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Generalized Laguerre approximation and its application to ordinary differential equation

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Abstract: In this paper, generalized Laguerre spectral method for ordinary differential equation is proposed, which is very efficient for long-time numerical simulations of dynamical systems. The global convergence of proposed algorithm is proved. Numerical results demonstrate the spectral accuracy of these new approache and coincide well with theoretical analysis.

Keywords: Generalized Laguerre approximation, spectral method, ordinary differential equation.

1. Introduction

Numerous problems in science and engineering are governed by ordinary differential equations. There have been fruitful results on their numerical solutions, see,e.g, Butcher [4,6], Hairer, Norsett and Wanner[16], Hairer and Wanner [17], Higham [18]and Humphries [24]. For Hamiltonian systems,we refer to the powerful symplectic difference method of Feng[7], also see [8,15] and the references therein.

In the past three decades, the spectral method has developed rapidly. Its merit advantage is the high accuracy. Many spectral algorithms have been proposed for the initial problem of ordinary differential equation. In the early investigations, one often used the Legendre-Radau interpolation to design the Runge-Kutta process. However, the Legendre-Radau interpolation is available for finite interval essentially. Conversely, if we use the Laguerre interpolation, we can approximate the exact solution on half line. Thereby, the related algorithm might be more appropriate for long-time calculations. In particular, the algorithm possess the global convergence. But the collocation method based on the Laguerre interpolation only computes the approximate value of exact solution at some interpolation points, which cannot reflect the global behavior of the exact solution. Recently, some authors has developed the generalized Laguerre approximation with successful applications to spatial approximations of various partial differential equations on the half line and a large class of other related problems,see, e.g., Funaro[7], Guo, Shen and Xu Cheng-long[14], Iranzo and Falques[19], Mastroianni and Monegate[21], which produces the possibility of building up a precise framework of generalized Laguerre approximation with its application to the initial problem of the ordinary differential equations on half line.

This paper is for new generalized Laguerre spectral method for the initial problem of the first-order ordinary equation. In the next section, we investigate the generalized Laguerre approximation. In section 3, we propose the new algorithm by using the generalized Laguerre approximation. This algorithm has several advantages. Firstly, it is easier to be implemented, especially for nonlinear systems. Next, it provides the global numerical solutions and the global convergence in certain weighted Sobolev space. Hence, it is very appropriate to long-time calculations. The numerical solution is represented by function form, so it can simulate more entirely the global property of exact solution, which provides more information above the structures of exact solution. Furthermore, by adjusting a parameter involved in the method, we may weaken the conditions on the underlying problems, the range of applications of the spectral method is enlarged essentially. We present numerical results in Section 4, which demonstrate the spectral accuracy of proposed method and coincide well with the theoretical analysis. The final section is for concluding remarks.

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2. Orthogonal Approximation

In this section, we investigate some results about the generalized Laguerre approximation. Let $\Lambda = \{\rho \mid 0 < \rho < \infty\}$. We define

$$L^2_\chi(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{\chi,\Lambda} < \infty\},$$

with the following inner product and norm

$$(u,v)_{\chi,\Lambda} = \int_\Lambda u(\rho)v(\rho)\chi(\rho)d\rho, \qquad ||v||_{\chi,\Lambda} = (v,v)^{\frac{1}{2}}_{\chi,\Lambda}.$$

For any integer $m \geq 0$, we define the space

$$H^m_\chi(\Lambda) = \{v \mid \frac{d^k v}{d\rho^k} \in L^2_\chi(\Lambda),\ 0 \leq k \leq m\},$$

equipped with the following inner product, semi-norm and norm

$$(u,v)_{m,\chi,\Lambda} = \sum_{0\leq k\leq m} (\frac{d^k u}{d\rho^k}, \frac{d^k v}{d\rho^k})_{\chi,\Lambda},$$

$$|v|_{m,\chi,\Lambda} = \|\frac{d^m u}{d\rho^m}\|_{\chi,\Lambda}, \qquad \|v\|_{m,\chi,\Lambda} = (v,v)^{1/2}_{m,\chi,\Lambda}.$$

For any $r > 0$, the space $H^r_\chi(\Lambda)$ and its norm $\|v\|_{r,\chi,\Lambda}$ are defined by space interpolation as in [2]. In particular, ${}_0H^1_\chi(\Lambda) = \{v \in H^1_\chi(\Lambda) \mid v(0) = 0\}$.

Let $\omega_{\alpha,\beta}(\rho) = \rho^\alpha e^{-\beta\rho}, \alpha > -1, \quad \beta > 0$. The corresponding generalized Laguerre polynomials of degree l are defined by

$$\mathscr{L}^{(\alpha,\beta)}_l(\rho) = \frac{1}{l!}\rho^{-\alpha}e^{\beta\rho}\frac{d^l}{d\rho^l}(\rho^{l+\alpha}e^{-\beta\rho}), \quad l = 0,1,2,\cdots$$

They are the eigenfunctions of the Sturm– Liouville problem

$$\frac{d}{d\rho}(\omega_{\alpha+1,\beta}(\rho)\frac{dv(\rho)}{d\rho}) + \lambda^{(\beta)}_l \omega_{\alpha,\beta}(\rho)v(\rho) = 0, \quad 0 < \rho < \infty, \tag{2.1}$$

with the corresponding eigenvalues $\lambda^{(\beta)}_l = \beta l$. They fulfill the following recurrence relations

$$\mathscr{L}^{(\alpha,\beta)}_l(\rho) = \frac{1}{\beta}(\frac{d\mathscr{L}^{(\alpha,\beta)}_l(\rho)}{d\rho} - \frac{d\mathscr{L}^{(\alpha,\beta)}_{l+1}(\rho)}{d\rho}), \tag{2.2}$$

$$\frac{d\mathscr{L}^{(\alpha,\beta)}_l(\rho)}{d\rho} = -\beta\sum_{k=0}^{l-1}\mathscr{L}^{(\alpha,\beta)}_k(\rho), \tag{2.3}$$

$$\frac{d\mathscr{L}^{(\alpha,\beta)}_l(\rho)}{d\rho} = -\beta\mathscr{L}^{(\alpha+1,\beta)}_{l-1}(\rho). \tag{2.4}$$

$$(l+1)\mathscr{L}^{(\alpha,\beta)}_{l+1}(\rho) - (2l+\alpha+1-\beta\rho)\mathscr{L}^{(\alpha,\beta)}_l(\rho) + (l+\alpha)\mathscr{L}^{(\alpha,\beta)}_{l-1}(\rho) = 0, \tag{2.5}$$

