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A Spectral Estimation of Discrete Harmonizable Process

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Abstract: In this paper, we develop the non-parametric spectral analysis for non-stationary discrete-time stochastic processes. This development will be investigated by using the multi-tapering and averaging technique. In particular, we obtain an estimator for the spectral density function of a harmonizable time series. This estimator is constructed by dividing the available time series into a number of overlapped and non-overlapped segments and then a multi-tapering technique is applied for each segment. Also, we obtain an estimator for the auto-covariance function and another estimator for the spectral distribution function of these processes, based on the spectral density estimator. Statistical properties of these estimators are investigated, including the asymptotic behaviour of the bias and covariance.

Keywords: Harmonizable process, discrete time series, nonstationary processes, unbiased estimator, multi-tapering

1 Introduction

Harmonizable process constitute an important class of non-stationary stochastic processes, Lii and Rosenblaat [1] proposed the spectral representation of discrete-and continuous-time harmonizable processes, and discussed the asymptotic behaviour of the bias and covariance of a number of spectral estimators in the continuous-time case. Fuentes [2] introduced a non-stationary second-order periodogram to estimate the spectral density function of a non-stationary continuous-time stochastic process, and then studied its statistical properties.

Many authors, as Brillinger [3], and Brillinger and Rosen-Blatt [4]; Dahlhaus [5]; Ghazal and Farag [6], have studied the asymptotic expressions of the first and second-order moments of spectral estimates via untapered and tapered data.

Ghazal [7] studied the statistical properties, of the spectral density estimate on non-overlapped intervals.

Ghazal and Farag [8] studied the statistical analysis of the spectral density estimate on over lapped intervals.

Teamah and Bakouch [9] studied the statistical properties of the spectral estimates on non-overlapped interval via different tapers for the continuous time process.

H. S. Bakouch, Mohie El-Din & et al. [10] introduced the spectral estimates for short discrete-time series on non-overlapped intervals.

Vasily L. Kazakov, Dmitry O. Moskaletz, Oleg D. Moskaletz, Mikhail A. Vaganov [11] studied transformation of a harmonized random process by spectral devices that perform instantaneous spectrum analysis.

The aim of this paper, is to develop the non-parametric spectral analysis for harmonizable processes. This development will be investigated by using the multi-tapering and averaging technique. In particular, we obtain an estimator for the spectral density function of a harmonizable time series. This estimator is constructed by dividing the available time series into a number of overlapped and non-overlapped segments and then a multi-tapering technique is applied for each segment.

Also, we get an estimator for the auto-covariance function and another estimator for the spectral distribution function of these processes, based on the spectral density estimator. Statistical properties of these estimators are investigated, including the asymptotic behaviour of the bias and covariance.

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2 Spectral Density Estimator

Let $X(t)$, $-\infty < t < \infty$, be a real-valued second-order process and absolutely summable process with a zero mean and let $f_{XX}(\lambda_1,\lambda_2), F_{XX}(\lambda_1,\lambda_2)$ and $C_{XX}(\lambda_1,\lambda_2)$ are spectral density, spectral distribution and auto-covariance functions respectively.

Definition 1. *If $X(t)$, has the spectral representation*

$$X(t) = \sum_{\lambda=-\pi}^{\pi} e^{i\lambda t}, \qquad t = 0,1,2,\cdots, \tag{1}$$

and the stochastic process $Y_X(.)$ satisfies:

$$\begin{aligned} &E[dY_X(\lambda_1)dY_X^*(\lambda_2)] \\ &= d^2F_{XX}(\lambda_1,\lambda_2) \\ &= f_{XX}(\lambda_1,\lambda_2)d\lambda_1 d\lambda_2, \qquad -\pi \le \lambda_1,\lambda_2 \le \pi. \end{aligned} \tag{2}$$

Then $\{X(t)\}$ is called a harmonizable process, Bakouch [2].

