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Cubic Transmuted Fréchet Distribution

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Abstract: In this paper, we have introduced a new generalization of the Fréchet distribution named as the Cubic Transmuted Fréchet distribution (CTFD) based on cubic ranking transmutation map. Furthermore, we have derived some properties of the proposed distribution including survival and hazard functions, moments, moment generating function, quantile function and random number generation. The estimation of CTFD parameters has been done using Maximum Likelihood method. Finally, an application of the proposed distribution CTFD using two uncensored data is conducted to illustrate and compare with the base Fréchet distribution and Transmuted Freshet distribution. It has been observed that the proposed distribution CTFD provides a better fit for the two datasets as compared to the other distributions.

Keywords: Fréchet distribution , cubic transmutation , maximum likelihood estimation, inverse Weibull distribution , extreme value distribution

1 Introduction

Fréchet distribution also known as inverse Weibull distribution [1,2], or extreme value distribution of type II is named after French mathematician Maurice René Fréchet, who developed it in the 1925 as a maximum value distribution[3] and it used to model maximum value. Kotz and Nadarajah[4] described Fréchet distribution and discussed its wide applicability in different fields such as accelerated life testing, natural calamities, horse racing, rainfall, queues in supermarkets, sea currents wind speeds, track race records and so on.
The cumulative distribution function (cdf) and the probability density function (pdf) of Frechet random variable X are defined, respectively, as

$$G(x) = e^{-(s/(x-m))^a}; \quad a,s \in (0,\infty), m \in (-\infty,\infty), x > m \tag{1}$$

and

$$g(x) = \frac{a}{s}\left(\frac{s}{x-m}\right)^{1+a} e^{-\left(\frac{s}{x-m}\right)^a} \tag{2}$$

where s,m and a are the scale, location and shape parameters respectively.
Using the default value of the location parameter $m = 0$ (see [5]) Eq.(1) and Eq.(2) become

$$G(x) = e^{-(s/x)^a}; \quad a,s \in (0,\infty), x > 0 \tag{3}$$

and

$$g(x) = \frac{a}{s}(s/x)^{1+a} e^{-(s/x)^a} \tag{4}$$

Shaw and Buckley[6] used the rank transmutation map to propose a new method for generating family of distribution. According to them the cumulative distribution function of the ranking quadratic transformation (QRT) map is:

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x); \quad |\lambda| \leq 1 \tag{5}$$

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where $G(x)$ is the cumulative distribution function (cdf) of the base distribution. Observe that, when $\lambda = 0$, the new distribution becomes the original one. Using ORT of Eq.(5), Mahmoud and Mandouh[7] developed transmuted Fréchet distribution.
Abed Al-Kadim[8] proposed generalized formula for transmuted distribution proposed by Shaw and Buckley[6], the cumulative distribution function of the Cubic Ranking transformation (CRT) map is:

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x) + \lambda G^3(x); \quad |\lambda| \le 1 \tag{6}$$

This method used by Abed Al-Kadim and Mohammed[9] to propose cubic transmuted Weibull distribution. Another two classes of Cubic Transmuted distributions with two transmuted parameters have been developed, one by Granzottoa et al.[10], the other by Rahman et al.[11]. Rahman et al.[11] used method to develop new generalization for many distributions, for examples: Pareto distribution[12] and Weibull distribution[13].
Based on the cubic ranking transmutation map approach proposed by Rahman et al.[11], the cdf and pdf respectively given as

$$F(x) = (1+\lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x) \tag{7}$$

and

$$f(x) = g(x)[(1+\lambda_1) + 2(\lambda_2 - \lambda_1)G(x) - 3\lambda_2 G^2(x)] \tag{8}$$

where $\lambda_1 \in [-1,1], \lambda_2 \in [-1,1]$ and $-2 \le \lambda_1 + \lambda_2 \le 1$.
$G(x)$ and $g(x)$ is the cdf and pdf of the base distribution respectively.
In this article, cubic ranking transmutation map suggested by Rahman et al.[11] is used to propose a new distribution which generalizes the Fréchet distribution. This new version of the Fréchet distribution called Cubic Transmuted Fréchet Distribution (CTFD). Some statistical properties are studied and the model parameters are estimated using maximum likelihood method. Moreover, an application to two real datasets from rivers is illustrated and compared with the base Fréchet distribution, transmuted Fréchet and the others two versions of cubic transmuted Fréchet distributions.
The remainder of this paper is organized as follows: The new proposed distribution Cubic Transmuted Fréchet (CTFD) is presented in Section 2. We have investigated some statistical properties for CTFD such as survival and hazard functions, moments, moment generating function, quantile function and random number generation in Section 3. Section 4 provides parameter estimation of the CTFD. An application of the CTFD to two uncensored data for the purpose of illustration is conducted in Section 5. Finally, Section 6 gives some concluding remarks.
We are motivated the new Fréchet model because it exhibits a right skewed fat-tailed shape with leptokurtic distribution as illustrated in Figure 1 . The justification for the practicality of the new generalized extreme value (GEV) model is based on its ability for modeling the breaking stress of carbon fibers and the strengths of glass fibers data sets as illustrated in Section 5 . We used the Fréchet model since it has a wide ability for modeling different shapes of real data sets, this claim has been demonstrated in applications of Section 5.