Therefore, it is straightforward to derive the following property (cf. [3])

$$\mathscr{L}^{(\alpha,\beta)}_l(0) = \frac{\Gamma(l+\alpha+1)}{\Gamma(\alpha+1)\Gamma(l+1)}, \qquad l \geq 0 \tag{2.6}$$

The set of $\mathscr{L}^{(\alpha,\beta)}_l(\rho)$ is the complete $L^2_{\omega_{\alpha,\beta}}(\Lambda)$-orthogonal system, namely,

$$(\mathscr{L}^{(\alpha,\beta)}_l,\ \mathscr{L}^{(\alpha,\beta)}_m)_{\omega_{\alpha,\beta},\Lambda} = \begin{cases} \gamma^{(\alpha,\beta)}_l, & l = m, \\ 0, & l \neq m \end{cases}$$

where

$$\gamma^{(\alpha,\beta)}_l = \frac{\Gamma(l+\alpha+1)}{\beta^{\alpha+1}\Gamma(l+1)}. \tag{2.7}$$

Thus, for any $v \in L^2_{\omega_{\alpha,\beta}}(\Lambda)$,

$$v(\rho) = \sum_{l=0}^\infty \hat{v}^{(\alpha,\beta)}_l \mathscr{L}^{(\alpha,\beta)}_l(\rho)$$

with the coefficients

$$\hat{v}^{(\alpha,\beta)}_l = \frac{1}{\gamma^{(\alpha,\beta)}_l}(v, \mathscr{L}^{(\alpha,\beta)}_l)_{\omega_{\alpha,\beta},\Lambda}, \qquad l \geq 0.$$

Now, let N be any positive integer and $\mathscr{P}_N(\Lambda)$ be the set of all algebraic polynomials of degree at most N. Furthermore, ${}_0\mathscr{P}_N(\Lambda) = \{v \in \mathscr{P}_N(\Lambda) \mid v(0) = 0\}$.

In order to describe the approximation results, we introduce the weighted space $A^r_{\alpha,\beta}(\Lambda)$. For any integer $r \geq 0$,

$$A^r_{\alpha,\beta}(\Lambda) = \{v \mid v \text{ is measurable on } \Lambda \text{ and } \|v\|_{A^r_{\alpha,\beta,\Lambda}} < \infty\},$$

equipped with the following semi-norm and norm

$$|v|_{A^r_{\alpha,\beta,\Lambda}} = \|\partial^r_\rho v\|_{\omega_{\alpha+r,\beta},\Lambda}, \quad \|v\|_{A^r_{\alpha,\beta,\Lambda}} = (\sum_{k=0}^r |v|^2_{A^k_{\alpha,\beta,\Lambda}})^{\frac{1}{2}}.$$

For any $r > 0$, we define the space $A^r_{\alpha,\beta}(\Lambda)$ and its norm by space interpolation as in [2].

In forthcoming discussions, we shall use the following Lemma

Lemma 2.1. For any $v \in_0 H^1_{\omega_{\alpha,\beta}}$,

$$\|v\|_{\omega_{\alpha,\beta},\Lambda} \leq c\|\frac{dv}{d\rho}\|_{\omega_{\alpha,\beta},\Lambda}$$

The proof see (cf. [13])

The orthogonal projection $P_{N,\alpha,\beta} : L^2_{\omega_{\alpha,\beta}}(\Lambda) \longrightarrow \mathscr{P}_N(\Lambda)$ is defined by

$$(P_{N,\alpha,\beta}v - v, \phi)_{\omega_{\alpha,\beta},\Lambda} = 0, \quad \forall\phi \in \mathscr{P}_N(\Lambda).$$

and the orthogonal projection ${}_0P^1_{N,\alpha,\beta} :_0 H^1_{\omega_{\alpha,\beta}}(\Lambda) \longrightarrow_0 \mathscr{P}_N(\Lambda)$

$$(\frac{d}{d\rho}({}_0P^1_{N,\alpha,\beta}v - v), \frac{d\phi}{d\rho})_{\omega_{\alpha,\beta},\Lambda} = 0, \quad \forall\phi \in_0 \mathscr{P}_N(\Lambda).$$

Similarly, we define the orthogonal projection ${}_0P_{N,\alpha,\beta} :_0 L^2_{\omega_{\alpha,\beta}}(\Lambda) \longrightarrow_0 \mathscr{P}_N(\Lambda)$ as

$$({}_0P_{N,\alpha,\beta}v - v, \phi)_{\omega_{\alpha,\beta},\Lambda} = 0, \quad \forall\phi \in_0 \mathscr{P}_N(\Lambda).$$

where

$${}_0L^2_{\omega_{\alpha,\beta}}(\Lambda) = \{v \ \mid \ v \in L^2_{\omega_{\alpha,\beta}}(\Lambda) \ \text{ and } \ v(0) = 0\}$$

The following results characterize property of $P_{N,\alpha,\beta}$ and ${}_0P_{N,\alpha,\beta}$.

Lemma 2.2. For any integers $0 \le r$,

$$\|P_{N,\alpha,\beta}v - v\|_{\omega_{\alpha,\beta},\Lambda} \le cN^{-\frac{r}{2}}\|v\|_{A^r_{\alpha,\beta,\Lambda}}$$

The proof see (cf. [13])

Lemma 2.3. For any integers $1 \le r, v \in_0 H^r_{\omega_{\alpha,\beta}}$

$$|{}_0P^1_{N,\alpha,\beta}v - v|_{1,\omega_{\alpha,\beta},\Lambda} \le cN^{\frac{1-r}{2}}\|\frac{dv}{d\rho}\|_{A^{r-1}_{\alpha,\beta,\Lambda}}$$

The proof see (cf. [13])

Lemma 2.4. For any integers $1 \le r, v \in_0 H^r_{\omega_{\alpha,\beta}}$

$$\|{}_0P_{N,\alpha,\beta}v - v\|_{\omega_{\alpha,\beta},\Lambda} \le cN^{\frac{1-r}{2}}\|\frac{dv}{d\rho}\|_{A^{r-1}_{\alpha,\beta,\Lambda}}$$

