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Constant-Partially Accelerated Life Tests for Compound Rayleigh Distribution with Progressive First-Failure Censoring

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Abstract: The constant-partially accelerated life tests (PALTs) model under progressive first-failure censoring based on compound Rayleigh distribution is considered in this paper. For this model, the maximum likelihood estimates (MLEs) of its parameters, as well as the corresponding observed Fisher information matrix, are derived. The likelihood equations do not lead to closed form expressions for the MLE, and they need to be solved by using an iterative procedure, such as the Newton-Raphson method. We then evaluate the bias, and mean square error of these estimates; and provide asymptotic, and bootstrap confidence intervals for the parameters. The results in the cases of first-failure censoring, progressive Type II censoring, Type II censoring and complete sample are a special cases. One set of real data has been analyzed for illustrative purposes. Different methods have been compared using Monte Carlo simulations.

Keywords: Compound Rayleigh distribution, constant-partially accelerated life tests, Progressive first-failure censoring, Maximum likelihood estimator, Bootstrap, Monte Carlo simulation

1 Introduction

Accelerated life test sampling plans provide information quickly on the lifetime distribution of products by testing them at higher-than-usual stress level to induce early failures and reduce the testing efforts. The accelerated life-testing, which may be performed either at constant high stress level or linearly increasing stress levels, will enable one to examine the effects of stress factors such as load, pressure, temperature, and voltage on the lifetimes of experimental units. Data collected at such accelerated conditions are then extrapolated through a physically appropriate statistical model to estimate the lifetime distribution at normal use conditions. In PALTs items are tested at both accelerated and use conditions. There are two types of PALTs, one is called step PALTs and the other is called constant PALTs. For more extensive research on ALTs, see [1] and [2]. Constant-stress ALTs, in which the stress remains constant through the test, were studied by several authors. Among others, are [3] and [4]. [5,6] applied constant ALTs to finite mixtures of distributions using Bayes and maximum likelihood methods of estimation, respectively. Step-stress ALT is a special kind of ALT in which the test condition is changed at a given time or upon the occurrence of a specified number of failures; see [7], [8], [9], [10] and [11]. When a test involves two levels of stress with the first one being the normal level and a specific time point for changing the stress, it is referred to as a step PALT. Several authors have dealt with this type of ALT, including, [12] and [13]. Recently, the constant-partially accelerated life tests has received considerable interest among the statisticians. See for example, [14], [15] and [16].

There are many situations in life-testing and reliability studies in which the experimenter may be unable to obtain complete information on failure times of all experimental items. There are also situations wherein the removal of items prior to failure is pre-planned in order to reduce the cost and time associated with testing. The most common censoring schemes are Type-I and Type-II censoring, but the conventional Type-I and Type-II censoring schemes do not have the flexibility of allowing removal of items at points other than the terminal point of the experiment. A generalization of Type-II censoring is the progressive Type-II censoring which allows for units to be removed from the test at points other than the final termination point. Inference, sampling design and generalization based on progressively censored samples

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were studied by [17], [18], [19], [20], [21] and [22]. [23] described a life test in which the experimenter might decide to group the test units into several sets, each as an assembly of test units, and then run all the test units simultaneously until occurrence the first failure in each group. Such a censoring scheme is called a first-failure censoring scheme. If an experimenter desires to remove some sets of test units before observing the first failures in these sets this life test plan is called a progressive first-failure-censoring scheme (first-failure censoring scheme is combined with progressive censoring scheme which introduced by [24].

The two-parameter compound Rayleigh distribution which is denoted by CRD(α,β) provides a population model which is useful in several areas of statistics, including life testing and reliability. The probability density function (PDF), cumulative distribution function (CDF), survival function (SF), and hazard rate function (HRF) of the two-parameter compound Rayleigh distribution are given, respectively, by

$$f_1(t) = 2\alpha\beta^{\alpha} t(\beta+t^2)^{-(\alpha+1)},\ t>0,\ (\beta>0,\ \alpha>0), \tag{1}$$

$$F_1(t) = 1-\beta^{\alpha}(\beta+t^2)^{-\alpha},\ t>0, \tag{2}$$

$$S_1(t) = \beta^{\alpha}(\beta+t^2)^{-\alpha},\ t>0, \tag{3}$$

and

$$H_1(t) = 2\alpha t(\beta+t^2)^{-1},\ t>0, \tag{4}$$

where α and β are the shape and scale parameters respectively.

This paper considers the constant PALT applied to items whose lifetimes under design condition are assumed to follow CRD(α,β) distribution under the progressively first-failure-censored. In Section 2, a description of the model is presented. The MLEs of the involved parameters and approximate confidence intervals are derived in Section 3. In Section 4, the parametric bootstrap confidence intervals are discussed. A simulated data set from CRD(α,β) is analyzed in Section 5. In Section 6, the different methods are compared using Monte Carlo simulations. Some concluding remarks are finally made in Section 7.

2 Model Description and Basic Assumptions

In this section, first-failure censoring scheme is combined with progressive censoring scheme as in [24]. n_1 test independent groups with k_1 items within each group are randomly chosen among n test independent groups with k_1 items within each group is allocated to use condition and $n_2 = n-n_1$ remaining independent groups with k_2 items are subjected to an accelerated condition. progressive first-failure censoring scheme is applied as follows: R_{j1} groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure (say $t^{\mathbf{R}}_{j1:m_j:n_j:k_j}$) has occurred, R_{j2} groups and the group in which the second failure is observed are randomly removed from the test as soon as the second failure (say $t^{\mathbf{R}}_{j2:m_j:n_j:k_j}$) has occurred, and finally R_{jm} $(m_j \leq n_j)$ groups and the group in which the m_j-th failure is observed are randomly removed from the test when the m_j-th failure (say $t^{\mathbf{R}}_{jm_j:m_j:n_j:k_j}$) has occurred. The observations $t^{\mathbf{R}}_{j1:m_j:n_j:k_j} < t^{\mathbf{R}}_{j2:m_j:n_j:k_j} < \ldots < t^{\mathbf{R}}_{jm_j:m_j:n_j:k_j}$ are called progressively first-failure-censored order statistics with progressive censoring scheme $\mathbf{R} = (R_{j1}, R_{j2}, \ldots, R_{jm})$. It is clear that for $j=1,2$, m_j is the number of the first failure observed $(1 < m_j \leq n_j)$ and $n_j = m_j + R_{j1} + R_{j2} + \ldots + R_{jm}$. If the failure times of the $n_j \times k_j$ items originally in the test are from a continuous population with distribution function $F_j(x)$ and probability density function $f_j(x)$, the joint probability density function for $t^{\mathbf{R}}_{j1:m_j:n_j:k_j}, t^{\mathbf{R}}_{j2:m_j:n_j:k_j}, \ldots, t^{\mathbf{R}}_{jm_j:m_j:n_j:k_j}$ is given by

