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Characterizations and Testing NBRUL Class of Life Distributions Based on Laplace Transform Technique

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Abstract: In this paper, we investigate the probabilistic characteristics for the new better than renewal used in laplace transform order class, the closure properties under various reliability operations such as convolution, mixture, mixing, homogeneous Poisson shock model are studied. Moreover, a new hypothesis test is constructed to test exponentiality against the new better than renewal used in laplace transform order based on Laplace transform technique. Pitman's asymptptic efficiency of the test are studied and compared with other tests. Furthermore, the powers of these test are calculated for some commonly used distributions in reliability such as Linear Failure Rate, Gamma and Weibull distributions. Finally, sets of real data are used as a practical applications of the proposed test.

Keywords: Convolution; Mixture; Mixing; Poisson shock model; Laplace transform order.

ACRONYMS

$DLTTF$	Decreasing Laplace transform of time to failure
$HNBUE(HWBUE)$	Harmonic new better (worse) than used in expectation
$IFR(DFR)$	Increasing (decreasing) failure rate
$IFRA(DFRA)$	Increasing (decreasing) failure rate average
$NBAFR$	New better than average failure rate
$NBRU(NWRU)$	New better (worse) than renewal used
$NBRUE$	New better than renewal used in expectation
$NBRUL(NWRUL)$	New better (worse) than renewal used in Laplace transform
$NBU(NWU)$	New better (worse) than used
$NBUE(NWUE)$	New better (worse) than used in expectation
$NBURFR-t_0$	New better than used renewal failure rate at specific age
ODL_{lt}	Overall decreasing life in Laplace transform order
$RNBU_{mgf}$	Renewal new better than used in moment generating function

1 Introduction

Certain classes of life distributions and their different versions have been introduced in reliability, the applications of these classes of life distributions can be seen in biological science, biometrics, engineering, maintenance, and social, and also there are many aging criteria in reliability engineering that describe the deterioration of coherent engineering systems or their units. These criteria are useful for maintenance engineers and design in constructing optimal maintenance policies. Stochastic comparisons between probability distributions also play a basic role in probability, statistics, and some related areas, such as reliability theory, survival analysis, economics, and actuarial science. Therefore, statisticians and reliability analysts have shown a growing interest in modelling survival data using classifications of life distributions based on some aspects of ageing.

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During the past decades, various classes of life distributions have been proposed in order to model different aspects of aging. The best known of these classes are IFA, IFRA, NBU, NBUE, HNBUE, DMRL and NBRU. Bryson and Siddiqui [1] and Barlow and Proschan [2] present the properties for these aging concepts and their duals DFR, DFRA, NWE, NWUE, IFR and NBU. Klefsjo [3] studied properties for HNBUE and HNWUE classes. Abouammoh et al. [4] studied properties for $NBRU$ class. El-Arishy et al. [5,6] studied characterizations and testing hypotheses for $RNBU_{mgf}$ class and $DLTTF$ class.

Testing exponentiality against some classes of life distributions has been introduced by many researchers. For testing exponentiality versus NBU class see Kumazawa [7]. Hassan et al [8] for $NBRU_{mgf}$ class. Abu-Youssef et al. [9] for $UBAC(2)$ class. Mahmoud et al. [10,11] for NBRUL class and Mahmoud et al [12] for ODL_{lt} class.

Here we will use Laplace transform technique for the class of $NBRUL$, and apply it for testing. Many authors proposed tests for exponentiality versus some classes of life distributions based on Laplace transform order was studied by some authors, Atallah et al. [13], Mahmoud et al. [12], Gadallah [14], Abu-Youssef et al. [15].

The rest of this paper can be organized as follows. In Section 2, we discuss preservation under convolution, mixture, mixing, homogeneous poisson shock model for $NBRUL$ class of life distribution. In Section 3, we present testing exponentiality against NBRUL class based on Laplace transform technique. The Pitman asymptotic for several common alternatives is obtained in Section 4. In section 5, Monte Carlo null distribution critical points and the power estimates are simulated. In section 6, a proposed test is presented for right censored data. Finally, we discuss some application to demonstrate the utility of the proposed statistical test in Section 7.

1.1 Renewal classes

Let X be a random variable represents life time of a device (system or component) with a continuous life distribution $F(t)$. Upon arising the failure of the device, it can be substituted by a sequence of mutually independent devices which are identically distributed with the same life distribution $F(t)$. The following stationary renewal distribution constitutes the remaining life distribution of the device under operation at time t.

$$W_f(x) = \mu^{-1}\int_0^t \overline{F}(y)\,dy, \quad t \geq 0, \tag{1}$$

where $\mu = \int_0^\infty \overline{F}(u)\,du$.
It is easy to show that

$$\overline{W}_f(x) = \mu^{-1}\int_x^\infty \overline{F}(y)\,dy, \quad t \geq 0. \tag{2}$$

For extra details, see Barlow and Proschan [2] and Abouammoh and Ahmed [16]. Now we need to present the definitions of the NBRU (NWRU) and NBRUL (NWRUL) classes of life distributions.

Definition 1.*Abouammoh et al. [17]. If X is a random variable with survival function $\bar{F}(x)$, then X is said to have new better (worse) than renewal used property, denoted by NBRU (NWRU), if*

$$\overline{W_F}(x|t) \leq (\geq)\overline{F}(x|0), \quad x \geq 0, t \geq 0,$$

or

$$\overline{W_F}(x+t) \leq (\geq)\overline{W_F}(t)\,\overline{F}(x), \quad x \geq 0, t \geq 0. \tag{3}$$

Depending on the definition (1), Mahmoud et al. [10] defined a new class which is called new better (worse) than renewal used in Laplace transform order NBRUL (NWRUL) as follows

Definition 2.*X is said to be NBRUL (NWRUL) if*

$$\int_0^\infty e^{-sx}\overline{W_F}(x+t)\,dx \leq (\geq)\overline{W_F}(t)\int_0^\infty e^{-sx}\overline{F}(x)\,dx, \quad x,t,s \geq 0. \tag{4}$$

Using (2), then (4) is equivalent to

$$\int_0^\infty \int_{x+t}^\infty e^{-sx}\overline{F}(u)dudx \leq \int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x)\overline{F}(u)dudx, \tag{5}$$

and (5) is equivalent to

$$\int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x+y)dydx \leq \int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x)\overline{F}(y)dydx, \tag{6}$$

It is obvious that NBRU⇒NBRUL⇒NBRUE.