2 Cubic Transmuted Fréchet Distribution (CTFD)

In this section, the new proposed distribution CTFD is discussed. Including the cumulative distribution function (cdf), probability density function (pdf), survival and hazard function.

2.1 Cumulative and density functions for CTFD

Theorem 2.1 Let X be a random variable with the cubic transmuted Fréchet distribution. The cdf and pdf are defined, respectively, as

$$F(x) = e^{-(s/x)^a}[(1+\lambda_1) + (\lambda_2 - \lambda_1)e^{-(s/x)^a} - \lambda_2 e^{-2(s/x)^a}] \tag{9}$$

and

$$f(x) = \frac{a}{s}(s/x)^{1+a} e^{-(s/x)^a}[(1+\lambda_1) + 2(\lambda_2 - \lambda_1)e^{-(s/x)^a} - 3\lambda_2 e^{-2(s/x)^a}] \tag{10}$$

where $a, s, x > 0, \lambda_1 \in [-1,1], \lambda_2 \in [-1,1] and -2 \le \lambda_1 + \lambda_2 \le 1$.
Proof: Consider the cdf of cubic transmuted distribution of Eq.(7) namely

$$\begin{aligned} F(x) &= (1+\lambda_1)G(x) + (\lambda_2 - \lambda_1)G^2(x) - \lambda_2 G^3(x) \\ &= G(x)[(1+\lambda_1) + (\lambda_2 - \lambda_1)G(x) - \lambda_2 G^2(x)] \end{aligned}$$

substituting $G(x)$ of Eq.(3), we get

$$F(x)=e^{-(s/x)^a}[(1+\lambda_1)+(\lambda_2-\lambda_1)e^{-(s/x)^a}-\lambda_2 e^{-2(s/x)^a}]$$

Let $y=(s/x)^a \Rightarrow s/x=\sqrt[a]{y}$ and $x=\frac{s}{\sqrt[a]{y}}$ Using y in the above equation, we get

$$F(y)=e^{-y}[(1+\lambda_1)+(\lambda_2-\lambda_1)e^{-y}-\lambda_2 e^{-2y}]$$

and now, let's use the following derivative relation

$$f(x)=\frac{dF(x)}{dx}=\frac{dF(y)}{dy}\frac{dy}{dx} \tag{11}$$

$$\frac{dF(y)}{dy}=-e^{-y}[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y}-3\lambda_2 e^{-2y}] \tag{12}$$

$$\frac{dy}{dx}=-(\frac{a}{s})y\sqrt[a]{y}=-(\frac{a}{s})\sqrt[a]{y^{1+a}} \tag{13}$$

so, using the relations of Eq.(11), Eq.(12) and Eq. (13) we get

$$\begin{aligned}f(x)&=\frac{a}{s}\sqrt[a]{y^{1+a}}e^{-y}[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y}-3\lambda_2 e^{-2y}]\\&=\frac{a}{s}(s/x)^{1+a}e^{-(s/x)^a}[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-(s/x)^a}-3\lambda_2 e^{-2(s/x)^a}]\end{aligned}$$

Therefore, the theorem is proved.
Now we notice, using Eq.(13), that

$$\int_0^\infty f(x)dx=-\int_0^\infty f(y)dy=\int_0^\infty e^{-y}[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y}-3\lambda_2 e^{-2y}]dy \tag{14}$$

$$\begin{aligned}&=(1+\lambda_1)+(\lambda_2-\lambda_1)-\lambda_2\\&=1\end{aligned}$$

Figure 1 illustrates some of possible shapes of the pdf and cdf for CTFD for selected values of parameters λ_1 and λ_2 where $a=2$ and $s=2$

From plot of pdf of Figure 1, we can observe that for the positive values of both transmuted parameters λ_1 and λ_2, the distribution is a fat-tailed with more leptokurtic, this compare with the negative values of λ_1 and λ_2.

2.2 *Survival and Hazard function*

The survival function is defined as $s(x)=1-F(x)$ and for the CTFD is given as

$$s(y)=1-e^{-y}[(1+\lambda_1)+(\lambda_2-\lambda_1)e^{-y}-\lambda_2 e^{-2y}];\quad y=\left(\frac{s}{x}\right)^a$$

The hazard function is defined as $h(x)=f(x)/s(x)$ and for the CTFD is given as

$$h(y)=\frac{\frac{a}{s}\sqrt[a]{y^{1+a}}e^{-y}[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y}-3\lambda_2 e^{-2y}]}{e^{y}-[(1+\lambda_1)+(\lambda_2-\lambda_1)e^{-y}-\lambda_2 e^{-2y}]};\quad y=\left(\frac{s}{x}\right)^a$$

Figure 2 shows some possible shapes of the survival and hazard functions for the CTFD using different combination of model parameters λ_1 and λ_2 where $a=2$ and $s=2$

3 Statistical Properties

In this section, some statistical properties for the proposed distribution, CTFD is demonstrated. These properties involve moments, moment generating function, quantile function and simulation the random sample.