$$L(\alpha,\beta,\lambda|\underline{t}) = \prod_{j=1}^{2} C_j k_j^{m_j} \prod_{i=1}^{m_j} f_j(t^{\mathbf{R}}_{ji:m_j:n_j:k_j})(1-F(t^{\mathbf{R}}_{ji:m_j:n_j:k_j}))^{k_j(R_{ji}+1)-1}, \tag{5}$$

$$0 < t^{\mathbf{R}}_{j1:m_j:n_j:k_j} < t^{\mathbf{R}}_{j2:m_j:n_j:k_j} < \ldots < t^{\mathbf{R}}_{jm_j:m_j:n_j:k_j} < \infty,$$

where

$$C_j = n_j(n_j-R_{j1}-1)(n_j-R_{j1}-R_{j2}-1)\ldots(n_j-R_{j1}-R_{j2}-\ldots R_{j(m_j-1)}-m_j+1).$$

Special cases

It is clear from (5) that the progressive first-failure censored scheme containing the following censoring schemes as special cases:

1. The first-failure censored scheme when $\mathbf{R} = (0, 0, ..., 0)$.
2. The progressive Type II censored order statistics if $k_j = 1$.
3. Usually Type II censored order statistics when $k_j = 1$ and $\mathbf{R} = (0, 0, ..., n_j - m_j)$.
4. The complete sample case when $k = 1$ and $\mathbf{R} = (0, 0, ..., 0)$.

Also, It should be noted that $t^{\mathbf{R}}_{j1:m_j:n_j:k_j}, t^{\mathbf{R}}_{j2:m_j:n_j:k_j}, ..., t^{\mathbf{R}}_{jm_j:m_j:n_j:k_j}$ can be viewed as a progressive Type II censored sample from a population with distribution functions $1-(1-F_j(t))^{k_j}$. For this reason, results for progressive Type II censored order statistics can be extend to progressive first-failure censored order statistics easily. Also, the progressive first-failure-censored plan has advantages in terms of reducing the test time, in which more items are used, but only m_j of $n_j \times k_j$ items are failures.
For more application about progressive-first-failure censoring data the readers may refer to [25], [26], and [27], [28] and [29].

Suppose that the lifetime of an n_1 test independent groups with k_1 items tested at use condition follows a CRD(α, β) with PDF, CDF, SF and HRF, given in (1)-(4). The hazard rate of an item tested at accelerated condition is given by $H_2(t) = \lambda H_1(t)$, where λ is an acceleration factor satisfying $\lambda > 1$. Therefore the HRF, SF, CDF and PDF under accelerated condition are given, for $t > 0$, $(\alpha, \beta) > 0$, $\lambda > 1$, respectively, by

$$H_2(t) = 2\alpha\lambda t(\beta + t^2)^{-1}, \tag{6}$$

$$S_2(t) = \exp\left(-\int_0^t H_2(z)dz\right) = \beta^{\alpha\lambda}(\beta + t^2)^{-\alpha\lambda}, \tag{7}$$

$$F_2(t) = 1 - \beta^{\alpha\lambda}(\beta + t^2)^{-\alpha\lambda}, \tag{8}$$

and

$$f_2(t) = 2\alpha\lambda\beta^{\alpha\lambda}(\beta + t^2)^{-(\alpha\lambda+1)}. \tag{9}$$

3 Maximum Likelihood Estimation

In this section, we first estimate the parameters α, β and λ; by considering the maximum likelihood (ML) methods and then we compute the observed Fisher information based on the likelihood equations. These will enable us to develop pivotal quantities based on the limiting normal distribution, the resulting pivotal quantities can be used to develop approximate confidence interval for the parameters. Finally, using the ML estimates, we construct the parametric bootstrap confidence intervals.

3.1 MLEs

Let, for $j = 1, 2$, and $T^{\mathbf{R}j}_{j1;m_j,n_j} < T^{\mathbf{R}j}_{j2;m_j,n_j} < ... < T^{\mathbf{R}j}_{jm_j;m_j,n_j}$ denote two progressively Type-II censored samples from two populations whose *CDF*s and *PDF*s are as given in (2), (1) and (8), (9), with $\mathbf{R}_j = (R_{j1}, R_{j2}, ..., R_{jm_j})$. the log-likelihood function $\ell(\alpha, \beta, \lambda | \underline{t}) = \log L(\alpha, \beta, \lambda | \underline{t})$ without normalized constant is then given by

$$\begin{aligned}\ell(\alpha,\beta,\lambda|\underline{t}) &= (m_1+m_2)\log\alpha + m_2\log\lambda + \alpha(k_1n_1+\lambda k_2n_2)\log\beta \\ &- \sum_{i=1}^{m_1}(\alpha k_1(R_{1i}+1)+1)\log(\beta+t_{1i}^2) - \sum_{i=1}^{m_2}(\alpha\lambda k_2(R_{2i}+1)+1)\log(\beta+t_{2i}^2).\end{aligned} \tag{10}$$