2 Some Properties of The NBRUL Class

In this section we study the closure properties of the new better than renewal used in laplace transform order under some reliability operations such as convolution, mixture and the shock model in homogeneous case.

2.1 Convolution properties

The aim of this subsection is to discuss preservation under convolution properties of $NBRUL$ class.

Theorem 1.*The $NBRUL$ class is preserved under convolution.*

*Proof.*Suppose that F_1 and F_2 are two independent $NBRUL$ lifetime distribution and their convolution is given by

$$\overline{F}(z) = \int_0^\infty \overline{F_1}(z-u)dF_2(u).$$

Therefore

$$\begin{aligned}\int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x+y)dydx &= \int_0^\infty e^{-sx}\int_t^\infty \int_0^\infty \overline{F}_1(x+y-u)dF_2(u)dydx\\ &= \int_0^\infty \int_0^\infty \int_t^\infty e^{-sx}\overline{F}_1(x+y-u)dydxdF_2(u).\end{aligned}$$

Since $\bar{F}_1$ is $NBRUL$, then

$$\begin{aligned}\int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x+y)dydx &\leq \int_0^\infty \int_0^\infty \int_t^\infty e^{-sx}\overline{F}_1(x)\overline{F}_1(y-u)dydxdF_2(u)\\ &\leq \int_0^\infty e^{-sx}\overline{F}_1(x)\int_t^\infty \int_0^\infty \overline{F}_1(y-u)dF_2(u)dydx\\ &\leq \int_0^\infty e^{-sx}\overline{F}_1(x)\int_t^\infty \overline{F}(y)dydx,\end{aligned}$$

by using $\overline{F}_i(z) \leq \overline{F}(z)$ for $i = 1,2$ we get

$$\int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x+y)dydx \leq \int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x)\overline{F}(y)dydx,$$

which complete the proof.

The following example is presented to show that the $NWRUL$ class is not preserved under convolution.

*Example 1.*The convolution of the exponential distribution $F(u) = 1-e^{-u}$ with itself yields the gamma distribution of order 2: $G(u) = 1-(1+u)e^{-u}$, with strictly increasing failure rate. Thus $G(u)$ is not $NWRUL$.

2.2 Mixture properties

The following theorem is stated and proved to show that the $NWRUL$ class is preserved under mixture.

Theorem 2.*The $NWRUL$ class is preserved under mixture.*

*Proof.*Suppose that $F(x)$ is the mixture of F_α, where each F_α is $NWRUL$ since

$$\overline{F}(x) = \int_0^\infty \overline{F_\alpha}(x)dG(\alpha),$$

then

$$\begin{aligned}\int_0^\infty \int_t^\infty e^{-sx}\overline{F}(x+y)dydx &= \int_0^\infty \int_t^\infty \int_0^\infty e^{-sx}\overline{F}_\alpha(x+y)dG(\alpha)dydx\\ &= \int_0^\infty \int_0^\infty \int_t^\infty e^{-sx}\overline{F}_\alpha(x+y)dydxdG(\alpha),\end{aligned}$$

since F_α is $NWRUL$, then

$$\int_0^\infty\int_0^\infty\int_t^\infty e^{-sx}\overline{F}_\alpha(x+y)dydxdG(\alpha)\geq\int_0^\infty\{\int_0^\infty\int_t^\infty e^{-sx}\overline{F}_\alpha(x)\overline{F}_\alpha(y)dydx\}dG(\alpha).$$

Upon using Chebyshev inequality for similarity ordered functions, we get

$$\begin{aligned}\int_0^\infty\int_t^\infty e^{-sx}\overline{F}(x+y)dydx &\geq\int_0^\infty\int_0^\infty e^{-sx}\overline{F}_\alpha(x)dG(\alpha)dx.\int_t^\infty\int_0^\infty\overline{F}_\alpha(y)dG(\alpha)dy\\ &\geq\int_0^\infty e^{-sx}\overline{F}(x)dx.\int_t^\infty\overline{F}(y)dy\\ &\geq\int_0^\infty\int_t^\infty e^{-sx}\overline{F}(x)\overline{F}(y)dydx,\end{aligned}$$

which complete the proof.

The following example shows that the $NBRUL$ class is not preserved under mixtures.

*Example 2.*Let $\overline{F}_\alpha(x)=e^{-\alpha x}$ and $\overline{G}(x)=\int_0^\infty\overline{F}_\alpha(x)e^{-\alpha}d\alpha=(x+1)^{-1}$. Then the failure rate function is $r_g(x)=(x+1)^{-1}$, which is strictly decreasing thus $\overline{G}(x)$ is not $NBRUL$.

2.3 Mixing properties

The following example illustrates that the NBRUL class is not preserved under mixing.

*Example 3.*Let $\overline{F}_1=e^{-2x}$ and $\overline{F}_2=e^{-3x}$. Let $\overline{F}=\frac{1}{2}\overline{F}_1+\frac{1}{2}\overline{F}_2$,it follows that both $\overline{F}_1$ and $\overline{F}_2$ are $NBRUL$ but $\overline{F}$ is not $NBRUL$.