Proof. By the projection theorem

$$\|{}_0P_{N,\alpha,\beta}v - v\|_{\omega_{\alpha,\beta},\Lambda} \le \|\phi - v\|_{\omega_{\alpha,\beta},\Lambda} \qquad \forall \phi \in_0 \mathscr{P}_N(\Lambda)$$

Taking $\phi = \int_0^\rho P_{N-1,\alpha,\beta}v' d\xi$ in above. Clearly, $\phi \in_0 \mathscr{P}_N(\Lambda)$. According to Lemma (2.1), we have

$$\|{}_0P_{N,\alpha,\beta}v - v\|_{\omega_{\alpha,\beta},\Lambda} \le c\|P_{N-1,\alpha,\beta}\frac{dv}{d\rho} - \frac{dv}{d\rho}\|_{\omega_{\alpha,\beta},\Lambda}$$

A combination of Lemma 2.2 and this formula leads to the desired result.

Lemma 2.5. For any $\phi \in \mathscr{P}_N(\Lambda)$, integer $r \ge 0$

$$\|\phi\|^2_{r,\omega_{\alpha,\beta},\Lambda} \le c(\beta N)^{2r}\|\phi\|^2_{\omega_{\alpha,\beta},\Lambda}$$

The proof see (cf. [9])

Now we define the projection operator $P^{k,0}_N :_0 H^k_{\omega_{\alpha,\beta}}(\Lambda) \longrightarrow_0 \mathscr{P}^k_N(\Lambda)$ as

$$\int_0^{+\infty} \frac{d^k(v - P^{k,0}_N v)}{d\rho^k}\frac{d^k\phi}{d\rho^k}\omega_{\alpha,\beta}d\rho = 0, \qquad \forall \phi \in_0 \mathscr{P}^k_N(\Lambda)$$

where
${}_0\mathscr{P}^k_N(\Lambda) = \{\phi \ \mid \ \phi \in \mathscr{P}_N(\Lambda) \ \text{ and } \ \frac{d^l\phi(0)}{d\rho^l} = 0, \ 0 \le l \le k-1\}$

Let $\omega_\beta(\rho) = \omega_{0,\beta}(\rho) = e^{-\beta\rho}$, $\beta > 0$. We have the following result

Theorem 2.1. Let k be a positive integer.For any nonnegative real numbers r and s, $0 \le r \le k \le s$, there exists a positive constant c depending only on s such that, for any function φ in $A^s_{0,\beta}(\Lambda) \cap_0 H^k_{\omega_\beta}$, the following estimate holds

$$\|\varphi - P^{k,0}_N\varphi\|_{H^r_{\omega_\beta}} \le cN^{\frac{r-s}{2}}\|\varphi\|_{A^s_{0,\beta,\Lambda}}$$

Proof. We first assume that $r = k$. Clearly,

$$\forall \varphi \in_0 H^k_{\omega_\beta}, \quad P^{k,0}_N\varphi(\rho) = \int_0^\rho P^{k-1,0}_{N-1}\frac{d\varphi}{d\xi}d\xi \tag{2.8}$$

To check that it belongs to ${}_0\mathscr{P}^k_N(\Lambda)$,we note that it belongs to $\mathscr{P}_N(\Lambda)$,that it vanishes in 0 and that its derivatives up to order $k-1$ vanish in 0, which ends the proof of (2.8).

We also note that

$$P^{1,0}_N\varphi(\rho) =_0 P^1_{N,0,\beta}\varphi(\rho)$$

By virtue of (2.8),we obtain the following estimate from Lemma 2.3

$$\begin{aligned}
&|\varphi - P^{k,0}_N\varphi|_{k,\omega_\beta} = |\varphi' - P^{k-1,0}_{N-1}\varphi'|_{k-1,\omega_\beta} = \cdots \\
&= |\frac{d^{k-1}\varphi}{d\rho^{k-1}} - P^{1,0}_{N-k+1}\frac{d^{k-1}\varphi}{d\rho^{k-1}}|_{1,\omega_\beta} = |\frac{d^{k-1}\varphi}{d\rho^{k-1}} -_0 P^1_{N-k+1,0,\beta}\frac{d^{k-1}\varphi}{d\rho^{k-1}}|_{1,\omega_\beta} \\
&\le c(N-k+1)^{\frac{k-s}{2}}\|\frac{d^k\varphi}{d\rho^k}\|_{A^{s-k}_{0,\beta,\Lambda}} \le cN^{\frac{k-s}{2}}\|\varphi\|_{A^s_{0,\beta,\Lambda}}
\end{aligned} \tag{2.9}$$

Clearly,

$$\begin{aligned}
&\beta\int_0^\rho(\varphi - P^{k,0}_N\varphi)^2\omega_\beta dy + (\varphi - P^{k,0}_N\varphi)^2\omega_\beta \\
&= 2\int_0^\rho(\varphi - P^{k,0}_N\varphi)\frac{d(\varphi - P^{k,0}_N\varphi)}{dy}\omega_\beta dy \\
&\le \frac{\beta}{2}\int_0^\rho(\varphi - P^{k,0}_N\varphi)^2\omega_\beta dy + \frac{2}{\beta}\int_0^\rho(\frac{d(\varphi - P^{k,0}_N\varphi)}{dy})^2\omega_\beta dy
\end{aligned}$$

Let $\rho \longrightarrow \infty$,we obtain

$$\int_0^{+\infty}(\varphi - P^{k,0}_N\varphi)^2\omega_\beta dy \le \frac{4}{\beta^2}\int_0^{+\infty}(\frac{d(\varphi - P^{k,0}_N\varphi)}{dy})^2\omega_\beta dy$$

Combining this estimate with (2.9), we obtain that

$$\|\varphi - P^{k,0}_N\varphi\|_{H^k_{\omega_\beta}} \le cN^{\frac{k-s}{2}}\|\varphi\|_{A^s_{0,\beta,\Lambda}} \tag{2.10}$$

Next we will prove the case for r= 0. We note that

$$\|\varphi - P^{k,0}_N\varphi\|_{L^2_{\omega_\beta}(\Lambda)} = \sup_{g\in L^2_{\omega_\beta}}\frac{\int_0^{+\infty}(\varphi - P^{k,0}_N\varphi)g\omega_\beta d\rho}{\|g\|_{L^2_{\omega_\beta}(\Lambda)}} \tag{2.11}$$