Calculating the first partial derivatives of (10) with respect to α, β and λ and equating each to zero, we get the likelihood equations as

$$\begin{aligned}\frac{\partial\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\alpha} &= \frac{m_1+m_2}{\alpha} + (k_1n_1+\lambda k_2n_2)\log\beta - k_1\sum_{i=1}^{m_1}(R_{1i}+1)\log(\beta+t_{1i}^2) \\ &-\lambda k_2\sum_{i=1}^{m_2}(R_{2i}+1)\log(\beta+t_{2i}^2) = 0\,,\end{aligned} \tag{11}$$

$$\frac{\partial\ell(\alpha,\beta,\lambda|\underline{t})}{\partial\beta} = \frac{\alpha(k_1n_1+\lambda k_2n_2)}{\beta} - \sum_{i=1}^{m_1}\frac{\alpha k_1(R_{1i}+1)+1}{\beta+t_{1i}^2} - \sum_{i=1}^{m2}\frac{\alpha\lambda k_2(R_{2i}+1)+1}{\beta+t_{2i}^2} = 0, \tag{12}$$

and

$$\frac{\partial \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \lambda} = \frac{m_2}{\lambda} + \alpha k_2 n_2 \log\beta - \alpha k_2 \sum_{i=1}^{m_2} (R_{2i}+1)\log(\beta + t_{2i}^2) = 0. \tag{13}$$

From (11), (12) and (13) we obtain the ML estimates of λ, α and β as

$$\hat{\lambda}(\alpha,\beta) = \frac{m_2}{\alpha k_2 \eta_2(\underline{t},\beta)}, \tag{14}$$

$$\hat{\alpha}(\beta) = \frac{m_1}{k_1 \eta_1(\underline{t},\beta)}, \tag{15}$$

and

$$\frac{m_1}{\eta_1(\underline{t},\beta)} \sum_{i=1}^{m_1} \frac{(R_{1i}+1)t_{1i}^2}{\beta(\beta+t_{1i}^2)} + \frac{m_2}{\eta_2(\underline{t},\beta)} \sum_{i=1}^{m2} \frac{(R_{2i}+1)t_{2i}^2}{\beta(\beta+t_{2i}^2)} - \sum_{i=1}^{m_1} \frac{1}{\beta+t_{1i}^2} - \sum_{i=1}^{m_2} \frac{1}{\beta+t_{2i}^2} = 0, \tag{16}$$

where

$$\eta_1(\underline{t},\beta) = \sum_{i=1}^{m_1} (R_{1i}+1)\log(1+\frac{t_{1i}^2}{\beta}), \tag{17}$$

and

$$\eta_2(\underline{t},\beta) = \sum_{i=1}^{m_2} (R_{2i}+1)\log(1+\frac{t_{2i}^2}{\beta}). \tag{18}$$

Thus, the ML estimate $\hat{\beta}$ of the parameter β can be obtained by solving the nonlinear likelihood Equation (16) using, for example, the Newton–Raphson iteration scheme. The corresponding ML estimates $\hat{\alpha}$ and $\hat{\lambda}$ of the parameters α and λ are computed from Equations (15) and (14), respectively.

3.2 Approximate interval estimation

The asymptotic variances and covariances of the MLE for parameters α, β and λ are given by elements of the inverse of the Fisher information matrix. From the log-likelihood function in (10), we have

$$\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \alpha^2} = -\frac{m_1+m_2}{\alpha^2}, \tag{19}$$

$$\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \beta^2} = -\frac{\alpha(k_1 n_1 + \lambda k_2 n_2)}{\beta^2} + \sum_{i=1}^{m_1} \frac{\alpha k_1 (R_{1i}+1)+1}{(\beta+t_{1i}^2)^2} + \sum_{i=1}^{m2} \frac{\alpha\lambda k_2 (R_{2i}+1)+1}{(\beta+t_{2i}^2)^2}, \tag{20}$$

$$\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \lambda^2} = -\frac{m_2}{\lambda^2}, \tag{21}$$

$$\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \alpha \partial \beta} = \frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \beta \partial \alpha} = \frac{k_1 n_1 + \lambda k_2 n_2}{\beta} - k_1 \sum_{i=1}^{m_1} \frac{(R_{1i}+1)}{\beta+t_{1i}^2} - \lambda k_2 \sum_{i=1}^{m2} \frac{(R_{2i}+1)}{\beta+t_{2i}}, \tag{22}$$

$$\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \alpha \partial \lambda} = \frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \lambda \partial \alpha} = -\eta_2(\underline{t},\beta), \tag{23}$$

and

$$\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \beta \partial \lambda} = \frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \lambda \partial \beta} = -\alpha k_2 \sum_{i=1}^{m2} \frac{(R_{2i}+1)t_{2i}^2}{\beta\,(\beta+t_{2i}^2)}. \tag{24}$$

The observed Fisher information matrix $I(\alpha,\beta,\lambda)$, for the MLEs ($\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$), see [30], is the 3×3 symmetric matrix of negative second partial derivatives of the log-likelihood function with respect to (α, β and λ). In practice , we usually estimate $I^{-1}(\alpha,\beta,\lambda)$ by $I^{-1}(\hat{\alpha},\hat{\beta},\hat{\lambda})$

$$I_0^{-1}(\hat{\alpha},\hat{\beta},\hat{\lambda}) = \begin{bmatrix} -\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \alpha 2} & -\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \alpha \partial \beta} & -\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \alpha \partial \lambda} \\ -\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \beta 2} & -\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \beta \partial \lambda} \\ -\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \lambda \partial \alpha} & -\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \lambda \partial \beta} & -\frac{\partial^2 \ell(\alpha,\beta,\lambda|\underline{t})}{\partial \lambda 2} \end{bmatrix}^{-1}_{(\hat{\alpha},\hat{\beta},\hat{\lambda})}. \tag{25}$$

Table 1: Simulated progressively censored samples with constant PALTs.

R_i	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1
	1	1	1	1	1	1	1	1	1	1
t_1	0.1037	0.1141	0.1275	0.1359	0.1828	0.1934	0.2019	0.2460	0.2477	0.2690
	0.2850	0.3133	0.3489	0.3543	0.3558	0.3614	0.3750	0.3755	0.3849	0.4044
	0.4198	0.4277	0.4756	0.5249	0.5271	0.5688	0.7789	0.8191	0.9419	1.0810
R_i	2	2	2	2	2	2	2	2	2	2
	1	1	1	1	1	1	1	1	1	1
t_2	0.1121	0.1249	0.1443	0.1491	0.1512	0.2313	0.2339	0.2398	0.2430	0.2511
	0.2916	0.2998	0.3265	0.3641	0.4253	0.4294	0.4323	0.4389	0.4798	0.8150

Table 2: MLEs, MSEs, RABs and (90%-95%) approximate confidence intervales.