2.4 Homogeneous poisson shock model

Suppose that a device is subjected to a sequence of shocks occurring randomly in the time according to a Poisson process with intensity λ. Suppose further that the device has probability $\overline{P}_k$ of surviving the first k shock, where $1=\overline{P}_0\geq\overline{P}_1\geq\cdots\cdots$. Denote $p_j=\overline{P}_{j-1}-\overline{P}_j, j\geq 1$. Then the survival function of the device is given by

$$\overline{H}(t)=\sum_{k=0}^{\infty}\overline{P}_k\frac{(\lambda t)^k}{k!}e^{-\lambda t},\qquad t\geq 0. \tag{7}$$

This shock model has been studied by Esary et al. [18] for different ageing properties such as IFR, $IFRA$, NBU, and $NBUE$, Klefsjo [19] for $HNBUE$ and Mahmoud et al. [20] for $NBURFR-t_0$.

Definition 3.*A discrete distribution $p_k, k=0,1,\cdots\cdots,\infty$ is said to have discrete new better (worse) than renewal used in laplace transform order (NBRUL) (NWRUL) if*

$$\sum_{i=0}^{\infty}\sum_{j=l}^{\infty}z^i\overline{p}_{i+j}\leq(\geq)\sum_{i=0}^{\infty}\sum_{j=l}^{\infty}z^i\overline{p}_i\overline{p}_j\quad, 0\leq z\leq 1. \tag{8}$$

Theorem 3.*If $\overline{P}_k$ is discrete NBRUL, then $\overline{H}(t)$ given by (7) is NBRUL.*

*Proof.*It must be shown that

$$\int_0^\infty\int_t^\infty e^{-sx}\overline{H}(x+y)dydx\leq\int_0^\infty\int_t^\infty e^{-sx}\overline{H}(x)\overline{H}(y)dydx.$$

Upon using (7), we get

$$\begin{aligned}\int_0^\infty\int_t^\infty e^{-sx}\overline{H}(x+y)dydx &= \int_0^\infty\int_t^\infty e^{-sx}\sum_{m=0}^{\infty}\overline{P}_m\frac{[\lambda(x+y)]^m}{m!}e^{-\lambda(x+y)}dydx\\ &= \int_0^\infty e^{-sx}e^{-\lambda x}\int_t^\infty\sum_{m=0}^{\infty}\overline{P}_m\frac{\lambda^{m-n+n}}{m!}\sum_{n=0}^{m}\binom{m}{n}x^{m-n}y^n e^{-\lambda y}dydx\\ &= \int_0^\infty e^{-x(s+\lambda)}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\overline{P}_m\frac{(\lambda.x)^{m-n}}{n!(m-n)!}\int_t^\infty(\lambda.y)^n e^{-\lambda y}dydx.\\ &= \sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\overline{P}_m\frac{e^{-\lambda t}}{n!(m-n)!}.\frac{(m-n)!}{\lambda(s+\lambda)}.[\frac{\lambda}{s+\lambda}]^{m-n}\sum_{l=0}^{n}\frac{n!(\lambda.t)^l}{l!}\\ &= \sum_{l=0}^{n}\sum_{n=0}^{\infty}\sum_{m=n}^{\infty}\overline{P}_m\frac{e^{-\lambda t}}{\lambda(s+\lambda)}[\frac{\lambda}{s+\lambda}]^{m-n}.\frac{(\lambda.t)^l}{l!},\end{aligned}$$

let $i=m-n$,

$$\begin{aligned}\int_0^\infty\int_t^\infty e^{-sx}\overline{H}(x+y)dydx &= \sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\overline{P}_{i+n}\frac{e^{-\lambda t}}{\lambda(s+\lambda)}[\frac{\lambda}{s+\lambda}]^i.\frac{(\lambda.t)^l}{l!}\\ &= \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\overline{P}_{i+n}[\frac{\lambda}{s+\lambda}]^i\frac{e^{-\lambda t}}{\lambda(s+\lambda)}.\frac{(\lambda.t)^l}{l!},\end{aligned}$$

since F is $NBRUL$

$$\begin{aligned}\int_0^\infty\int_t^\infty e^{-sx}\overline{H}(x+y)dydx &\le \sum_{i=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\overline{P}_i\overline{P}_n[\frac{\lambda}{s+\lambda}]^i\frac{e^{-\lambda t}}{\lambda(s+\lambda)}.\frac{(\lambda.t)^l}{l!}\\ &= \sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{n}\overline{P}_i\overline{P}_n[\frac{\lambda}{s+\lambda}]^i\frac{e^{-\lambda t}}{\lambda(s+\lambda)}.\frac{(\lambda.t)^l}{l!}\\ &= \sum_{i=0}^{\infty}\overline{P}_i\sum_{n=0}^{\infty}\overline{P}_n\sum_{l=0}^{n}\frac{(\lambda.t)^l}{l!}.\frac{e^{-\lambda t}}{\lambda}.\frac{1}{(s+\lambda)}[\frac{\lambda}{s+\lambda}]^{m-n}\\ &= \int_0^\infty e^{-sx}\sum_{i=0}^{\infty}\overline{P}_i\frac{(\lambda.x)^{m-n}}{(m-n)!}e^{-\lambda x}.\int_t^\infty\sum_{n=0}^{\infty}\overline{P}_n\frac{(\lambda.y)^n}{n!}e^{-\lambda y}dydx\\ &= \int_0^\infty e^{-sx}\sum_{i=0}^{\infty}\overline{P}_i\frac{(\lambda.x)^i}{i!}e^{-\lambda x}.\int_t^\infty\sum_{n=0}^{\infty}\overline{P}_n\frac{(\lambda.y)^n}{n!}e^{-\lambda y}dydx\\ &= \int_0^\infty e^{-sx}\overline{H}(x).\int_t^\infty\overline{H}(y)dydx.\end{aligned}$$

The proof for the NWRUL class is obtained by reversing the inequality.