For any g in $L^2_{\omega_\beta}$, we consider the solution χ in ${}_0H^k_{\omega_\beta}$ of the problem

$$\int_0^{+\infty}\frac{d^k\chi}{d\rho^k}\frac{d^k\psi}{d\rho^k}\omega_\beta d\rho = \int_0^{+\infty}g\psi\omega_\beta d\rho$$

The existence and uniqueness of this solution follow from the Lax-milgram Theorem. Moreover, the regulalty tells us that χ belongs to $H^{2k}_{\omega_\beta}(\Lambda)$ and satisfies

$$\|\chi\|_{H^{2k}_{\omega_\beta}} \le c\|g\|_{L^2_{\omega_\beta}} \tag{2.12}$$

Then we compute

$$\int_0^{+\infty} g(\varphi - P_N^{k,0}\varphi)\omega_\beta d\rho = \int_0^{+\infty} \frac{d^k(\varphi - P_N^{k,0}\varphi)}{d\rho^k}\frac{d^k\chi}{d\rho^k}\omega_\beta d\rho$$
$$= \int_0^{+\infty} \frac{d^k(\varphi - P_N^{k,0}\varphi)}{d\rho^k}\frac{d^k(\chi - P_N^{k,0}\chi)}{d\rho^k}\omega_\beta d\rho$$
$$\leq |\varphi - P_N^{k,0}\varphi|_{H^k_{\omega_\beta}}|\chi - P_N^{k,0}\chi|_{H^k_{\omega_\beta}}$$

Applying twice the estimate for $r = k$,we obtain from (2.12) that

$$\int_0^{+\infty} g(\varphi - P_N^{k,0}\varphi)\omega_\beta d\rho \leq cN^{-\frac{s}{2}}\|\varphi\|_{A^s_{0,\beta,\Lambda}}\|g\|_{L^2_{\omega_\beta}} \tag{2.13}$$

Combining this result with (2.11) gives the case of $r = 0$. By space interpolation, we complete the proof.

Next, we introduce a set of polynomials $\chi_{k,l}$, $0 \leq l \leq k-1$. $\chi_{k,l}$ stands for the unique polynomial in $\mathscr{P}_{k-1}(\Lambda)$, which satisfies

$$\frac{d^l\chi_{k,l}(0)}{d\rho^l} = 1 \text{ and } \frac{d^m\chi_{k,l}(0)}{d\rho^m} = 0,\ \ 0 \leq m \leq k-1, m \neq l$$

For each function φ in $H^k_{\omega_\beta}(\Lambda)$, we define a function $\tilde{\varphi}$ in ${}_0H^k_{\omega_\beta}(\Lambda)$ by

$$\tilde{\varphi} = \varphi - \sum_{l=0}^{k-1}\frac{d^l\varphi(0)}{d\rho^l}\chi_{k,l}(\rho) \tag{2.14}$$

We note that for any $\rho \in [0, \frac{1}{\beta}]$,

$$\left|\frac{d^l\varphi(\rho)}{d\rho^l} - \frac{d^l\varphi(0)}{d\rho^l}\right| \leq \left(\int_0^{\frac{1}{\beta}} e^{\beta\rho}d\rho\right)^{\frac{1}{2}}\left\|\frac{d^{l+1}\varphi(\rho)}{d\rho^{l+1}}\right\|_{L^2_{\omega_\beta}(0,\frac{1}{\beta})}$$
$$\leq c\beta^{-\frac{1}{2}}\left\|\frac{d^{l+1}\varphi(\rho)}{d\rho^{l+1}}\right\|_{L^2_{\omega_\beta}(0,\frac{1}{\beta})}$$

Now, let

$$\frac{d^l\varphi(\rho^*)}{d\rho^l} = \min_{0\leq\rho\leq\frac{1}{\beta}}\left|\frac{d^l\varphi(\rho)}{d\rho^l}\right|$$

Clearly,

$$\left|\frac{d^l\varphi(\rho^*)}{d\rho^l}\right| \leq \beta\int_0^{\frac{1}{\beta}}\left|\frac{d^l\varphi(\rho)}{d\rho^l}\right|d\rho \leq c\beta^{\frac{1}{2}}\left\|\frac{d^l\varphi(\rho)}{d\rho^l}\right\|_{L^2_{\omega_\beta}(0,\frac{1}{\beta})}$$

The previous two formula gives

$$\left|\frac{d^l\varphi(0)}{d\rho^l}\right| \leq \left|\frac{d^l\varphi(\rho^*)}{d\rho^l}\right| + \left|\frac{d^l\varphi(\rho^*)}{d\rho^l} - \frac{d^l\varphi(0)}{d\rho^l}\right|$$
$$\leq c\left(\beta^{\frac{1}{2}}\left\|\frac{d^l\varphi(\rho)}{d\rho^l}\right\|_{L^2_{\omega_\beta}(0,\frac{1}{\beta})} + \beta^{-\frac{1}{2}}\left\|\frac{d^{l+1}\varphi(\rho)}{d\rho^{l+1}}\right\|_{L^2_{\omega_\beta}(0,\frac{1}{\beta})}\right)$$
$$\leq c\|\varphi\|_{H^k_{\omega_\beta}(\Lambda)}.\quad 0 \leq l \leq k-1$$

Next, we define

$$\|\varphi\|_{B^s_{\omega_{0,\beta}}(\Lambda)} = \|\varphi\|_{A^s_{0,\beta,\Lambda}} + \|\varphi\|_{H^s_{\omega_\beta}(\Lambda)}$$

According to previous formula and (2.14), for any real number s,

$$\|\tilde{\varphi}\|_{A^s_{0,\beta,\Lambda}} \leq c\|\varphi\|_{B^s_{\omega_{0,\beta}}(\Lambda)}$$

Next, set

$$\tilde{P}^k_N\varphi = P_N^{k,0}\tilde{\varphi} + \sum_{l=0}^{k-1}\frac{d^l\varphi(0)}{d\rho^l}\chi_{k,l}(\rho) \tag{2.15}$$