Parameters	$(.)_{ML}$	RAB	MSE	90%	95%
$\alpha = 0.5$	0.4781	0.0437	0.0219	(-0.1995, 1.1558)	(0.2003, 0.7560)
$\beta = 0.8$	0.8083	0.0104	0.0083	(-0.5533, 2.1699)	(0.2500, 1.3666)
$\lambda = 1.5$	1.4681	0.0212	0.0319	(0.7497, 2.1866)	(1.1735, 1.7628)

Table 3: Percentile bootstrap CIs and Bootstrap-t CIs based on 10000 replications.

Parameters	Bootstrap-p				Bootstrap-t			
	$(.)_{Boot\_P}$	RAB	MSE	95%	$(.)_{Boot\_t}$	RAB	MSE	95%
$\alpha = 0.5$	0.4641	0.0718	0.0359	(0.3628, 0.5450)	0.3887	0.2226	0.1113	(0.3015, 0.5258)
$\beta = 0.8$	0.7658	0.0427	0.0342	(0.5895, 0.9269)	0.6661	0.1675	0.1340	(0.4852, 0.8927)
$\lambda = 1.5$	1.5422	0.0281	0.0422	(1.2168, 1.7923)	1.4099	0.0601	0.0901	(1.1610, 1.7287)

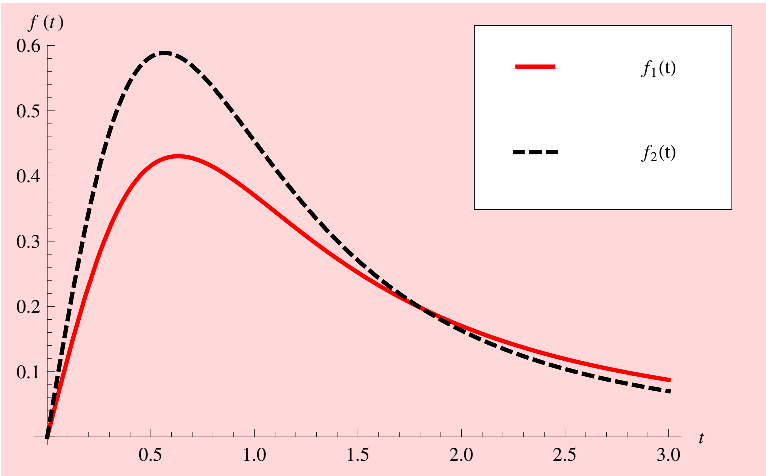

Fig. 1: Plot the density functions f_1(t) and f_2(t).

Thus, the $100(1-\gamma)\%$ approximate confidence intervals for α, β and λ are

$$\hat{\alpha} \mp z_{\frac{\gamma}{2}}\sqrt{v_{11}},\ \hat{\beta} \mp z_{\frac{\gamma}{2}}\sqrt{v_{22}} \text{ and } \hat{\lambda} \mp z_{\frac{\gamma}{2}}\sqrt{v_{33}}, \tag{26}$$

where v_{11}, v_{22} and v_{33} are the elements on the main diagonal of the covariance matrix $I^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ and $z_{\frac{\gamma}{2}}$ is the percentile of the standard normal distribution with right-tail probability $\frac{\gamma}{2}$.

Table 4: MLEs, MSEs and RABs for the parameters (α ,β ,λ) at (0.7, 0.3, 1.5) with $k_1 = k_2 = k$.

(k,n,m)	CS	AVG			MSE			RAB		
		$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$
(1,40,30)	I	0.889	0.379	1.434	0.346	0.186	0.362	0.269	0.263	0.144
	II	0.892	0.382	1.532	0.356	0.193	0.433	0.273	0.317	0.197
	III	0.921	0.454	1.664	0.467	0.275	0.537	0.3167	0.514	0.199
(1,40,35)	I	0.833	0.304	1.404	0.225	0.204	0.287	0.172	0.246	0.147
	II	0.821	0.398	1.534	0.240	0.217	0.326	0.181	0.296	0.153
	III	0.831	0.385	1.573	0.312	0.224	0.459	0.197	0.382	0.169
(1,50,30)	I	0.727	0.333	1.630	0.209	0.162	0.464	0.039	0.111	0.137
	II	0.842	0.376	1.637	0.243	0.187	0.497	0.191	0.381	0.148
	III	0.874	0.421	1.676	0.328	0.202	0.523	0.248	0.402	0.162
(1,50,40)	I	0.784	0.362	1.543	0.156	0.165	0.329	0.120	0.206	0.112
	II	0.798	0.371	1.570	0.184	0.174	0.346	0.139	0.319	0.132
	III	0.810	0.396	1.590	0.213	0.186	0.454	0.157	0.386	0.143
(1,70,50)	I	0.642	0.290	1.562	0.096	0.130	0.338	0.056	0.065	0.043
	II	0.651	0.312	1.578	0.129	0.158	0.342	0.068	0.087	0.058
	III	0.698	0.318	1.597	0.156	0.161	0.352	0.084	0.099	0.065
(1,70,60)	I	0.657	0.221	1.422	0.112	0.104	0.227	0.032	0.030	0.022
	II	0.719	0.247	1.475	0.136	0.112	0.231	0.042	0.070	0.037
	III	0.757	0.321	1.513	0.145	0.128	0.280	0.082	0.071	0.039
(5,40,30)	I	0.818	0.368	1.574	0.334	0.179	0.449	0.169	0.226	0.130
	II	0.987	0.437	1.617	0.429	0.234	0.453	0.342	0.347	0.114
	III	1.001	0.455	1.707	0.608	0.286	0.469	0.441	0.517	0.158
(5,40,35)	I	1.011	0.442	1.499	0.504	0.215	0.294	0.385	0.306	0.046
	II	1.023	0.450	1.509	0.534	0.249	0.0349	0.391	0.341	0.064
	III	0.997	0.487	1.637	0.636	0.348	0.404	0.446	0.424	0.035
(5,50,30)	I	0.707	0.294	1.493	0.284	0.107	0.378	0.101	0.021	0.044
	II	0.702	0.311	1.532	0.325	0.128	0.342	0.192	0.108	0.057
	III	0.852	0.388	1.721	0.458	0.219	0.455	0.217	0.295	0.147
(5,50,40)	I	0.896	0.337	1.377	0.391	0.181	0.343	0.324	0.315	0.035
	II	0.901	0.348	1.398	0.432	0.201	0.354	0.329	0.323	0.041
	III	0.935	0.401	1.481	0.660	0.253	0.307	0.336	0.337	0.044
(5,70,50)	I	0.797	0.409	1.453	0.371	0.177	0.272	0.282	0.332	0.025
	II	0.783	0.410	1.480	0.375	0.181	0.291	0.327	0.363	0.031
	III	0.966	0.425	1.472	0.427	0.197	0.314	0.380	0.417	0.038
(5,70,60)	I	0.826	0.368	1.301	0.321	0.134	0.277	0.180	0.225	0.027
	II	0.875	0.375	1.371	0.431	0.147	0.0219	0.172	0.391	0.029
	III	0.893	0.479	1.472	0.507	0.298	0.307	0.562	0.595	0.029

