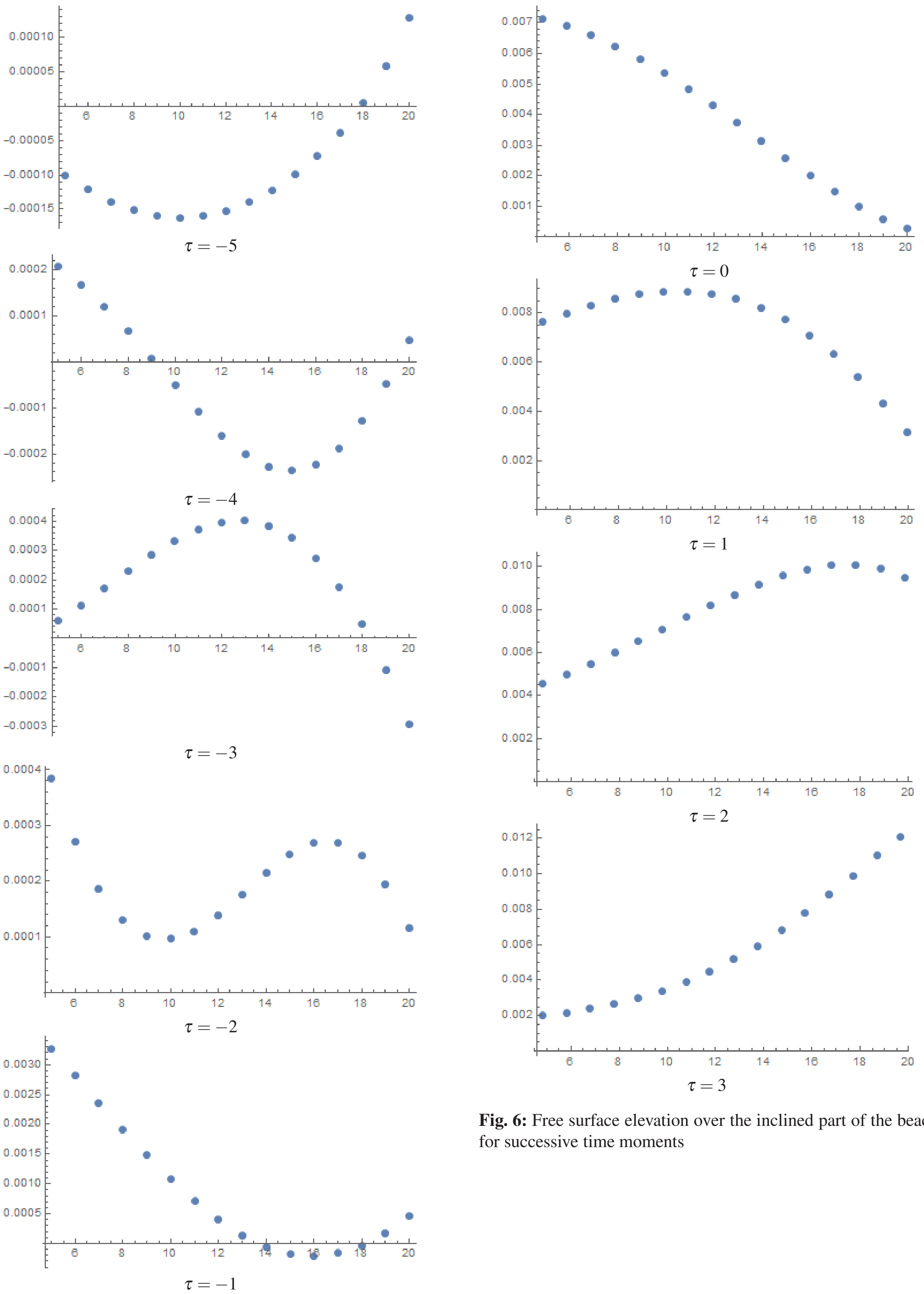
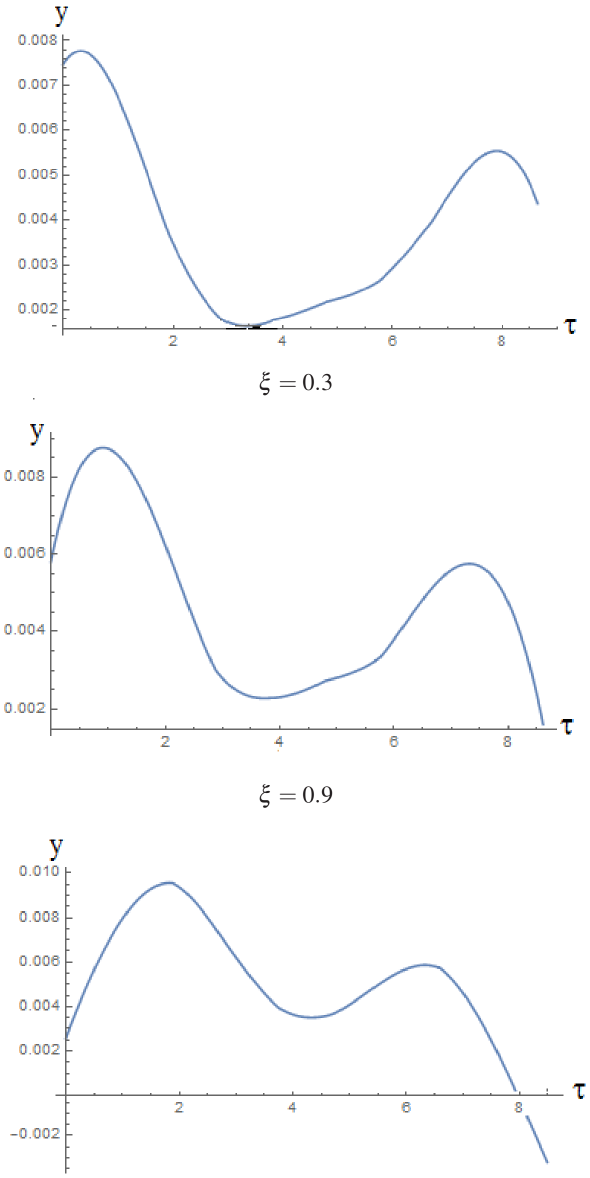