Obviously,

$$\varphi - \tilde{P}^k_N\varphi = \tilde{\varphi} - P_N^{k,0}\tilde{\varphi}$$

Using Theorem 2.1 leads to

$$\|\varphi - \tilde{P}^k_N\varphi\|_{H^r_{\omega_\beta}(\Lambda)} = \|\tilde{\varphi} - P_N^{k,0}\tilde{\varphi}\|_{H^r_{\omega_\beta}(\Lambda)} \leq cN^{\frac{r-s}{2}}\|\varphi\|_{B^s_{\omega_{0,\beta}}(\Lambda)} \tag{2.16}$$

Next, we define $\widehat{P}^1_N$ as

$$\widehat{P}^1_N\varphi =_0 P_{N,0,\beta}\tilde{\varphi} + \varphi(0) \tag{2.17}$$

By using Lemma (2.4) with $\alpha = 0$, we obtain

$$\|\varphi - \widehat{P}^1_N\varphi\|_{\omega_\beta,\Lambda} = \|\tilde{\varphi} -_0 P_{N,0,\beta}\tilde{\varphi}\|_{\omega_\beta,\Lambda} \leq cN^{\frac{1-r}{2}}\|\varphi\|_{A^r_{0,\beta,\Lambda}} \tag{2.18}$$

3. Generalized Laguerre Spectral Method

In this section, we apply generalized Laguerre approximation to ordinary differential equation.

Let $\omega_\beta(\rho) = \omega_{0,\beta}(\rho) = e^{-\beta\rho}$, $\beta > 0$. The corresponding generalized Laguerre polynomials

$$\mathscr{L}^{(\beta)}_l(\rho) = \mathscr{L}^{(0,\beta)}_l(\rho), \qquad l \geq 0$$

The set of $\mathscr{L}^{(\beta)}_l(\rho)$ is the complete $L^2_{\omega_\beta}(\Lambda)$-orthogonal system, namely,

$$(\mathscr{L}^{(\beta)}_l(\rho),\ \mathscr{L}^{(\beta)}_m(\rho))_{\omega_\beta,\Lambda} = \begin{cases} \frac{1}{\beta}, & l = m, \\ 0, & l \neq m \end{cases}$$

Thus, for any $v \in L^2_{\omega_\beta}(\Lambda)$,

$$v(\rho) = \sum_{l=0}^{\infty}\hat{v}^{(\beta)}_l\mathscr{L}^{(\beta)}_l(\rho)$$

with the coefficients

$$\hat{v}^{(\beta)}_l = \beta(v, \mathscr{L}^{(\beta)}_l)_{\omega_\beta,\Lambda}, \qquad l \geq 0.$$

Using (2.6), we derive that

$$\mathscr{L}^{(\beta)}_l(0) = 1, \quad l \geq 0 \tag{3.1}$$

Thus,

$$\psi_k = \mathscr{L}^{(\beta)}_k(\rho) - \mathscr{L}^{(\beta)}_{k+1}(\rho), \qquad 0 \leq k \leq N-1 \tag{3.2}$$

form the basis of ${}_0\mathscr{P}_N(\Lambda)$.

We consider the following problem

$$\begin{cases} \dfrac{dv}{d\rho} = f(v(\rho),\rho), \quad \rho \geq 0 \\ v(0) = v_0 \end{cases} \tag{3.3}$$

Next, we construct the numerical scheme. To do this,we approximate $v(\rho)$ by $u_N(\rho)$. where $u_N(\rho) \in \mathscr{P}_N(\Lambda)$ and $u_N(0) = v_0$.

$u_N(\rho)$ can be expanded to

$$u_N(\rho) = \sum_{l=0}^{N} \tilde{u}_l \mathscr{L}_l^{(\beta)}$$

By virtue of (2.3),

$$\begin{aligned} \frac{d}{d\rho}u_N(\rho) &= \sum_{l=1}^{N} \tilde{u}_l \frac{d}{d\rho}\mathscr{L}_l^{(\beta)}(\rho) = -\beta \sum_{l=1}^{N} \tilde{u}_l (\sum_{m=0}^{l-1} \mathscr{L}_m^{(\beta)}) \\ &= -\beta \sum_{m=0}^{N-1} \mathscr{L}_m^{(\beta)} (\sum_{l=m+1}^{N} \tilde{u}_l) \end{aligned} \tag{3.4}$$

Due to the orthogonality of $\mathscr{L}_l^{(\beta)}$,we deduce that

$$\begin{aligned} (\mathscr{L}_l^{(\beta)}, \psi_k)_{\omega_\beta} &= (\mathscr{L}_l^{(\beta)}, \mathscr{L}_k^{(\beta)})_{\omega_\beta} - (\mathscr{L}_l^{(\beta)}, \mathscr{L}_{k+1}^{(\beta)})_{\omega_\beta} \\ &= \begin{cases} \frac{1}{\beta} & l = k \\ -\frac{1}{\beta} & l = k+1 \\ 0 & \text{other.} \end{cases} \end{aligned} \tag{3.5}$$

A combination of (3.4) and (3.5) leads to

$$\begin{aligned} (\frac{d}{d\rho}u_N(\rho), \psi_k)_{\omega_\beta} &= -\sum_{l=k+1}^{N} \tilde{u}_l + \sum_{l=k+2}^{N} \tilde{u}_l, \quad 0 \leq k \leq N-1 \\ &= \begin{cases} -\tilde{u}_{k+1} & 0 \leq k \leq N-2 \\ -\tilde{u}_N & k = N-1. \end{cases} \end{aligned} \tag{3.6}$$

We note that $\tilde{u}_N = v_0 - \sum_{l=0}^{N-1} \tilde{u}_l$, then

$$(\frac{d}{d\rho}u_N(\rho), \psi_k)_{\omega_\beta} = \begin{cases} -\tilde{u}_{k+1} & 0 \leq k \leq N-2 \\ \sum_{l=0}^{N-1} \tilde{u}_l - v_0 & k = N-1. \end{cases} \tag{3.7}$$

Let

$$a_{k,j} = \begin{cases} -1 & j = k+1, \quad 0 \leq k \leq N-2 \\ 1 & j = 0, \cdots, N-1, \quad k = N-1. \end{cases}$$

$$A^N = (a_{k,j})_{N\times N}, \qquad B^N = (0,0,\cdots,0,1)^\top$$

$$u^N = (\tilde{u}_0, \tilde{u}_1, \cdots, \tilde{u}_{N-2}, \tilde{u}_{N-1})^\top$$

$$\ddot{f}_k = (f(u_N(\rho),\rho),\psi_k)_{\omega_\beta}, \ \mathbf{F}^N(u_N) = (\ddot{f}_0, \ddot{f}_1, \cdots, \ddot{f}_{N-2}, \ddot{f}_{N-1})^\top$$