3 Testing Against NBRUL Alternatives

In this section, we test the null hypothesis $H_0 : F$ is exponential against $H_1 : F$ is NBRUL class and not exponential. The following lemma is needed.

Lemma 1. *Let X be a NBRUL random variable with distribution function F, then, based on Laplace transform technique,*

$$(s^2-s\beta)\phi(\beta)\phi(s)+(s^2\beta-s\beta^2)\mu\phi(s)+(s\beta-s^2)\phi(s)+s\beta\phi(\beta)-s\beta\ge\beta^2\phi(s)-\beta^2, s,\beta\ge 0, s\ne\beta. \quad (9)$$

where

$$\phi(s)=Ee^{-sX}=-\int_0^\infty e^{-sx}d\overline{F}(x)$$

*Proof.*since F is NBRUL then

$$\int_0^\infty e^{-sx}\overline{W_F}(x+t)\,dx \le \overline{W_F}(t)\int_0^\infty e^{-sx}\overline{F}(x)\,dx,, \quad x,t \ge 0.$$

Consider the following integral

$$\int_0^\infty \int_0^\infty e^{-\beta t} e^{-sx}\overline{W_F}(x+t)\,dxdt \le \int_0^\infty e^{-\beta t}\overline{W_F}(t)\int_0^\infty e^{-sx}\overline{F}(x)dxdt. \tag{10}$$

Setting

$$I_1 = \int_0^\infty \int_0^\infty e^{-\beta t} e^{-sx}\overline{W_F}(x+t)\,dxdt,$$

hence

$$\begin{aligned} I_1 &= \int_0^\infty \int_v^\infty e^{-\beta v} e^{-s(u-v)}\overline{W_F}(u)\,dudv \\ &= \int_0^\infty \int_0^v e^{-\beta u} e^{-s(v-u)}\overline{W_F}(v)\,dudv \\ &= \frac{1}{\beta - s}[\int_0^\infty e^{-sv}\overline{W_F}(v)\,dv - \int_0^\infty e^{-\beta v}\overline{W_F}(v)\,dv], s \ne \beta \end{aligned}$$

Note that

$$\begin{aligned} \int_0^\infty e^{-sv}\overline{W_F}(v)\,dv &= \mu_F^{-1}\int_0^\infty e^{-sv}\int_v^\infty \overline{F}(y)\,dydv \\ &= \frac{\mu_F^{-1}}{s}[\mu - \frac{1}{s}(1-\phi(s))], \end{aligned}$$

therefore

$$I_1 = \frac{1}{\beta - s}\{\frac{\mu_F^{-1}}{s}[\mu - \frac{1}{s}(1-\phi(s))] - \frac{\mu_F^{-1}}{\beta}[\mu - \frac{1}{\beta}(1-\phi(\beta))]\}, s \ne \beta. \tag{11}$$

Setting

$$\begin{aligned} I_2 &= \int_0^\infty e^{-\beta t}\overline{W_F}(t)\int_0^\infty e^{-sx}\overline{F}(x)dxdt, \quad \text{gives} \\ I_2 &= E\int_0^\infty e^{-\beta t}\overline{W_F}(t)\int_0^\infty e^{-sx} I(X > x)dxdt. \\ &= \frac{1}{s}(1-\phi(s))\int_0^\infty e^{-\beta t}\overline{W_F}(t)\,dt \end{aligned}$$

therefore

$$I_2 = \mu_F^{-1}[\frac{1}{\beta}\mu - \frac{1}{\beta^2}(1-\phi(\beta))][\frac{1}{s}(1-\phi(s))]. \tag{12}$$

Substituting (11) and (12) into (10), we get

$$(s^2 - s\beta)\phi(\beta)\phi(s) + (s^2\beta - s\beta^2)\mu\phi(s) + (s\beta - s^2)\phi(s) + s\beta\phi(\beta) - s\beta \ge \beta^2\phi(s) - \beta^2.$$

This completes the proof.

Let $X_1, X_2, \cdots, X_n$ be a random sample from a population with distribution F. Using the pervious Lemma 3.1 and $\delta(s,\beta)$, as a measure of departure from exponentiality as follows:

$$\begin{aligned} \delta(s,\beta) &= (s^2 - s\beta)\phi(\beta)\phi(s) + (s^2\beta - s\beta^2)\mu\phi(s) \\ &\quad + (s\beta - s^2 - \beta^2)\phi(s) + s\beta\phi(\beta) + \beta^2 - s\beta. \end{aligned} \tag{13}$$

Note that under $H_0, \delta(s,\beta) = 0$, while under $H_1, \delta(s,\beta) > 0$. The empirical estimate $\hat{\delta}(s,\beta)$ of $\delta(s,\beta)$ can be obtained as

$$\begin{aligned} \hat{\delta}(s,\beta) &= \frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n [(s^2 - s\beta)e^{-\beta X_i}e^{-sX_j} + (s^2\beta - s\beta^2)X_i e^{-sX_j} \\ &\quad + (s\beta - s^2 - \beta^2)e^{-sX_i} + s\beta e^{-\beta X_i} + \beta^2 - s\beta]. \end{aligned}$$

To make the test invariant, let $\Delta(s,\beta)=\frac{\delta(s,\beta)}{\mu}$ which estimated by $\hat{\Delta}(s,\beta)=\frac{\hat{\delta}(s,\beta)}{\overline{X}}$ where $\overline{X}$ is the sample mean. Then

$$\hat{\Delta}(s,\beta)=\frac{1}{n^2\overline{X}}\sum_{i=1}^{n}\sum_{j=1}^{n}[(s^2-s\beta)e^{-\beta X_i}e^{-sX_j}+(s^2\beta-s\beta^2)X_ie^{-sX_j}$$
$$+(s\beta-s^2-\beta^2)e^{-sX_i}+s\beta e^{-\beta X_i}+\beta^2-s\beta].$$

One can note that $\hat{\Delta}(s,\beta)$ is an unbiased estimator of $\delta(s,\beta)$.