4 Bootstrap Confidence Intervals

The bootstrap is a resampling method for statistical inference. It is commonly used to estimate confidence intervals, but it can also be used to estimate bias and variance of an estimator or calibrate hypothesis tests. More survey of the nonparametric and parametric bootstrap methods [31], [32]. In this section, the two confidence intervals based on the parametric bootstrap methods are proposed: percentile bootstrap method (Boot-p) based on the idea of [33]. (ii) bootstrap-t method (Boot-t) based on the idea of [34]. The algorithms for estimating the confidence intervals of parameters using both methods are illustrated below.

1. Based on the original progressively first-failure censored sample, $(t_{j1;m_j,n_j,k_j} < t_{j2;m_j,n_j,k_j} < \ldots < t_{jm_j;m_j,n_j,k_j})$, obtain $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\lambda}$, $j = 1,2$.
2. Based on the values of n_j, m_j and $k_j(1 < m_j < n_j)$ with the same values of R_{ji}, $(i = 1, 2, \ldots, m_j)$, $j = 1,2$, generate two independent random samples of sizes m_1 and m_2 from compound Rayleigh distribution, $\underline{t}^* = (t^*_{j1;m_j,n_j} < t^*_{j2;m_j,n_j} < \ldots < t^*_{jm_j;m_j,n_j})$ by using the algorithm described in [35] with distribution functions $1-(1-F_j(t))^{k_j}$.
3. As in step 1 based on $\underline{t}^*$ compute the bootstrap sample estimates of $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\lambda}$ say $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{\lambda}^*$.
4. Repeat the above steps 2 and 3 N^* times representing N^* different bootstrap samples. The value of N^* has been taken to be 1000.
5. Arrange all $\hat{\alpha}^*$, $\hat{\beta}^*$ and $\hat{\lambda}^*$ in an ascending order to obtain the bootstrap sample $(\hat{\varphi}_\ell^{*[1]}, \hat{\varphi}_\ell^{*[2]}, \ldots, \hat{\varphi}_\ell^{*[B]})$, $\ell = 1,2,3$ where $(\varphi_1^* = \alpha^*, \varphi_2^* = \beta^*, \varphi_3^* = \lambda^*)$.

Percentile bootstrap confidence interval:

Let $G(z) = P(\hat{\varphi}_\ell^* \leqslant z)$ be cumulative distribution function of $\hat{\varphi}_\ell^*$. Define $\hat{\varphi}_{\ell boot}^* = G^{-1}(z)$ for given z. The approximate bootstrap $100(1-\gamma)\%$ confidence interval of $\hat{\varphi}_\ell^*$ given by

$$\left[\hat{\varphi}_{\ell boot}^*(\frac{\gamma}{2}), \hat{\varphi}_{\ell boot}^*(1-\frac{\gamma}{2})\right]. \tag{27}$$

Bootstrap-t confidence interval

Table 5: Comparisons of (AC) and (CP) of 95% confidence intervals (α,β,λ) at $(0.7,0.3,1.5)$ with $k_1 = k_2 = k$.