We derive the following spectral scheme for (3.3)

$$\begin{cases} A^N u^N = \mathbf{F}^N(u_N) + v_0 B^N \\ \tilde{u}_N = v_0 - \sum_{l=0}^{N-1} \tilde{u}_l. \end{cases} \tag{3.8}$$

Obviously, the system (3.8)is equivalent to

$$\begin{cases} \dfrac{du_N(\rho)}{d\rho} = f(u_N(\rho),\rho), \quad \rho \geq 0 \\ u_N(0) = v_0 \end{cases} \tag{3.9}$$

Next, we analyze the numerical error of (3.8). To do this, let $E_N = u_N - \widehat{P}_N^1 v$. We suppose that $\dfrac{dv}{d\rho}$ is continuous for $\rho \geq 0$.Let

$$G_{\beta,1} = \frac{d}{d\rho}\widehat{P}_N^1 v(\rho) - \widehat{P}_N^1 \frac{dv}{d\rho}. \tag{3.10}$$

Then, we have that

$$(\frac{d}{d\rho}\widehat{P}_N^1 v(\rho), \phi)_{\omega_\beta} = (\widehat{P}_N^1 \frac{dv}{d\rho}, \phi)_{\omega_\beta} + (G_{\beta,1}, \phi)_{\omega_\beta} \quad \forall \phi \in_0 \mathscr{P}_N(\Lambda) \tag{3.11}$$

Subtracting (3.11) from (3.9) yields that

$$\begin{cases} (\dfrac{d}{d\rho}E_N(\rho), \phi)_{\omega_\beta} = (G_{\beta,2}, \phi)_{\omega_\beta} - (G_{\beta,1}, \phi)_{\omega_\beta} \quad \forall \phi \in_0 \mathscr{P}_N(\Lambda), \\ E_N(0) = 0 \end{cases} \tag{3.12}$$

where

$$G_{\beta,2} = f(u_N(\rho),\rho) - \widehat{P}_N^1 \frac{dv}{d\rho} \quad \text{and} \quad E_N(\rho) \in_0 \mathscr{P}_N(\Lambda)$$

Taking $\phi = 2E_N$ in (3.12) leads to

$$\begin{aligned} 2(E_N, \frac{d}{d\rho}E_N)_{\omega_\beta} &= 2(G_{\beta,2}, E_N)_{\omega_\beta} - 2(G_{\beta,1}, E_N)_{\omega_\beta} \\ &= A_{\beta,2} + A_{\beta,1} \end{aligned} \tag{3.13}$$

Where

$$A_{\beta,1} = -2(G_{\beta,1}, E_N)_{\omega_\beta} \quad \text{and} \quad A_{\beta,2} = 2(G_{\beta,2}, E_N)_{\omega_\beta}$$

Since $E_N(0) = 0$, integration by parts

$$2(E_N, \frac{d}{d\rho}E_N)_{\omega_\beta} = \beta \|E_N\|_{\omega_\beta}^2 \tag{3.14}$$

By using the Cauchy inequality, we derive that

$$|A_{\beta,1}| \leq 2\|G_{\beta,1}\|_{\omega_\beta} \|E_N\|_{\omega_\beta} \leq \varepsilon \|E_N\|_{\omega_\beta}^2 + \frac{1}{\varepsilon}\|G_{\beta,1}\|_{\omega_\beta}^2 \tag{3.15}$$

Next, we assume that there exists a real number γ such that

$$(f(z_1,\rho) - f(z_2,\rho))(z_1 - z_2) \leq \gamma (z_1 - z_2)^2 \tag{3.16}$$

Then

$$\begin{aligned} A_{\beta,2} &= 2(f(u_N,\rho) - \widehat{P}_N^1 \frac{dv}{d\rho}, E_N)_{\omega_\beta} \\ &= 2(f(u_N,\rho) - f(\widehat{P}_N^1 v,\rho), E_N)_{\omega_\beta} \\ &+2(f(\widehat{P}_N^1 v,\rho) - f(v,\rho), E_N)_{\omega_\beta} + 2(\frac{dv}{d\rho} - \widehat{P}_N^1 \frac{dv}{d\rho}, E_N)_{\omega_\beta} \\ &\leq 2\gamma \|E_N\|_{\omega_\beta}^2 + 2\gamma(\widehat{P}_N^1 v - v, E_N)_{\omega_\beta} \\ &+2(\frac{dv}{d\rho} - \widehat{P}_N^1 \frac{dv}{d\rho}, E_N)_{\omega_\beta} \end{aligned}$$

According to the above formula, we obtain that

$$
\begin{aligned}
|A_{\beta,2}| &\le 2\gamma\|E_N\|_{\omega_\beta}^2 + 2\gamma\|\widehat{P}_N^1 v - v\|_{\omega_\beta}\|E_N\|_{\omega_\beta} \\
&+2\|\frac{dv}{d\rho} - \widehat{P}_N^1\frac{dv}{d\rho}\|_{\omega_\beta}\|E_N\|_{\omega_\beta} \\
&\le (2\gamma+\varepsilon+\varepsilon)\|E_N\|_{\omega_\beta}^2 + \frac{1}{\varepsilon}\|\frac{dv}{d\rho} - \widehat{P}_N^1\frac{dv}{d\rho}\|_{\omega_\beta}^2 \\
&+\frac{\gamma}{\varepsilon}\|\widehat{P}_N^1 v - v\|_{\omega_\beta}^2
\end{aligned}
\tag{3.17}
$$

Substituting (3.14),(3.15),(3.17) into (3.13), we assert that

$$
\begin{aligned}
\beta\|E_N\|_{\omega_\beta}^2 &\le (2\gamma+3\varepsilon)\|E_N\|_{\omega_\beta}^2 + \tfrac{1}{\varepsilon}\|G_{\beta,1}\|_{\omega_\beta}^2 \\
&+\frac{1}{\varepsilon}\|\frac{dv}{d\rho} - \widehat{P}_N^1\frac{dv}{d\rho}\|_{\omega_\beta}^2 + \frac{1}{\varepsilon}\|\widehat{P}_N^1 v - v\|_{\omega_\beta}^2
\end{aligned}
\tag{3.18}
$$