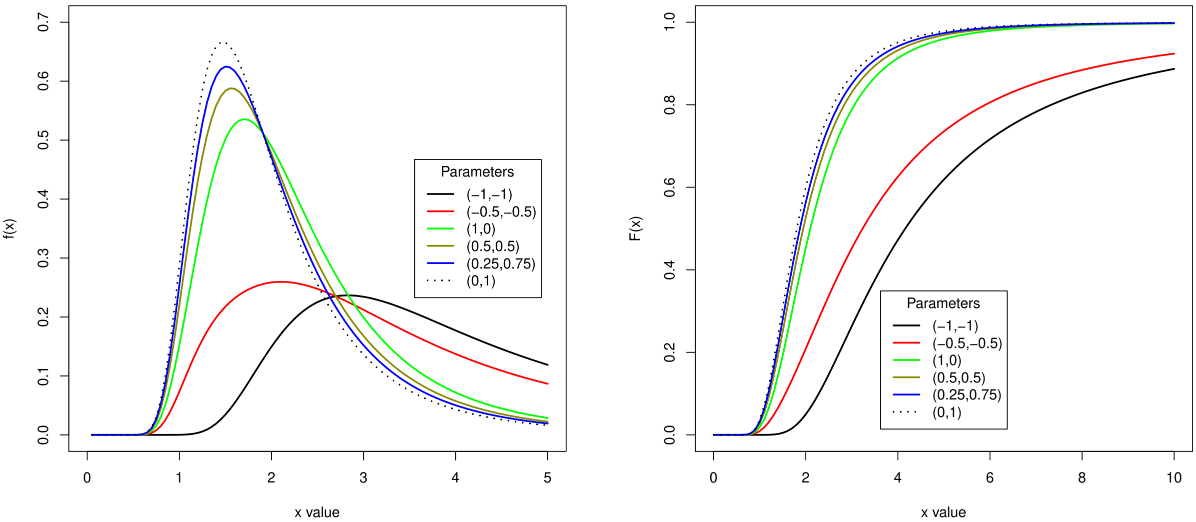

Fig. 1: The pdf and cdf of CTFD for different value of λ_1 and λ_2 where $a = 2$ and $s = 2$.

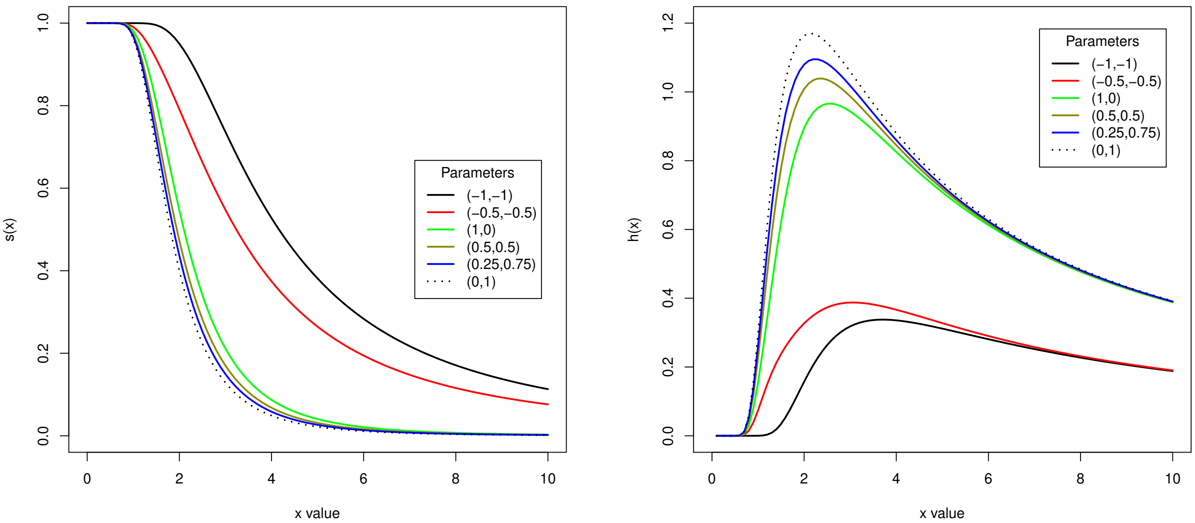

Fig. 2: The $s(x)$'s and $h(x)$'s of CTFD for different value of λ_1 and λ_2 where $a = 2$ and $s = 2$.