Now, set

$$\phi(X_i,X_j)=(s^2-s\beta)e^{-\beta X_i}e^{-sX_j}+(s^2\beta-s\beta^2)X_ie^{-sX_j}$$
$$+(s\beta-s^2-\beta^2)e^{-sX_i}+s\beta e^{-\beta X_i}+\beta^2-s\beta,$$

and define the symmetric kernel

$$\psi_s(X_i,X_j)=\frac{1}{2!}\sum\phi_s(X_i,X_j),$$

where the summation over all arrangements of X_i,X_j, then $\hat{\Delta}(s,\beta)$ in (14) is equivalent to the U_n statistic given by

$$U_n=\frac{1}{\binom{n}{2}}\sum_{i<j}^{n}\psi_s(X_i,X_j). \tag{14}$$

The asymptotic normality of $\hat{\Delta}(s,\beta)$ can be summarized in the following theorem.

Theorem 4.*(i) As $n\to\infty$, $\sqrt{n}(\hat{\Delta}(s,\beta)-\Delta(s,\beta))$ is asymptotically normal with mean* 0 *and variance $\sigma^2(s,\beta)$, where*

$$\begin{aligned}\sigma^2(s,\beta)=Var\{&(s^2-s\beta)e^{-\beta X}\phi(s)+(s^2\beta-s\beta^2)X\phi(s)+(s\beta-s^2-\beta^2)e^{-sX}\\&+s\beta e^{-\beta X}+(s^2-s\beta)e^{-sX}\phi(\beta)+(s^2\beta-s\beta^2)\mu e^{-sX}\\&+(s\beta-s^2-\beta^2)\phi(s)+s\beta\phi(\beta)+2\beta^2-2s\beta\}.\end{aligned} \tag{15}$$

(ii) Under H_0, the variance $\sigma_0^2(s,\beta)$ is

$$\sigma_0^2(s,\beta)=\frac{s^4(s-\beta)^2\beta^4(5+s+7\beta+2s\beta+2\beta^2)}{(1+s)^2(1+2s)(1+\beta)^2(1+s+\beta)(1+2\beta)}. \tag{16}$$

*Proof.*Using standard U-statistics theory, see Lee [21], yields

$$\sigma^2=V\{E[\phi(X_1,X_2)\mid X_1]+E[\phi(X_1,X_2)\mid X_2]\}.$$

Using (15) we can find $E[\phi(X_1,X_2)\mid X_1]$ and $E[\phi(X_1,X_2)\mid X_2]$ as follows

$$E(\phi(X_1,X_2)\mid X_1)=(s^2-s\beta)e^{-\beta X}\int_0^\infty e^{-sx}dF(x)+(s^2\beta-s\beta^2)X\int_0^\infty e^{-sx}dF(x)$$
$$+(s\beta-s^2-\beta^2)e^{-sX}+s\beta e^{-\beta X}+\beta^2-s\beta,$$

and

$$E(\phi(X_1,X_2)\mid X_2)=(s^2-s\beta)e^{-sX}\int_0^\infty e^{-\beta x}dF(x)+(s^2\beta-s\beta^2)e^{-sX}\int_0^\infty xdF(x)$$
$$+(s\beta-s^2-\beta^2)\int_0^\infty e^{-sx}dF(x)+s\beta\int_0^\infty e^{-\beta x}dF(x)+\beta^2-s\beta,$$

therefore

$$\begin{aligned}\sigma^2(s,\beta)=Var\{&(s^2-s\beta)e^{-\beta X}\phi(s)+(s^2\beta-s\beta^2)X\phi(s)+(s\beta-s^2-\beta^2)e^{-sX}\\&+s\beta e^{-\beta X}+(s^2-s\beta)e^{-sX}\phi(\beta)+(s^2\beta-s\beta^2)\mu e^{-sX}\\&+(s\beta-s^2-\beta^2)\phi(s)+s\beta\phi(\beta)+2\beta^2-2s\beta\}.\end{aligned}$$

Under H_0

$$\sigma_0^2(s,\beta)=\frac{s^4(s-\beta)^2\beta^4(5+s+7\beta+2s\beta+2\beta^2)}{(1+s)^2(1+2s)(1+\beta)^2(1+s+\beta)(1+2\beta)}.$$

4 The PAE of $\hat{\Delta}(s,\beta)$

To verdict on the quality of this procedure, the PAEs are computed and compared with some other tests based on the following alternative distributions:

(i)Weibull distribution: $\bar{F}_1(x) = e^{-x^{\theta}}, x \geq 0, \theta \geq 1.$
(ii)Linear failure rate distribution: $\bar{F}_2(x) = e^{-x-\frac{\theta}{2}x^2}, x \geq 0, \theta \geq 0.$
(iii)Makeham distribution: $\bar{F}_3(x) = e^{-x-\theta\left(x+e^{-x}-1\right)}, x \geq 0, \theta \geq 0.$

Note that For $\theta = 1, \bar{F}_1(x)$ reduces to exponential distribution while for $\theta = 0, \bar{F}_2(x)$ and $\bar{F}_3(x)$ reduce to exponential distribution. The PAE of $\hat{\Delta}(s,\beta)$ is defined by

$$PAE(\Delta(s,\beta)) = \frac{1}{\sigma_\circ(s,\beta)} \left| \frac{d}{d\theta} \Delta(s,\beta) \right|_{\theta\to\theta_\circ}. \tag{17}$$