Then it remains to estimate $\|G_{\beta,1}\|_{\omega_\beta}^2$.

$$
\begin{aligned}
\|G_{\beta,1}\|_{\omega_\beta}^2 &\le \|\frac{d(\widehat{P}_N^1 v - v)}{d\rho}\|_{\omega_\beta}^2 + \|\frac{dv}{d\rho} - \widehat{P}_N^1\frac{dv}{d\rho}\|_{\omega_\beta}^2 \\
&\le \|\widehat{P}_N^1 v - v\|_{1,\omega_\beta}^2 + \|\frac{dv}{d\rho} - \widehat{P}_N^1\frac{dv}{d\rho}\|_{\omega_\beta}^2
\end{aligned}
$$

With the aid of the above formula, we obtain that

$$
\begin{aligned}
(\beta-\delta)\ \|E_N\|_{\omega_\beta}^2 &\le c(\|\widehat{P}_N^1 v - v\|_{\omega_\beta}^2 + |\widehat{P}_N^1 v - v|_{1,\omega_\beta}^2 \\
&+\|\frac{dv}{d\rho} - \widehat{P}_N^1\frac{dv}{d\rho}\|_{\omega_\beta}^2) \\
&\le c(\|\widehat{P}_N^1 v - v\|_{\omega_\beta}^2 + \|\frac{dv}{d\rho} - \widehat{P}_N^1\frac{dv}{d\rho}\|_{\omega_\beta}^2 \\
&+|\widehat{P}_N^1 v - \tilde{P}_N^1 v|_{1,\omega_\beta}^2 + |\tilde{P}_N^1 v - v|_{1,\omega_\beta}^2)
\end{aligned}
\tag{3.19}
$$

where

$$\delta = 2\gamma + 3\varepsilon$$

By virtue of Lemma 2.5 with $\alpha = 0$, we derive that

$$
\begin{aligned}
|\widehat{P}_N^1 v - \tilde{P}_N^1 v|_{1,\omega_\beta}^2 &\le cN^2\|\widehat{P}_N^1 v - \tilde{P}_N^1 v\|_{\omega_\beta}^2 \\
&\le cN^2(\|\widehat{P}_N^1 v - v\|_{\omega_\beta}^2 + \|v - \tilde{P}_N^1 v\|_{\omega_\beta}^2)
\end{aligned}
$$

Substituting this formula into (3.19), we obtain the following Theorem

Theorem 3.1. If v belongs to $A_{\omega_{0,\beta}}^s(\Lambda)\cap H_{\omega_\beta}^s(\Lambda)$, taking β such that $\beta > \delta$,by (2.16) with $r = 1$ and (2.18), then

$$\|E_N\|_{\omega_\beta} \le cN^{\frac{2-s}{2}}(\|v\|_{B_{\omega_{0,\beta}}^s(\Lambda)} + \|v\|_{A_{0,\beta,\Lambda}^{s-1}} + \|\frac{dv}{d\rho}\|_{A_{0,\beta,\Lambda}^{s-1}})$$

Remark 3.1. The algorithm with fixed parameter β is still applicable, even if $\beta < \delta$. For example, we assume that for certain real number γ

$$(f(z_1,\rho) - f(z_2,\rho))(z_1 - z_2) \le \gamma(z_1 - z_2)^2$$

Then in this case, we take α such that $\beta > 2(\gamma-\alpha)+3\varepsilon = 2\gamma_1 + 3\varepsilon$ and make the variable transformation
$v(\rho) = e^{\alpha\rho}U(\rho),\quad F(U(\rho),\rho) = e^{-\alpha\rho}f(e^{\alpha\rho}U(\rho),\rho) - \alpha U(\rho)$

$$
\begin{cases}
\dfrac{dU(\rho)}{d\rho} = F(U(\rho),\rho) & \rho > 0 \\
U(0) = v_0
\end{cases}
\tag{3.20}
$$

We may use (3.8) to resolve (3.24) and obtain the numerical solution $U_N(\rho)$. Moreover, the condition (3.16) ensures the global accuracy of $U_N(\rho)$.The numerical solution of (3.3) is given by $u_N(\rho) = e^{\alpha\rho}U_N(\rho)$

Remark 3.2. The scheme (3.8)is an implicit scheme. If f(z; t) is nonlinear function for z, this system can be solved by nonlinear iteration. Suppose that $f(z,t)$ fulfills the following Lipschitz condition

$$|f(z_1,\rho) - f(z_2,\rho)| \le L|z_1 - z_2|,\quad L > 0$$

Then iteration process is convergent.

Proof. Since the scheme (3.8) is equivalently to (3.9), we consider the following iteration process

$$
\begin{cases}
\dfrac{du_N^{(m)}(\rho)}{d\rho} = f(u_N^{(m-1)}(\rho),\rho), & \rho \ge 0 \\
u_N^{(m)}(0) = v_0
\end{cases}
$$

Where

$$u_N^{(m)}(\rho) = \sum_{l=0}^{N}\tilde{u}_l^{(m)}\mathscr{L}_l^{(\beta)}$$

Let $\bar{u}_N^{(m)}(\rho) = u_N^{(m)}(\rho) - u_N^{(m-1)}(\rho)$, then

$$
\begin{cases}
(\dfrac{d}{d\rho}\bar{u}_N^{(m)}(\rho),\phi)_{\omega_\beta} = (f(u_N^{(m-1)}(\rho),\rho) - f(u_N^{(m-2)}(\rho),\rho),\phi)_{\omega_\beta}, \\
\qquad\qquad\qquad\qquad \forall\phi \in_0 \mathscr{P}_N(\Lambda), \\
\bar{u}_N^{(m)}(0) = 0
\end{cases}
$$

Taking $\phi = 2\bar{u}_N^{(m)}(\rho)$ in the above equation,we derive that

$$
\begin{aligned}
\beta\|\bar{u}_N^{(m)}(\rho)\|_{\omega_\beta}^2 &\le 2(|f(u_N^{(m-1)}(\rho),\rho) - f(u_N^{(m-2)}(\rho),\rho)|, \\
&|\bar{u}_N^{(m)}(\rho)|)_{\omega_\beta} \\
&\le 2L(|u_N^{(m-1)}(\rho) - u_N^{(m-2)}(\rho)|,|\bar{u}_N^{(m)}(\rho)|)_{\omega_\beta} \\
&\le 2L\|\bar{u}_N^{(m-1)}\|_{\omega_\beta}\|\bar{u}_N^{(m)}\|_{\omega_\beta}
\end{aligned}
$$

This inequality implies

$$\|\bar{u}_N^{(m)}(\rho)\|_{\omega_\beta} \le \frac{2L}{\beta}\|\bar{u}_N^{(m-1)}\|_{\omega_\beta}.$$

Thus, for $\beta > 2L$, the above iteration process is convergent. This fact implies the existence of solution of (3.8).We can prove the uniqueness of solution of (3.8) easily.