(k,n,m)	CS	MLE			Boot-P			Boo-t		
		α	β	λ	α	β	λ	α	β	λ
(1,40,30)	*I*	1.622	0.925	1.715	1.367	0.637	1.555	1.023	0.521	1.615
		(0.953)	(0.392)	(0.962)	(0.942)	(0.962)	(0.963)	(0.915)	(0.931)	(0.923)
	II	1.543	1.018	1.754	1.427	0.641	1.645	1.216	0.563	1.742
		(0.923)	(0.903)	(0.933)	(0.931)	(0.943)	(0.925)	(0.930)	(0.889)	(0.925)
	III	1.978	1.247	1.789	1.525	0.666	1.969	1.371	0.684	1.865
		(0.923)	(0.906)	(0.921)	(0.89)	(0.944)	(0.923)	(0.902)	(0.932)	(0.954)
(1,40,35)	*I*	0.852	0.677	1.338	0.912	0.421	1.355	0.697	0.513	1.398
		(0.934)	(0.952)	(0.961)	(0.923)	(0.962)	(0.944)	(0.934)	(0.946)	(0.907)
	II	0.948	0.732	1.432	0.954	0.498	1.428	0.743	0.564	1.436
		(0.943)	(0.927)	(0.918)	(0.933)	(0.936)	(0.938)	(0.942)	(0.947)	(0.934)
	III	1.079	0.806	1.547	1.031	0.653	1.712	0.835	0.664	1.574
		(0.945)	(0.930)	(0.955)	(0.947)	(0.958)	(0.932)	(0.908)	(0.939)	(0.961)
(1,50,30)	*I*	0.778	0.606	1.458	0.936	0.610	1.609	0.703	0.432	1.513
		(0.950)	(0.943)	(0.961)	(0.928)	(0.962)	(0.959)	(0.951)	(0.963)	(0.963)
	II	0.845	0.786	1.564	0.972	0.651	1.732	0.864	0.479	1.564
		(0.937)	(0.935)	(0.961)	(0.945)	(0.953)	(0.952)	(0.949)	(0.954)	(0.946)
	III	1.903	1.405	1.775	1.351	0.670	1.923	1.084	0.523	1.787
		(0.945)	(0.960)	(0.966)	(0.933)	(0.950)	(0.925)	(0.919)	(0.931)	(0.948)
(1,50,40)	*I*	0.721	0.606	1.222	0.822	0.572	1.416	0.660	0.463	1.443
		(0.950)	(0.966)	(0.967)	(0.952)	(0.955)	(0.962)	(0.965)	(0.957)	(0.971)
	II	0.843	0.649	1.320	0.876	0.611	1439	0.764	0.514	1.397
		(0.944)	(0.963)	(0.954)	(0.964)	(0.947)	(0.962)	(0.946)	(0.939)	(0.941)
	III	1.021	0.772	1.391	1.051	0.631	1.517	0.811	0.540	1.451
		(0.965)	(0.790)	(0.966)	(0.962)	(0.950)	(0.963)	(0.974)	(0.963)	(0.960)
(1,70,50)	*I*	0.541	0.403	1.287	0.635	0.461	1.395	0.522	0.391	1.212
		(0.967)	(0.958)	(0.972)	(0.964)	(0.942)	(0.956)	(0.963)	(0.960)	(0.964)
	II	0.654	0.511	1.302	0.739	0.487	1.412	0.543	0.412	1.259
		(0.953)	(0.961)	(0.949)	(0.944)	(0.957)	(0.968)	(0.950)	(0.964)	(0.958)
	III	0.836	0.612	1.312	0.907	0.589	1.429	0.623	0.433	1.311
		(0.953)	(0.960)	(0.953)	(0.954)	(0.956)	(0.966)	(0.937)	(0.971)	(0.963)
(1,70,60)	*I*	0.567	0.403	1.028	0.659	0.454	1.139	0.542	0.389	1.058
		(0.976)	(0.968)	(0.956)	(0.929)	(0.962)	(0.953)	(0.956)	(0.978)	(0.958)
	II	0.539	0.434	1.104	0.743	0.513	1.154	0.563	0.399	1.119
		(0.959)	(0.974)	(0.966)	(0.973)	(0.959)	(0.964)	(0.957)	(0.948)	(0.966)
	III	0.672	0.468	1.117	0.807	0.556	1.262	0.593	0.420	1.177
		(0.965)	(0.976)	(0.975)	(0.967)	(0.952)	(0.945)	(0.958)	(0.965)	(0.965)

First, find the order statistics $\delta_\ell^{*[1]} < \delta_\ell^{*[2]} < \ldots < \delta_\ell^{*[B]}$, where

$$\delta_\ell^{*[j]} = \frac{\hat{\varphi}_\ell^{*[j]} - \hat{\varphi}_\ell}{\sqrt{\text{var}\left(\hat{\varphi}_\ell^{*[j]}\right)}}, \ j = 1,2,\ldots,B, \ \ell = 1,2,3, \tag{28}$$

where $\hat{\varphi}_1 = \hat{\alpha}$, $\hat{\varphi}_2 = \hat{\beta}$, $\hat{\varphi}_3 = \hat{\lambda}$.

Let $H(z) = P(\delta_\ell^* < z)$ be the cumulative distribution function of δ_ℓ^*. For a given z, define

$$\hat{\varphi}_{\ell\text{boot}-t} = \hat{\varphi}_\ell + \sqrt{\text{Var}(\hat{\varphi}_\ell)}H^{-1}(z). \tag{29}$$

The approximate $100(1-\gamma)\%$ confidence interval of $\hat{\varphi}_\ell$ is given by

$$\left(\hat{\varphi}_{\ell\text{boot}-t}(\frac{\gamma}{2}), \hat{\varphi}_{\ell\text{boot}-t}(1-\frac{\gamma}{2})\right). \tag{30}$$

5 Illustrative Example

Let us consider the simulated data presented in Table 1 with sample size $m_1 = 30$ and $m_2 = 20$ of $n_1 = 60$ and $n_2 = 50$ with $k_1 = k_2 = 5$ are generated from compound Rayleigh distribution with parameters $(\alpha,\beta,\lambda) = (0.5,0.8,1.5)$ and two progressive censoring scheme R_1 and R_2. Figure 1 shows the probability density functions under normal conditions and accelerate conditions. By substituting from Eq. (14) and Eq. (15) in Eq. (10), we plot the profile log-likelihood function of β as in Figure 2. It is a unimodal function. We can use any iteration procedure such as quasi-Newton Raphson or fixed