At $s = 1.2, \beta = 1.65$,

$$\delta_\theta(s,\beta) = (s^2 - s\beta)\phi_\theta(\beta)\phi_\theta(s) + (s^2\beta - s\beta^2)\mu_\theta\phi_\theta(s) + (s\beta - s^2 - \beta^2)\phi_\theta(s) + s\beta\phi_\theta(\beta) + \beta^2 - s\beta,$$

where

$$\mu_\theta = \int_0^\infty \overline{F}_\theta(x)dx, \phi_\theta(s) = \int_0^\infty e^{-sx} dF_\theta(x).$$

Hence,

$$\frac{d}{d\theta}\delta_\theta(s,\beta) = (s^2 - s\beta)[\phi_\theta(\beta)\phi'_\theta(s) + \phi'_\theta(\beta)\zeta_\theta(s)] + (s^2\beta - s\beta^2)[\mu_\theta\phi'_\theta(s) + \mu'_\theta\phi_\theta(s)]$$
$$+ (s\beta - s^2 - \beta^2)\phi'_\theta(s) + s\beta\phi'_\theta(\beta),$$

where

$$\mu'_\theta = \int_0^\infty \overline{F}'_\theta(x)dx, \phi'_\theta(s) = -\int_0^\infty e^{-sx} d\overline{F}'_\theta(x).$$

Upon using the definition of the PAE in (19), we obtain

$$PAE(\delta(s,\beta) = \frac{1}{\sigma_0} \left| \begin{array}{c} (s^2 - s\beta)[\phi_\theta(\beta)\phi'_\theta(s) + \phi'_\theta(\beta)\zeta_\theta(s)] + (s^2\beta - s\beta^2)[\mu_\theta\phi'_\theta(s) + \mu'_\theta\phi_\theta(s)] \\ +(s\beta - s^2 - \beta^2)\phi'_\theta(s) + s\beta\phi'_\theta(\beta) \end{array} \right|_{\theta\to\theta_0}.$$

When $s = 1.2, \beta = 1.65$, this leads to:

$$PAE\ [\Delta(1.2,1.65), Weibull] = 1.15842,\ PAE[\Delta(1.2,1.65), LFR] = 0.901371 \text{ and}$$

$$PAE\ [\Delta(1.2,1.65), Makeham] = 0.286589,\ \text{where } \sigma_0(1.2,1.65) = 0.210182.$$

Table 1: Comparison between the PAE of our test and some other tests

Test	Weibull	LFR	Makeham
Kango [22]	0.132	0.433	0.144
Mugdadi and Ahmad [23]	0.170	0.408	0.039
Abdel - Aziz [24]	0.223	0.535	0.184
Our test $\hat{\Delta}(1.2,1.65)$	1.158	0.901	0.287

It is obvious that $\hat{\Delta}(1.2,1.65)$ is better than the other tests based on the PAEs.

5 Monte Carlo Null Distribution Critical Points

In this section, the Monte Carlo null distribution critical points of $\hat{\Delta}(s,\beta)$ are simulated based on 10000 generated samples of size $n = 5(5)50, 29, 43$. From the standard exponential distribution by using Mathematica 8 program. Table 2 gives the upper percentile points of statistic $\hat{\Delta}(1.2, 1.65)$ for different confidence levels 90%, 95% and 99%.

Table 2: Critical values of the statistic $\hat{\Delta}(1.2, 1.65)$

n	90%	95%	99%
5	0.143557	0.167444	0.212644
10	0.092752	0.111232	0.143396
15	0.075286	0.088315	0.114424
20	0.066082	0.079013	0.101269
25	0.057624	0.070131	0.089956
29	0.053816	0.064950	0.085194
30	0.052566	0.064135	0.082944
35	0.048597	0.058774	0.079260
40	0.045069	0.054272	0.071701
43	0.044186	0.053492	0.069444
45	0.042472	0.051877	0.069252
50	0.041004	0.049830	0.065399

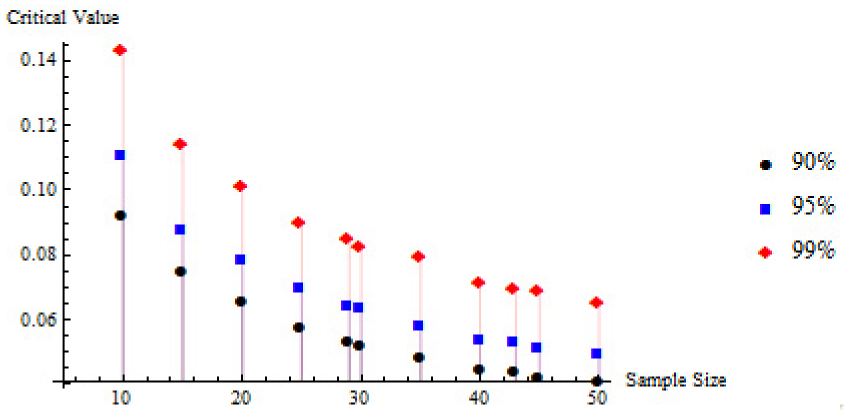

Fig. 1: Relation between critical values, sample size and confidence levels

It can be noticed from Table 2 and Fig. 1 that the critical values are increasing as the confidence level increases and are almost increasing as the sample size increases.

5.1 *Power estimates of the test* $\hat{\Delta}(s,\beta)$

In this section the power of our test $\hat{\Delta}(s,\beta)$ will be estimated at $(1-\alpha)\%$ confidence level, $\alpha = 0.05$ with suitable parameters values of θ at $n = 10, 20$ and 30 for some commonly used distributions such as Weibull and Gamma distributions based on 10000 samples.

Table 3 shows that the power estimates of our test $\hat{\Delta}(1.2, 1.65)$ are good power for all alternatives and increases when the value of the parameter θ and the sample sizes increasing.