3.1 The Moments

Theorem 3.1 Let X be a random variable has the CTFD, then the r^{th} moment of X about the origin is

$$E(X^r) = s^r \Gamma(1 - r/a)\ \ [(1+\lambda_1) + \sqrt[a]{2^r}(\lambda_2 - \lambda_1) - \sqrt[a]{3^r}\lambda_2]; \quad r = 0, 1, 2, \ldots \tag{15}$$

where $\Gamma(1 - r/a)$ is the gamma function.

Proof. we know that

$$E(X^r) = \int_0^{\infty} x^r f(x)dx = \int_{\infty}^{0} (s/\sqrt[a]{y})^r f(y)(\frac{-s}{a})\frac{1}{\sqrt[a]{y^{1+a}}}dy$$

using Eq.(14) we get

$$\begin{aligned} E(X^r) &= \int_0^{\infty} (sy^{-1/a})^r e^{-y}[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y}-3\lambda_2 e^{-2y}]dy \\ &= s^r \int_0^{\infty} y^{-r/a} e^{-y}[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y}-3\lambda_2 e^{-2y}]dy \\ &= s^r \left[(1+\lambda_1)\int_0^{\infty} y^{-r/a}e^{-y}dy + 2(\lambda_2-\lambda_1)\int_0^{\infty} y^{-r/a}e^{-2y}dy - 3\lambda_2 \int_0^{\infty} y^{-r/a}e^{-3y}dy\right] \end{aligned}$$

Using the relation $\int_0^{\infty} t^b e^{-at}dt = \frac{\Gamma(1+b)}{a^{(1+b)}}$ we get

$$E(X^r) = s^r \Gamma(1-r/a) \ \ [(1+\lambda_1) + \sqrt[a]{2^r}(\lambda_2-\lambda_1) - \sqrt[a]{3^r}\lambda_2]$$

Therefore, the theorem is proved.
The mean and variance can be easily obtained by using $r = 1,2$ in Eq.(15).
The mean and variance of CTFD for various combinations of model parameters are given in Table 1 and Table 2. From

Table 1: Mean of the CTFD for various combinations of the parameters

			$\lambda_1=-1$	$\lambda_1=-0.5$	$\lambda_1=1$	$\lambda_1=0.5$	$\lambda_1=0$
$a=4$	$s=2$	$\lambda_2=-1$	3.2255	2.9936	2.298	2.5299	2.7618
		$\lambda_2=-0.5$	3.07	2.8382	2.1426	2.3744	2.6063
		$\lambda_2=0$	2.9145	2.6827	1.9871	2.219	2.4508
		$\lambda_2=0.5$	2.7591	2.5272	-	2.0635	2.2954
		$\lambda_2=1$	2.6036	2.3718	-	-	2.1399
	$s=4$	$\lambda_2=-1$	6.451	5.9872	4.5961	5.0598	5.5235
		$\lambda_2=-0.5$	6.14	5.6763	4.2852	4.7489	5.2126
		$\lambda_2=0$	5.8291	5.3654	3.9742	4.438	4.9017
		$\lambda_2=0.5$	5.5182	5.0545	-	4.127	4.5907
		$\lambda_2=1$	5.2072	4.7435	-	-	4.2798
$a=6$	$s=2$	$\lambda_2=-1$	2.7112	2.573	2.1583	2.2965	2.4347
		$\lambda_2=-0.5$	2.6226	2.4844	2.0697	2.2079	2.3462
		$\lambda_2=0$	2.534	2.3958	1.9811	2.1193	2.2576
		$\lambda_2=0.5$	2.4455	2.3072	-	2.0308	2.169
		$\lambda_2=1$	2.3569	2.2186	-	-	2.0804
	$s=4$	$\lambda_2=-1$	5.4224	5.1459	4.3165	4.593	4.8695
		$\lambda_2=-0.5$	5.2452	4.9688	4.1394	4.4158	4.6923
		$\lambda_2=0$	5.0681	4.7916	3.9622	4.2387	4.5151
		$\lambda_2=0.5$	4.8909	4.6145	-	4.0615	4.338
		$\lambda_2=1$	4.7138	4.4373	-	-	4.1608

Table 1 and Table 2 , regarding the CTFD, it is observed that, holding s constant as the shape parameter a and transmuted parameters λ_1 and λ_2 increase the mean and variance decrease. Whilst, for the scale parameter s, holding other parameters constants,the mean and variance increase.