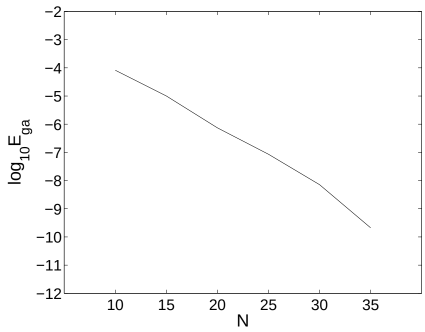

Figure 1 Convergence rates of absolute errors.

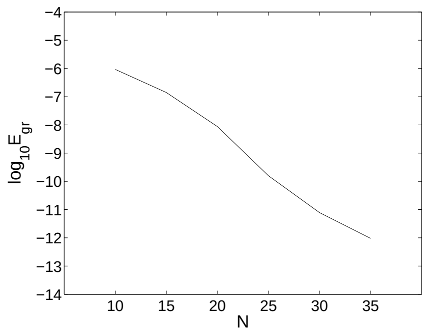

Figure 2 Convergence rates of relative errors.

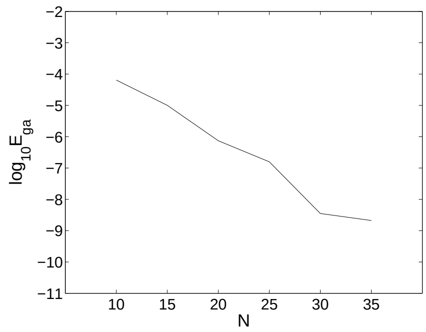

Figure 3 Convergence rates of absolute errors.

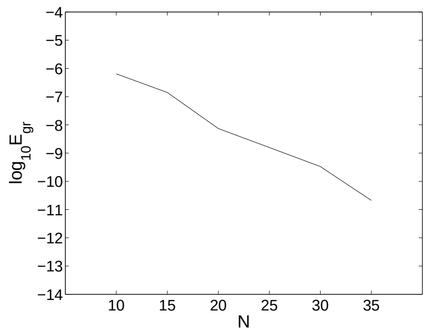

Figure 4 Convergence rates of relative errors.

4. Numerical results

In this section, we present some numerical results. We use scheme (3.8) to solve problem (3.3) with the test function $v(\rho) = (1+\rho)^5 e^{-\rho}$. The corresponding right term at (3.3) is

$$f(v(\rho),\rho) = \tfrac{1}{4}v(\rho) + 5(1+\rho)^4 e^{-\rho} - (1+\rho)^5 e^{-\rho} - \tfrac{1}{4}(1+\rho)^5 e^{-\rho}$$

which fulfills the condition (3.16) with $\gamma = \frac{1}{4}$. Therefore, as predicted by Theorem (3.1), for any $\beta > 2\gamma + 3\varepsilon \simeq 0.5$, the global numerical error $\|u_N - v\|_{\omega_\beta}$ decays exponentially as $N \longrightarrow +\infty$.

To describe the numerical errors, we introduce the global absolute error $E_{ga} = \|u_N - v\|_{\omega_\beta}$ and the global relative error $E_{gr} = \|\frac{u_N - v}{v}\|_{\omega_\beta}$. In Figure1 and Figure2, we plot the global absolute errors $\lg_{10}$ of E_{ga} and the global relative errors $\lg_{10}$ of E_{gr} with various values of N. They indicate that the global errors decay exponentially as N increases, They coincide very well with theoretical analysis.

We next use scheme (3.8) to solve the problem:

$$\begin{cases} \dfrac{dv}{d\rho} = \dfrac{1}{6}\exp(\cos(v(\rho)) + F(\rho), \quad \rho \geq 0, \\ v(0) = v_0, \end{cases}$$

which fulfills the condition (3.16) with $\gamma = \frac{e}{6}$. Take the test function $v(\rho) = e^{-\rho}(\rho+1)^5$. Then a direct computation shows that

$$F(\rho) = 5e^{-\rho}(\rho+1)^4 - e^{-\rho}(\rho+1)^5 - \tfrac{1}{6}\exp(\cos(e^{-\rho}(\rho+1)^5)).$$

Since $f(v,\rho)$ is a nonlinear function for v, we need a nonlinear iteration to solve this system.

In Figure3 and Figure4, we plot the global absolute errors $\log_{10}$ of E_{ga} and the global relative errors $\log_{10}$ of E_{gr} with various values of N. Therefore, as predicted by Theorem (3.1), for any $\beta > 2\gamma \simeq 1$, the global numerical error $\|u_N - v\|_{\omega_\beta}$ decays exponentially as $N \longrightarrow +\infty$.

5. Conclusion

In this paper, we propose a new generalized Laguerre spectral method for the initial problem of the first-order ordinary differential equations, which has fascinating advantages.

• The suggested generalized Laguerre spectral method is based on the modified generalized Laguerre polynomial approximation on the half line. It provides the global numerical solution and the global convergence naturally, thus it is available for long-time numerical simulations of dynamical systems.

• The numerical solution is represented by function form, so it can simulate more entirely the global property of exact solution.

• The numerical results demonstrate that the new generalized Laguerre spectral method possesses the spectral accuracy, which coincides with theoretical analysis very well.

• In this paper, we also develop a powerful framework for analyzing various spectral methods of initial value problems of ODES.

Although we only consider a model problem, the suggested method and technique are also applicable to many other problems, such as various evolutionary partial differential equations and infinite-dimensional nonlinear dynamical system.

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