Table 3: The Power Estimates of $\hat{\Delta}(1.2, 1.65)$

n	θ	Weibull	Gamma
10	2	0.5841	0.5641
	3	0.9287	0.8654
	4	0.9918	0.9462
20	2	0.9315	0.7713
	3	0.9999	0.9865
	4	1.0000	0.9989
30	2	0.9944	0.8960
	3	1.0000	0.9988
	4	1.0000	1.0000

6 Testing for Censored Data

In this section, a test statistic is proposed to test $H_\circ$ versus H_1 with randomly right-censored data. Such a censored data is usually the only information available in a life-testing model or in a clinical study where patients may be lost (censored) before the completion of a study. This experimental situation can formally be modeled as follows. Suppose n objects are put on test, and $X_1, X_2, \cdots, X_n$ denote their true life time. We let that $X_1, X_2, \cdots, X_n$ be independent and identically distributed (i.i.d.) according to a continuous life distribution F. Let $Y_1, Y_2, \cdots, Y_n$ be (i.i.d.) according to a continuous life distribution G . Also we assume that X's and Y's are independent. In the randomly right-censored model, we observe the pairs (Z_j, δ_j), $j = 1, \cdots, n$, where $Z_j = \min(Xj, Y_j)$ and

$$\delta_j = \begin{cases} 1, \text{ if } Z_j = X_j \text{ (j-th observation is uncensored)} \\ 0, \text{ if } Z_j = Y_j \text{ (j-th observation is censored)} \end{cases}$$

Let $Z_{(0)} = 0 < Z_{(1)} < Z_{(2)} < \cdots . < Z_{(n)}$ denote the orderd Z's and $\delta_{(j)}$ is δ_j corresponding to $Z_{(j)}$. Using the censored data $(Z_j, \delta_j), j = 1, \cdots, n$. Kaplan and Meier [25] proposed the product limit estimator,

$$\bar{F}_n(X) = \prod_{[j:Z(j)\le X]} \{(n-j)/(n-j+1)\}^{\delta_{(j)}}, X \in [0, Z_{(n)}].$$

Now, for testing $H_\circ : \hat{\phi}_c = 0$ against $H_1 : \hat{\phi}_c > 0$, using the randomly right-censored data, we propose the following test statistic

$$\hat{\phi}_c = (s^2 - s\beta)\phi(\beta)\phi(s) + (s^2\beta - s\beta^2)\mu\phi(s) + (s\beta - s^2 - \beta^2)\phi(s) + s\beta\phi(\beta) + \beta^2 - s\beta.$$

where $\phi(s) = \int_0^\infty e^{-sx} dF_n(x)$. For computational purpose, $\hat{\phi}_c$ may be rewritten as

$$\hat{\phi}_c = (s^2 - s\beta)\tau\eta + (s^2\beta - s\beta^2)\Omega\eta + (s\beta - s^2 - \beta^2)\eta + s\beta\tau + \beta^2 - s\beta,$$

where

$$\Omega = \sum_{k=1}^{n} [\prod_{m=1}^{k-1} C_m^{\delta(m)} \left(Z_{(k)} - Z_{(k-1)}\right)],$$

$$\eta = \sum_{j=1}^{n} e^{-sZ_{(j)}} [\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}],$$

$$\tau = \sum_{j=1}^{n} e^{-\beta Z_{(j)}} [\prod_{p=1}^{j-2} C_p^{\delta(p)} - \prod_{p=1}^{j-1} C_p^{\delta(p)}],$$

and

$$dF_n(Z_j) = \bar{F}_n(Z_{j-1}) - \bar{F}_n(Z_j), c_k = [n-k]\,[n-k+1]^{-1}.$$

To make the test invarient, let

$$\hat{\Delta}_c = \frac{\hat{\phi}_c}{\bar{Z}}, \text{where } \bar{Z} = \sum_{i=1}^{n} \frac{Z_{(i)}}{n}.$$

Table (4) below gives the upper percentile of $\hat{\Delta}_c$ test for sample sizes n=10(10)80,51,81. The Monte Carlo null distribution critical values of $\hat{\Delta}_c$ at $s = 1.2, \beta = 1.65$ for samples sizes n=10(10)80,51,81 with 10000 replications are simulated from the standard exponential distribution by using Mathematica 8 program. Table 4 gives the critical values percentiles points of the statistic $\hat{\Delta}_c$.

Table 4: The upper percentile of $\hat{\Delta}_c$ at $s = 1.2, \beta = 1.65$

n	90%	95%	99%
10	0.337507	0.402862	0.534374
20	0.241152	0.290811	0.384251
30	0.196601	0.239162	0.318664
40	0.160513	0.195560	0.258967
50	0.144987	0.176517	0.235611
51	0.138862	0.169872	0.230025
60	0.129742	0.157761	0.215909
70	0.118407	0.143743	0.190618
80	0.110184	0.135442	0.182909
81	0.107611	0.131956	0.181564

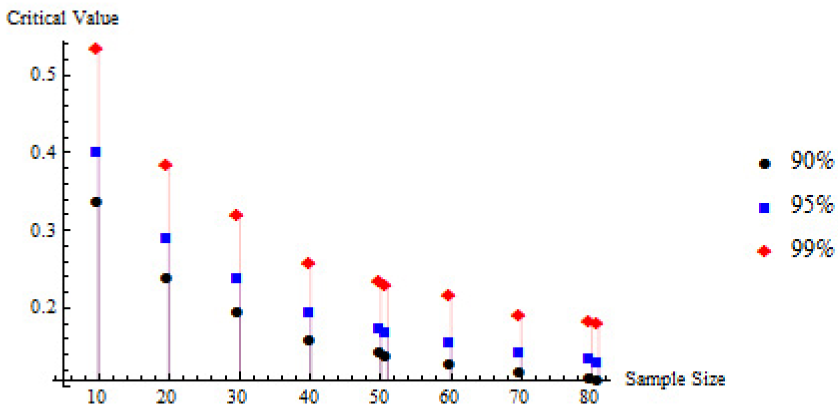

Fig. 2: Relation between critical values, sample size and confidence levels

It can be noticed from Table 4 and Fig. 2 that the critical values are increasing as the confidence level increases and decreasing as the sample size increases.