Table 2: Variance of the CTFD for various combinations of the parameters

			$\lambda_1=-1$	$\lambda_1=-0.5$	$\lambda_1=1$	$\lambda_1=0.5$	$\lambda_1=0$
$a=4$	$s=2$	$\lambda_2=-1$	1.8762	1.8498	1.1255	1.4745	1.7159
		$\lambda_2=-0.5$	1.7282	1.6297	0.6892	1.1102	1.4237
		$\lambda_2=0$	1.5319	1.3613	0.2045	0.6976	1.0832
		$\lambda_2=0.5$	1.2873	1.0446	-	0.2367	0.6944
		$\lambda_2=1$	0.9943	0.6795	-	-	0.2572
	$s=4$	$\lambda_2=-1$	7.5048	7.3992	4.502	5.8978	6.8636
		$\lambda_2=-0.5$	6.9129	6.519	2.7566	4.4408	5.6949
		$\lambda_2=0$	6.1277	5.4453	0.8179	2.7905	4.3329
		$\lambda_2=0.5$	5.1491	4.1784	-	0.9467	2.7776
		$\lambda_2=1$	3.9771	2.718	-	-	1.0289
$a=6$	$s=2$	$\lambda_2=-1$	0.4613	0.4878	0.3381	0.4262	0.4761
		$\lambda_2=-0.5$	0.44	0.442	0.2188	0.3314	0.4058
		$\lambda_2=0$	0.403	0.3805	0.0838	0.2209	0.3198
		$\lambda_2=0.5$	0.3503	0.3033	-	0.0948	0.2182
		$\lambda_2=1$	0.2819	0.2104	-	-	0.1008
	$s=4$	$\lambda_2=-1$	1.8451	1.9512	1.3523	1.7048	1.9044
		$\lambda_2=-0.5$	1.7599	1.768	0.8752	1.3257	1.6233
		$\lambda_2=0$	1.6118	1.522	0.3353	0.8838	1.2793
		$\lambda_2=0.5$	1.4011	1.2133	-	0.3791	0.8726
		$\lambda_2=1$	1.1275	0.8418	-	-	0.4031

3.2 *The Moment Generating Function*

Theorem 3.2 Let X be a random variable has the CTFD, then the (MGF) of X is

$$M_x(t) = a\sum_{r=0}^{\infty}\Gamma[1-a(1+r)]Q_r \tag{16}$$

where $\Gamma[1-a(1+r)]$ is the gamma function and

$$Q_r = \left(\frac{(-1)^{a+ar+r}(st)^{a(r+1)}}{r!}\right)\left[(1+\lambda_1)+2^{(r+1)}(\lambda_2-\lambda_1)-3^{(r+1)}\lambda_2\right]$$

Proof. We know that

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)dx$$

Substitute $f(x)$ of Eq.(10) in above equation and use the alternative formula of Eq. (14) to obtain

$$\begin{aligned} M_X(t) &= \int_0^{\infty} e^{\left(\frac{st}{\sqrt[a]{y}}\right)} e^{-y}[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y}-3\lambda_2 e^{-2y}]dy \\ &= \left[(1+\lambda_1)\int_0^{\infty} e^{\left(\frac{st}{\sqrt[a]{y}}\right)} e^{-y}dy + 2(\lambda_2-\lambda_1)\int_0^{\infty} e^{\left(\frac{st}{\sqrt[a]{y}}\right)} e^{-2y}dy - 3\lambda_2\int_0^{\infty} e^{\left(\frac{st}{\sqrt[a]{y}}\right)} e^{-3y}dy\right] \\ &= (1+\lambda_1)I_1 + 2(\lambda_2-\lambda_1)I_2 - 3\lambda_2 I_3 \end{aligned} \tag{17}$$

where $I_1 = \int_0^\infty e^{(\frac{st}{\sqrt[a]{y}})} e^{-y} dy, I_2 = \int_0^\infty e^{(\frac{st}{\sqrt[a]{y}})} e^{-2y} dy$ and $I_3 = \int_0^\infty e^{(\frac{st}{\sqrt[a]{y}})} e^{-3y} dy$
Now, we get the value of I_1 using Maclaurin expansion as

$$\begin{aligned}
I_1 &= \int_0^\infty e^{(\frac{st}{\sqrt[a]{y}})} e^{-y} dy \\
&= \int_0^\infty e^{(\frac{st}{\sqrt[a]{y}})} \Big(\sum_{r=0}^{\infty} \frac{(-y)^r}{r!} \Big) dy \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \Big(\int_0^\infty e^{(\frac{st}{\sqrt[a]{y}})} y^r dy \Big) \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} I_r
\end{aligned}$$

where $I_r = \int_0^\infty e^{(\frac{st}{\sqrt[a]{y}})} y^r dy$
Putting $-z = (\frac{st}{\sqrt[a]{y}}) \Rightarrow dy = a(-1)^{(a+1)}(st)^a z^{(-(a+1))} dz$ and then there are two cases:
Case I: If $t > 0$ then $(y : 0 \to \infty \Rightarrow z : -\infty \to 0)$, and so

$$\begin{aligned}
I_r &= \int_0^\infty e^{(\frac{st}{\sqrt[a]{y}})} y^r dy \\
&= \int_{-\infty}^0 e^{-z} \Big[(\frac{-z}{st})^{-a} \Big]^r a(-1)^{(a+1)} (st)^a z^{(-(a+1))} dz \\
&= -a(-st)^{a(r+1)} \int_{-\infty}^0 z^{-a(r+1)} e^{-z} dz \\
&= a(-st)^{a(r+1)} \int_0^{-\infty} z^{-a(r+1)} e^{-z} dz \\
&= -a(st)^{a(r+1)} \int_0^\infty z^{-a(r+1)} e^{-z} dz \\
&= a(-st)^{a(r+1)} \Gamma[1 - a(1+r)]
\end{aligned}$$