Fig. 6: Free surface elevation over the inclined part of the beach for successive time moments



$\xi = 0.3$

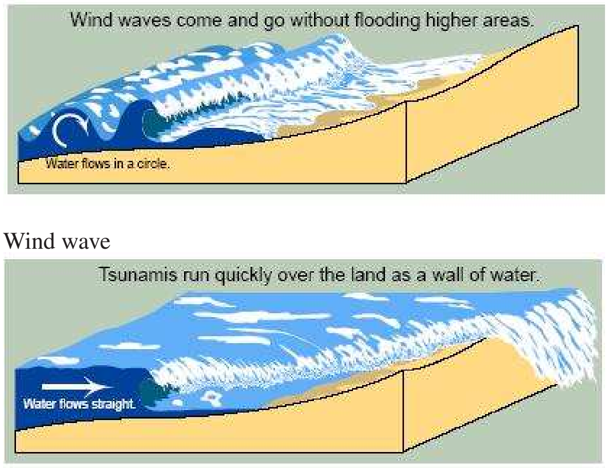
$\xi = 0.9$

$\xi = 1.5$

Fig. 7: Free surface elevation at three locations at the inclined part of the beach as functions of time.

4 Discussion

This last section is devoted to a brief discussion on modeling tsunamis. The tsunami wave differs from the surface wave because at some stage it looks like a wall that attacks the beach. Away from the shore, however, tsunamis may have long wavelengths, reaching hundreds of kilometers, while water depth is much smaller, a few kilometers at most. Much smaller are the wave amplitudes, a few meters at most. Thus, the horizontal length scale is much larger than the vertical one, the usual conditions for shallow water are satisfied in this case.



Wind wave

Tsunami near a beach

Fig. 8: Wind waves vs. Tsunamis near a beach.

In the final phases of a tsunami, i.e. close to the shore, more complicated phenomena must be taken into account, as illustrated on Fig. 8.

It is now widely recognized that the propagation of tsunamis has to be described mathematically by nonlinear equations, which is compatible with the shallow water approximation (C.f. [57]). For long time, there was a strong conviction among physicists and mathematicians that tsunamis can be described by the KdV equation. But the data of Sumatra's tsunami of 2004 did not match with the outcomes of the KdV equation. Since that time, researchers could not decide on a concrete model that describes tsunamis. The Green-Naghdi equations and different types of Boussinecq equations are now being put forward as the most convincing models for tackling the problem of propagation of tsunamis [45,49,54,58]. It is thought that more work still needs to be done in this field, especially in what concerns the nonlinear aspects of tsunami propagation.

Accordingly, the modeling of tsunamis far away from the beach can be performed through the approach presented above. The use of shallow water theory in conjunction with the Lagrangian description of the motion brings in nonlinear ingredients needed for an efficient modeling.

5 Conclusions

In this work we have investigated the propagation of waves on a beach consisting of a horizontal bed and a uniformly sloping bottom within a Lagrangian description performed along the guidelines formulated by Germain (c.f. [17, 18]). The model relies on the expansion of the solution in asymptotic double series combining a small parameter and local oscillations localized at the vertical line separating the horizontal and the variable-depth parts of the flow bed. The nonlinearity of the free-surface condition is fully retained.

The case of a beach of appreciable inclination to the horizontal has shown that the incident wave is fully reflected and no local oscillations are formed. For a uniformly sloping beach of small inclination to the horizontal, local oscillations are created and therefore the energy of the reflected wave is less than that of the incoming wave. Explicit formulae are obtained for the reflected wave amplitude and for the local oscillations. The numerical application concerns the deformation of a solitary wave approaching a beach. The given figures show the dependence of the reflected wave characteristics on the beach slope, and the deformation of the free surface over the beach. The free surface height as observed at three locations on the beach are also given. A brief discussion of tsunami modeling is included in a final section, showing that the proposed approach can be used for tsunami modeling far away from the beach.

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A Useful summations

The following summations should be understood in a generalized sense.

$$\sum_{m=1}^{\infty} \sin mx = \frac{1}{2} \cot \frac{x}{2}, \tag{A.1}$$

$$\sum_{m=1}^{\infty} \cos mx = \pi \sum_{k=-\infty}^{\infty} \delta(x - 2\pi k) - \frac{1}{2} \tag{A.2}$$