6.1 Power estimates of the test $\hat{\Delta}_c(s,\beta)$

In this section the power of our test $\hat{\Delta}_c(s,\beta)$ will be estimated at $(1-\alpha)\%$ confidence level, $\alpha = 0.05$ with suitable parameters values of θ at $n = 10, 20$ and 30 for some commonly used distributions such as Weibull and Gamma distributions based on 10000 samples.

Table 5: The Power Estimates of $\hat{\Delta}_c(1.2,1.65)$

n	θ	Weibull	Gamma
10	2	1.0000	1.0000
	3	1.0000	1.0000
	4	1.0000	1.0000
20	2	1.0000	1.0000
	3	1.0000	1.0000
	4	1.0000	1.0000
30	2	0.9996	0.9971
	3	1.0000	1.0000
	4	1.0000	1.0000

Table 5 shows that the power estimates of our test $\hat{\Delta}_c(1.2,1.65)$ are good power for all alternatives and increases when the value of the parameter θ and the sample sizes increasing.

7 Applications to Real Data

In this section, we apply our test to some real data-sets in the both non censored and censored data at 95% confidence level.

7.1 Non censored data

*Example 4.*Consider the following data set is from Kotz and Johnson [26] and represents the survival times (in years) after diagnosis of 43 patient with certain kind of leukemia.

Since $\hat{\Delta}(1.2,1.65) = 0.0554508$ and this value greater than the corresponding critical value in Table 2. Then we conclude that this data set have NBRUL property and not exponential.

*Example 5.*Consider the data in Abouammoh et al. [17]. These data represent set of 40 patients suffering from blood cancer (Leukemia) from one ministry of health hospital in Saudi Arabia

Since $\hat{\Delta}(1.2,1.65) = 0.138068$ and this value greater than the corresponding critical value in Table 2. Then we conclude that this data set have NBRUL property and not exponential.

*Example 6.*We consider a classical real data in Keating et al. [27] set on the times, in operating days, between successive failures of air conditioning equipment in an aircraft.

In this case, $\hat{\Delta}(1.2,1.65) = 0.0700803$ which is greater than the corresponding critical value in Table 2. Then we accept H_1which states that the data set have NBRUL property.

7.2 Censored data

*Example 7.*The following data sets are associated with 101 patients with advanced acute myelogenous leukemia reported to the International Bone Marrow Transplant Registry (see Ghitany and Al-Awadhi [28]). Fifty of these patients had an allogeneic bone marrow transplant where marrow from an HLA (Histocompatibility Leukocyte Antigen) matched sibling was used to replenish their immune systems. Fifty-one patients had an autologous bone marrow transplant in which, after high doses of chemotherapy, their own marrow was reinfused to replace their destroyed immune system. The leukemia free-survival times (in months) for the 50 allogeneic transplant patients (+ indicates censored observations)are:

0.030	0.493	0.855	1.184	1.283	1.480	1.776	2.138
2.500	2.763	2.993	3.224	3.421	4.178	4.441+	5.691
5.855+	6.941+	6.941	7.993+	8.882	8.882	9.145+	11.480
11.513	12.105+	12.796	12.993+	13.849+	16.612+	17.138+	20.066
20.329+	22.368+	26.776+	28.717+	28.717+	32.928+	33.783+	34.221+
34.770+	39.539+	41.118+	45.033+	46.053+	46.941+	48.289+	57.401+
58.322+	60.625+						

Taking into account the whole set of survival data (both censored and uncensored).
We get $\hat{\Delta}_c(1.2, 1.65) = -0.0337921$ which is less than the critical value of the Table 4. Then, H_1 which states that the set of data have NBRUL property is rejected.
The leukemia free-survival times (in months) for the 51 autologous transplant patients are:

0.658	0.822	1.414	2.500	3.322	3.816	4.737	4.836+
4.934	5.033	5.757	5.855	5.987	6.151	6.217	6.447+
8.651	8.717	9.441+	10.329	11.480	12.007	12.007+	12.237
12.401+	13.059+	14.474+	15.000+	15.461	15.757	16.480	16.711
17.204+	17.237	17.303+	17.664+	18.092	18.092+	18.750+	20.625+
23.158	27.730+	31.184+	32.434+	35.921+	42.237+	44.638+	46.480+
47.467+	48.322+	56.086					

Taking into account the whole set of survival data (both censored and uncensored).
We get $\hat{\Delta}_c(1.2, 1.65) = -0.0113224$ which is less than the critical value of the Table 4. Then, H_1 which states that the set of data have NBRUL property is rejected.

*Example 8.*Consider the data in Susarla and Vanryzin [29]. These data represent 46 survival times of patients of melanoma. Of them 35 represent whole life times(non-censored data). The ordered censored observations are:

16	21	44	50	55	67	73	76	80	81	86	93
100	108	114	120	124	125	129	130	132	134	140	147
148	151	152	152	158	181	190	193	194	213	215	

Taking into account the whole set of survival data (both censored and uncensored).
We get $\hat{\Delta}_c(1.2, 1.65) = -6.08646 \times 10^{-10}$ which is less than the critical value of the Table 4. Then, H_1 which states that the set of data have NBRUL property is rejected.

8 Conclusion

We studied the probabilistic characteristics for NBRUL class, the closure properties under various reliability operations such as convolution, mixture, mixing, homogeneous Poisson shock model. A testing hypothesis is proposed to test exponentiality against this new class based on Laplace transform. We calculated PAEs and compared them to some old tests of all alternatives used. Critical values and the powers of our test are calculated in case of censored and non censored. At the end of the article, our test was applied to some real engineering and medical data to show the usefulness of the test.

Conflict of interest

The authors declare that they have no conflict of interest.

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