Case II: If $t < 0$ then $(y : 0 \to \infty \Rightarrow z : \infty \to 0)$, and so

$$\begin{aligned}
I_r &= \int_0^\infty e^{(\frac{st}{\sqrt[a]{y}})} y^r dy \\
&= \int_\infty^0 e^{-z} \Big[(\frac{-z}{st})^{-a} \Big]^r a(-1)^{(a+1)} (st)^a z^{-(a+1)} dz \\
&= a(-st)^{a(r+1)} \int_0^\infty z^{-a(r+1)} e^{-z} dz \\
&= a(-st)^{a(r+1)} \Gamma[1 - a(1+r)]
\end{aligned}$$

from two cases we get

$$I_r = a(-st)^{a(r+1)} \Gamma[1 - a(1+r)]$$

and

$$I_1 = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} I_r = a \sum_{r=0}^{\infty} \frac{(-1)^r (-st)^{a(r+1)}}{r!} \Gamma[1 - a(1+r)] \tag{18}$$

following the same way to obtain

$$I_2 = \sum_{r=0}^{\infty} \frac{(-2)^r}{r!} I_r = a \sum_{r=0}^{\infty} \frac{(-2)^r (-st)^{a(r+1)}}{r!} \Gamma[1 - a(1+n)] \tag{19}$$

and

$$I_3 = \sum_{r=0}^{\infty} \frac{(-3)^r}{r!} I_r = a \sum_{r=0}^{\infty} \frac{(-3)^r (-st)^{a(r+1)}}{r!} \Gamma[1 - a(1+n)] \tag{20}$$

by substituting Eq.s(18), (19) and (20) in Eq. (17), we get

$$M_x(t) = a\sum_{r=0}^{\infty}\Gamma[1-a(1+r)]Q_r$$

3.3 Quantile function

The quantile function for CTFD is derived by finding the value of Q for which $F_x(Q) = p$

$$Q(p,a) = \frac{s}{\sqrt[a]{-\ln(z)}} \tag{21}$$

where

$$z = \left(\frac{\sqrt[3]{\theta_2+\sqrt{h}}}{3\sqrt[3]{2}k}\right) - \left(\frac{\sqrt[3]{2}\theta_1}{3k\sqrt[3]{\theta_2+\sqrt{h}}}\right) - \frac{b}{3k} \tag{22}$$

$\theta_1 = 3kc - b^2, \theta_2 = -2b^3 + 9kbc - 27k^2d$ and $h = 4(\theta_1)^3 + (\theta_2)^2$
$k = \lambda_2, b = (\lambda_1 - \lambda_2), c = -(1+\lambda_1), d = p$

3.4 Simulating the Random Sample

Random numbers from the CTFD can be obtained by equating cdf of the distribution in Eq.(9) with a uniform random number and inverting the expression, that is the random number from CTFD is obtained by solving $F(x) = u$ for x. The random sample from CTFD can be further expressed as

$$x = \frac{s}{\sqrt[a]{-\ln(z)}} \tag{23}$$

where z is given in Eq.(22) with $d = u$ and u is an arbitrary continuous uniform point over (0,1).

4 Parameters estimation

This section pertains to discuss the maximum likelihood estimation (MLE) for parameters of CTFD. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from CTFD. Then the likelihood function is given by

$$\begin{aligned}
L &= \prod_{i=1}^{n} f(y_i; a, s) \\
&= \prod_{i=1}^{n}\left[\frac{a}{s}\sqrt[a]{y_i^{1+a}}e^{-y_i}[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y_i}-3\lambda_2e^{-2y_i}\right]; \quad y_i = \left(\frac{s}{x_i}\right)^a \\
&= (\frac{a}{s})^n e^{-(\Sigma_{i=1}^{n} y_i)}\sqrt[a]{\prod_{i=1}^{n} y_i^{1+a}}\prod_{i=1}^{n}[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y_i}-3\lambda_2e^{-2y_i}]
\end{aligned}$$

so, the log likelihood function is

$$l = \ln L = n\ln(\frac{a}{s}) - \sum_{i=1}^{n}\left[y_i - \frac{1+a}{a}\ln y_i - \ln[(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y_i}-3\lambda_2e^{-2y_i}]\right] \tag{24}$$

by differentiating the log-likelihood function in Eq. (24) with respect to the unknown parameters $a, s, \lambda_1, \lambda_2$ we obtain