Table 6: Continue

(k,n,m)	CS	MLE α	MLE β	MLE λ	Boot-P α	Boot-P β	Boot-P λ	Boo-t α	Boo-t β	Boo-t λ
(5,40,30)	*I*	1.375	1.511	1.567	1.561	0.616	1.627	1.307	0.416	1.643
		(0.920)	(0.934)	(0.944)	(0.921)	(0.932)	(0.954)	(0.934)	(0.922)	(0.946)
	II	1.409	1.563	1.654	1.627	0.634	1.733	1.345	0.457	1.709
		(0.947)	(0.926)	(0.935)	(0.947)	(0.954)	(0.942)	(0.957)	(0.945)	(0.937)
	III	1.502	1.557	1.675	1.687	0.647	1.767	1.461	0.543	1.772
		(0.945)	(0.943)	(0.945)	(0.963)	(0.954)	(0.964)	(0.919)	(0.954)	(0.964)
(5,40,35)	*I*	1.627	1.715	1.549	1.478	0.631	1.537	1.364	0.454	1.476
		(0.922)	(0.932)	(0.944)	(0.929)	(0.948)	(0.974)	(0.965)	(0.956)	(0.971)
	II	1.754	1.869	1.501	1.543	0.639	1.576	1.437	0.501	1.522
		(0.945)	(0.935)	(0.965)	(0.954)	(0.955)	(0.945)	(0.938)	(0.945)	(0.966)
	III	1.868	2.971	1.512	1.656	0.649	1.606	1.557	0.531	1.542
		(0.935)	(0.961)	(0.934)	(0.933)	(0.941)	(0.946)	(0.908)	(0.964)	(0.933)
(5,50,30)	*I*	1.275	1.037	1.567	1.616	0.514	1.698	1.028	0.323	1.614
		(0.922)	(0.942)	(0.921)	(0.942)	(0.955)	(0.976)	(0.943)	(0.932)	(0.953)
	II	1.341	1.231	1.458	1.637	0.534	1.736	1.114	0.354	1.774
		(0.955)	(0.957)	(0.939)	(0.946)	(0.953)	(0.963)	(0.961)	(0.925)	(0.944)
	III	1.458	1.482	1.898	1.668	0.658	1.941	1.239	0.492	1.896
		(0.929)	(0.901)	(0.932)	(0.943)	(0.966)	(0.954)	(0.928)	(0.973)	(0.953)
(5,50,40)	*I*	1.454	1.512	1.341	1.432	0.555	1.332	1.245	0.476	1.282
		(0.965)	(0.963)	(0.967)	(0.959)	(0.962)	(0.986)	(0.973)	(0.954)	(0.962)
	II	1.453	1.537	1.351	1.543	0.618	1.453	1.321	0.488	1.297
		(0.971)	(0.963)	(0.956)	(0.966)	(0.954)	(0.964)	(0.939)	(0.965)	(0.971)
	III	1.567	1.559	1.355	1.574	0.636	1.534	1.379	0.585	1.343
		(0.976)	(0.965)	(0.948)	(0.965)	(0.975)	(0.949)	(0.967)	(0.970)	(0.961)
(5,70,50)	*I*	1.509	1.346	1.223	1.411	0.627	1.198	1.084	0.424	1.180
		(0.965)	(0.974)	(0.977)	(0.956)	(0.971)	(0.965)	(0.973)	(0.953)	(0.966)
	II	1.675	1.447	1.221	1.522	0.634	1.227	1.118	0.476	1.231
		(0.968)	(0.975)	(0.973)	(0.959)	(0.978)	(0.953)	(0.976)	(0.968)	(0.955)
	III	1.739	1.748	1.238	1.542	0.645	1.258	1.126	0.501	1.314
		(0.967)	(0.954)	(0.971)	(0.954)	(0.975)	(0.968)	(0.973)	(0.934)	(0.963)
(5,70,60)	*I*	1.349	1.017	0.972	1.362	0.684	1.155	1.034	0.409	1.126
		(0.972)	(0.964)	(0.966)	(0.982)	(0.975)	(0.967)	(0.967)	(0.956)	(0.965)
	II	1.364	1.231	0.984	1.408	0.698	1.219	1.117	0.432	1.131
		(0.977)	(0.968)	(0.980)	(0.975)	(0.969)	(0.967)	(0.985)	(0.982)	(0.967)
	III	1.416	1.415	1.093	1.501	0.705	1.236	1.291	0.444	1.133
		(0.967)	(0.955)	(0.972)	(0.973)	(0.959)	(0.958)	(0.947)	(0.971)	(0.962)

Table 7: Comparisons of (AC) and (CP) of 95% confidence intervals (α,β,λ) at (1, 0.7, 2) with $k_1 = k_2 = k$.

(k,n_1,m_1,n_2,m_2)	CS	MLE α	MLE β	MLE λ	Boot-P α	Boot-P β	Boot-P λ	Boo-t α	Boo-t β	Boo-t λ
(1,60,40,50,30)	*I*	0.999	1.056	1.792	0.844	0.663	1.583	0.844	0.762	1.554
		(0.934)	(0.921)	(0.954)	(0.923)	(0.933)	(0.941)	(0.962)	(0.937)	(0.937)
	II	1.249	1.273	1.865	0.867	0.669	1.612	0.965	0.798	1.637
		(0.934)	(0.929)	(0.917)	(0.943)	(0.908)	(0.922)	(0.931)	(0.932)	(0.908)
	III	1.827	1.439	1.978	0.934	0.675	1.679	1.054	0.815	1.822
		(0.925)	(0.907)	(0.944)	(0.939)	(0.941)	(0.938)	(0.960)	(0.938)	(0.932)
(1,60,50,50,40)	*I*	0.915	0.914	1.668	0.827	0.619	1.571	0.799	0.676	1.504
		(0.952)	(0.949)	(0.964)	(0.972)	(0.933)	(0.948)	(0.965)	(0.955)	(0.947)
	II	1.129	1.218	1.733	0.843	0.637	1.645	0.863	0.743	1.660
		(0.939)	(0.954)	(0.950)	(0.944)	(0.967)	(0.964)	(0.946)	(0.955)	(0.948)
	III	1.337	1.381	1.789	0.858	0.651	1.719	0.977	0.835	1.770
		(0.943)	(0.948)	(0.955)	(0.959)	(0.964)	(0.976)	(0.954)	(0.972)	(0.965)
(5,60,40,50,30)	*I*	1.544	1.654	1.567	0.801	0.453	1.673	1.262	0.656	1.542
		(0.964)	(0.943)	(0.957)	(0.982)	(0.966)	(0.954)	(0.954)	(0.967)	(0.957)
	II	1.672	1.774	1.870	0.883	0.511	1.708	1.331	0.697	1.654
		(0.949)	(0.955)	(0.938)	(0.973)	(0.954)	(0.960)	(0.939)	(0.966)	(0.950)
	III	1.896	1.970	1.925	0.931	0.522	1.806	1.519	0.719	1.721
		(0.964)	(0.966)	(0.948)	(0.953)	(0.955)	(0.967)	(0.976)	(0.947)	(0.956)
(5,60,50,50,40)	*I*	1.507	1.597	1.659	0.714	0.508	1.560	0.884	0.648	1.519
		(0.975)	(0.977)	(0.967)	(0.980)	(0.971)	(0.965)	(0.973)	(0.968)	(0.972)
	II	1.602	1.609	1.763	0.811	0.533	1.640	0.973	0.674	1.640
		(0.974)	(0.980)	(0.966)	(0.949)	(0.958)	(0.977)	(0.981)	(0.972)	(0.964)
	III	1.736	1.719	1.866	0.934	0.653	1.691	1.223	0.761	1.747
		(0.972)	(0.965)	(0.945)	(0.939)	(0.957)	(0.963)	(0.974)	(0.967)	(0.960)