$$\frac{\partial l}{\partial a} = 2n\frac{1+a}{a^2} - \frac{1}{a}\sum_{i=1}^{n}\frac{y_i(1+\lambda_1-3\lambda_2e^{-2y_i})}{(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y_i}-3\lambda_2e^{-2y_i}}$$

$$\frac{\partial l}{\partial s} = \frac{an}{s} - \sum_{i=1}^{n}\left[y_i + \frac{2(\lambda_2-\lambda_1)e^{-y_i}-6\lambda_2e^{-2y_i}}{(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y_i}-3\lambda_2e^{-2y_i}}\right]$$

$$\frac{\partial l}{\partial \lambda_1} = \sum_{i=1}^{n} \left[\frac{1-2e^{-y_i}}{(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y_i}-3\lambda_2 e^{-2y_i}} \right]$$

$$\frac{\partial l}{\partial \lambda_2} = \sum_{i=1}^{n} \left[\frac{2e^{-y_i}-3e^{-2y_i}}{(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y_i}-3\lambda_2 e^{-2y_i}} \right]$$

Putting $\frac{\partial l}{\partial a} = 0, \frac{\partial l}{\partial s} = 0, \frac{\partial l}{\partial \lambda_1} = 0$, and $\frac{\partial l}{\partial \lambda_2} = 0$ to get respectively the first, second, third and fourth likelihood equations

$$\sum_{i=1}^{n} \frac{y_i(1+\lambda_1-3\lambda_2 e^{-2y_i})}{(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y_i}-3\lambda_2 e^{-2y_i}} = 2n\frac{1+a}{a} \tag{25}$$

$$\sum_{i=1}^{n} \left[y_i + \frac{2(\lambda_2-\lambda_1)e^{-y_i}-6\lambda_2 e^{-2y_i}}{(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y_i}-3\lambda_2 e^{-2y_i}} \right] = \frac{an}{s} \tag{26}$$

$$\sum_{i=1}^{n} \left[\frac{1-2e^{-y_i}}{(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y_i}-3\lambda_2 e^{-2y_i}} \right] = 0 \tag{27}$$

$$\sum_{i=1}^{n} \left[\frac{2e^{(-y_i)}-3e^{(-2y_i)}}{(1+\lambda_1)+2(\lambda_2-\lambda_1)e^{-y_i}-3\lambda_2 e^{-2y_i}} \right] = 0 \tag{28}$$

Solutions of the equations (25), (26), (27) and (28) are called maximum likelihood estimates (**MLEs**). However, the equations must be solved with numerical methods such as Newton Raphson or iteratively Reweighting algorithm.

5 Application of CTFD

In this section, the CTFD is applied to two uncensored data. The first data in Table 3 is on the breaking stress of carbon fibers in Gba (BSC). The second data in Table 4 is generated data to simulate the strengths of glass fibers (SGF). Summary statistics of the two datasets is reported in Table 5. The maximum likelihood estimates, the log-likelihood value (-Log(L)), the Kolmogorov–Smirnov (k–s) test statistic and the p-value for the k–s statistic for the fitted distributions are demonstrated in Table 6 and 7. Recently, Mahmoud and Mandouh[7] developed transmuted Fréchet distribution fitted the data in Table 3 and 4 compared the results with the Fréchet distribution. By comparing the

Table 3: Breaking stress of carbon fibers (BSC) data

0.92	0.928	0.997	0.9971	1.061	1.117	1.162	1.183	1.187	1.192
1.196	1.213	1.215	1.2199	1.22	1.224	1.225	1.228	1.237	1.24
1.244	1.259	1.261	1.263	1.276	1.31	1.321	1.329	1.331	1.337
1.351	1.359	1.388	1.408	1.449	1.4497	1.45	1.459	1.471	1.475
1.477	1.48	1.489	1.501	1.507	1.515	1.53	1.5304	1.533	1.544
1.5443	1.552	1.556	1.562	1.566	1.585	1.586	1.599	1.602	1.614
1.616	1.617	1.628	1.684	1.711	1.718	1.733	1.738	1.743	1.759
1.777	1.794	1.799	1.806	1.814	1.816	1.828	1.83	1.884	1.892
1.944	1.972	1.984	1.987	2.02	2.0304	2.029	2.035	2.037	2.043
2.046	2.059	2.111	2.165	2.686	2.778	2.972	3.504	3.863	5.306

Table 4: The strength glass fibers (SGF) data

1.014	1.081	1.082	1.185	1.223	1.248	1.267	1.271	1.272	1.275
1.276	1.278	1.286	1.288	1.292	1.304	1.306	1.355	1.361	1.364
1.379	1.409	1.426	1.459	1.46	1.476	1.481	1.484	1.501	1.506
1.524	1.526	1.535	1.541	1.568	1.579	1.581	1.591	1.593	1.602
1.666	1.67	1.684	1.691	1.704	1.731	1.735	1.747	1.748	1.757
1.800	1.806	1.867	1.876	1.878	1.91	1.916	1.972	2.012	2.456
2.592	3.197	4.121							

Table 5: Summary Statistics for BSC and SGF datasets

	n	Min	Max	Mean	Median	Skewness	kurtosis
BSC	100	0.92	5.306	1.657838	1.54415	3.231	15.234
SGF	63	1.014	4.121	1.615635	1.526	3.01186	12.5632

Table 6: Parameters estimates, -log (L), k-s test value and p-value for Fréchet, Transmuted Fréchet, and Cubic Transmuted Fréchet for BSC data

Distribution	**Parameters estimates**				**-log(L)**	**k-s**	**P-value**
Fréchet	$s = 1.397$	$a = 4.3738$			53.592	0.0875	0.4287
TF	$s = 1.5912$	$a = 3.3636$	$\lambda = 0.8517$		52.412	0.0755	0.6189
CTF	$s = 1.172$	$a = 5.0803$	$\lambda_1 = -0.7817$	$\lambda_2 = -1$	51.865	**0.066**	**0.7767**

Table 7: Parameters estimates, -log (L), AIC, BIC and k-s test value for Fréchet, Transmuted Fréchet, and Cubic Transmuted Fréchet for SGF data.

Distribution	**Parameters estimates**				**-log(L)**	**k-s**	**P-value**
Fréchet	$s = 1.4108$	$a = 5.4379$			20.064	0.0772	0.8187
TF	$s = 1.5491$	$a = 4.3142$	$\lambda = 0.7778$		19.369	0.0635	0.9472
CTF	$s = 1.4591$	$a = 4.6668$	$\lambda_1 = 0.001$	$\lambda_2 = 0.652$	19.25	0.0643	**0.9416**

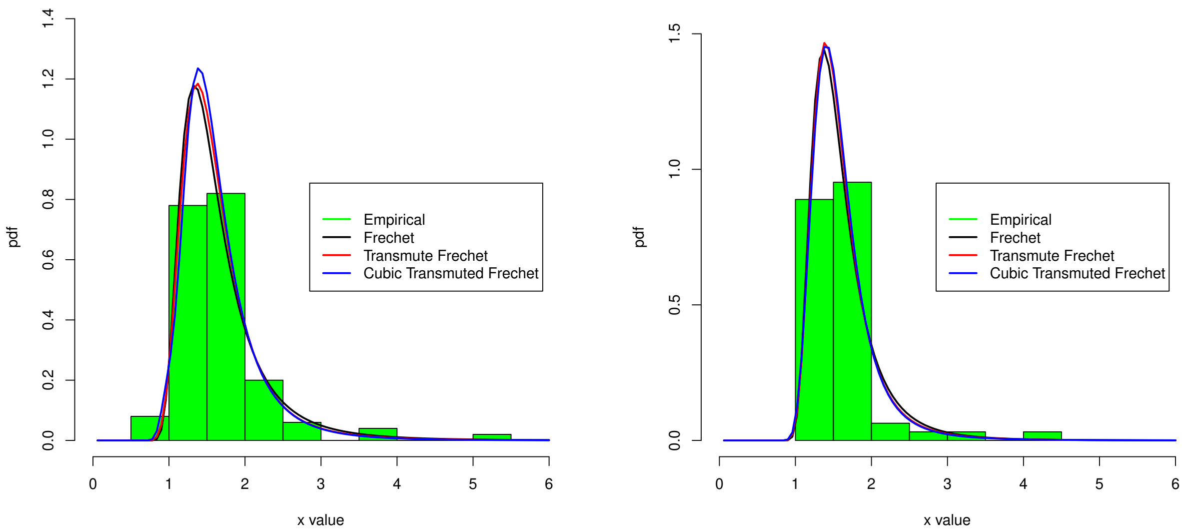

Fig. 3: The pdf of Fréchet, TF and CTF for BSC and SGF datasets

goodness of fit statistics in Table 6 among the three distributions, it is clear that all distributions are competitors and fit the breaking stress of carbon fibers data well but the proposed distribution CTFD fits it best (see Figure 3 on the left) . Moreover, basing on -2log (L) criteria (the smaller the better), CTFD performs better than other distributions. Regarding the SGF data, from Table 7 basing on -2log (L) criteria among the three distributions, it is observed that the CTFD performs better than the other distributions. While the Transmuted Fréchet fits the data best (see Figure 3 on the right).

6 Conclusions

In this paper, a new generalization of the Fréchet distribution called the Cubic Transmuted Fréchet (CTFD) distribution is suggested. Furthermore, some properties of the CTFD including survival and hazard functions, mean, variance, quantile function and random number generation are derived. The estimation of the distribution parameters is performed using maximum Likelihood method. In order to test a goodness of fit for CTFD, the distribution is fitted to a two uncensored data and compared with some related distributions. It is observed that CTFD works better than these distributions.

Conflict of interest

The authors declare that they have no conflict of interest.

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