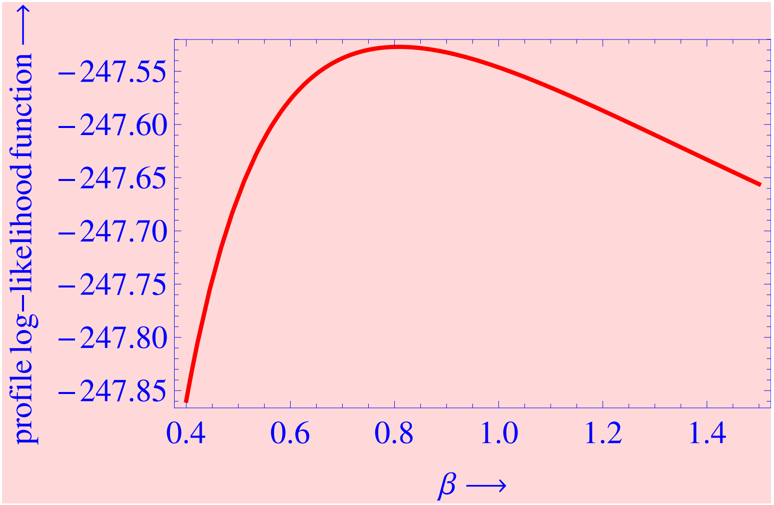

Fig. 2: Profile log-likelihood function of β.

point algorithm to compute the MLE with the initial guess of 0.73. The point estimates and relate relative absolute biases (RABs) and mean squared errors (MSEs) of the parameters as well as (90% and 95%) approximate confidence intervals are presented in Table 2. Also the point estimates and relate (RABs and MSEs) of the parameters as well as (90% and 95%) percentile bootstrap and bootstrap-t confidence intervals are presented in Table 3. We, observed that the percentile bootstrap and bootstrap-t confidence intervals always include the population parameter values.

6 Simulation Studies

In order to obtain the MLEs of (α,β,λ) and study the properties of their estimates through the MSEs and RABs. A Monte Carlo simulation study is carried out in order to calculate the MLEs, MSEs, RABs and 90% approximate confidence intervals of the model parameters, based on N 1000 Monte Carlo simulations. Based on N^* 1000 bootstrap replications, then the average of MLE, MSE and RAB of ψ_ℓ, $\ell = 1,2,3$ (Where $\psi_1 \equiv \alpha$, $\psi_2 \equiv \beta$, $\psi_3 \equiv \lambda$) over the N samples are given, respectively, by

$$\overline{\hat{\psi}}_\ell = \frac{1}{N}\sum_{i=1}^{N}\hat{\psi}_{\ell i},$$

$$\mathrm{MSE}(\hat{\psi}_\ell) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\psi}_{\ell i}-\psi_\ell)^2,$$

$$\mathrm{RAB}(\hat{\psi}_\ell) = \frac{|\overline{\hat{\psi}}_\ell-\psi_\ell|}{\psi_\ell}.$$

The different confidence intervals, namely the confidence intervals obtained by using asymptotic distributions of the MLEs and the two different bootstrap confidence intervals in terms of the average confidence lengths (ACL) and coverage percentages (CP) are calculated and compared. For each simulated sample under a particular setting, we computed 95% confidence intervals and checked whether the true value lay within the interval and recorded the length of the confidence interval. In our study we have used three different censoring schemes (C.S), namely:

Scheme I: $R_1 = n-m, R_i = 0$ for $i \neq 1$.

Scheme II: $R_{\frac{m}{2}} = n-m, R_i = 0$ for $i \neq \frac{m}{2}$ if m even

and $R_{\frac{m+1}{2}} = n-m, R_i = 0$ for $i \neq \frac{m+1}{2}$ if m odd.

Scheme III: $R_m = n-m, R_i = 0$ for $i \neq m$.

We consider two cases separately to draw inference on parameters, namely: (i) The parameter values $(\alpha = 0.7, \beta = 0.3, \lambda = 1.5)$, for different choices of sample sizes $(n_1 = n_2 = n)$, and observed failure times $(m_1 = m_2 = m)$, with $k_1 = k_2 = k$ and (ii) The parameter values $(\alpha = 1.0, \beta = 0.7, \lambda = 2)$, based on different values of n_1, n_2,m_1 and m_2, with $k_1 = k_2 = k$.

7 Perspective

In this article, we have considered the constant-partially accelerated life tests with progressive first-failure censoring when the observed data comes from compound Rayleigh distribution. We have obtained the MLEs and parametric bootstrap methods are used for estimating the unknown parameters of compound Rayleigh distribution. The progressive first-failure censored sampling plan has an advantage in terms of shorter test-time, a saving of resources, and in which a specific fraction of individuals at risk may be removed from the experiment at each of several ordered failure times. The familiar complete, Type II right censored, first-failure censored and progressively Type II right censored samples are special cases of the progressive first-failure censored sampling plan. From empirical evidence in Tables 4 and 5, we have:

(i)The censoring scheme I namely, ($R=(n-m,...,0)$, in the sense for fixed n and m, $n-m$ items are removed at the time of the first failure) is most efficient for all choices, it seems to usually provide the smallest MSEs and RABs for all estimators.
(ii)For fixed values of the sample size, by increasing the failure times the MSEs and RABs of the considered parameters decrease.
(iii)We observe that from Tables 5 and 6, in most cases the estimated coverage probability is close to the nominal level of 0.95 based on different effective sample sizes m, different k.
(iv)The bootstrap confidence intervales give more accurate results than the approximate confidence intervales since the lengths of the former are less than the lengths of latter, for different sample sizes, observed failures and schemes.
(v)For fixed k, when the effective sample proportion m/n increases, the MSEs and the average confidence interval lengths of the ML and parametric bootstrap estimators are reduced.
(vi)The MSEs and average confidence interval lengths for the estimates of the parameters and for the proposed progressive first-failure censoring ($k=5$) are similar to those for progressive Type-II censoring ($k=1$).

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Conflict of interest

The authors declare that they have no conflict of interest.

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