Using Definition 1, the auto covariance function is given by

$$\begin{aligned} &C_{XX}(t_1,t_2) \\ &= E[X(t_1)X(t_2)] \\ &= E\left[\sum_{\lambda_1=-\pi}^{\pi}\sum_{\lambda_2=-\pi}^{\pi} e^{i\{t_1\lambda_1+t_2\lambda_2\}}\right] \\ &= \sum_{\lambda_1=-\pi}^{\pi}\sum_{\lambda_2=-\pi}^{\pi} e^{i\{t_1\lambda_1-t_2\lambda_2\}} f_{XX}(\lambda_1,\lambda_2), \\ &\quad t_1,t_2 = 0,\pm1,\pm2,\cdots. \end{aligned} \tag{3}$$

Provided that:

$$\sum_{\lambda_1=-\pi}^{\pi}\sum_{\lambda_2=-\pi}^{\pi}\left|f_{XX}(\lambda_1,\lambda_2)\right| < \infty.$$

Inversely, the spectral density function is given by

$$f_{XX}(\lambda_1,\lambda_2) = \frac{1}{(2\pi)^n}\sum_{t_1=-\infty}^{\infty}\sum_{t_2=-\infty}^{\infty} C_{XX}(t_1,t_2)e^{i\{t_1\lambda_1-t_2\lambda_2\}}, \tag{4}$$

such that

$$\sum_{t_1=-\infty}^{\infty}\sum_{t_2=-\infty}^{\infty}\left|C_{XX}(t_1,t_2)\right| < \infty.$$

Moreover, $f_{XX}(\lambda_1,\lambda_2)$ and all its partial derivatives are bounded.

Let $X(t)$ be a realization from the zero mean harmonizable processes $X(t)$, $-\infty < t < \infty$. Divide this realization into L overlapped and non-overlapped segments, where $X^{(j)}(t)$ the set of observations in the j^{th} segment,

$$\begin{aligned} &X^{(j)}(t) = X[(j-1)q+t], \\ &j = 1,2,\cdots,L, \quad 1 \le t \le M, \quad M,q < N. \end{aligned} \tag{5}$$

If $q \neq M$ and $N-M$ is multiples of q, then the segments are overlap and $L-1 = \dfrac{(N-M)}{q}$. And, if $q = M$, then the number of non-overlapped segments $L = \dfrac{N}{M}$, where

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–M is the number of obsevations in each interval,
–q is the number of overlaping interval,
–N is the total number of obsevations,
–L is the number of intervals.

Definition 2. *The tapered Fourier transform of the j^{th} segment, $X^{(j)}(.)$, is defined by*

$$J_X^{j,k}(\lambda_1) = \sum_{t_1=1}^{M} h_{j,k}(t_1) X^{(j)}(t_1) e^{-i\lambda_1 t_1}, \quad k = 1,2,\cdots,K, \tag{6}$$

where K is the number of tapers in each segment and $h_{j,k}(.)$, is the k^{th} tapers corresponding to the j^{th} segment. Moreover, $h_{j,k}(.)$ is bounded function and equal zero outside the interval $[1,M]$.

Remark. Using formula (6), we can deduce that

$$E(J_X^{(j,k)}(\lambda_1)) = 0.$$

Definition 3. *The multi-segment multi-taper spectral density estimator of $f_{XX}(\lambda_1,\lambda_2)$ is defined as*

$$\begin{aligned} \hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2) &= \sum_{k=1}^{K}\sum_{j=1}^{L} \gamma_{j,k}^{-1} J_X^{j,k}(\lambda_1) J_X^{*(j,k)}(\lambda_2), \\ -\pi &\le \lambda_1,\lambda_2 \le \pi, \end{aligned} \tag{7}$$

where,

$$\gamma_{j,k} = KL \sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi} H_{j,k}(w_1) H_{j,k}^{*}(w_2), \tag{8}$$

and

$$H_{j,k}(w) = \sum_{t=1}^{M} h_{j,k}(t) e^{-iwt},$$

is a bounded function.

When $L = 1$, then formula (7) reduces to

$$\hat{f}_{XX}^{(mt)}(\lambda_1,\lambda_2) = \sum_{k=1}^{K} \gamma_k^{-1} J_X^{k}(\lambda_1) J_X^{*(k)}(\lambda_2),$$

where

$$J_X^{k}(\lambda_1) = \sum_{t_1=1}^{N} h_k(t_1) X(t_1) e^{-i\lambda_1 t_1}, \qquad k = 1,2,\cdots,K,$$

and

$$\gamma_k = K \sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi} H_k(w_1) H_k^{*}(w_2),$$

where,

$$H_k(w) = \sum_{t=1}^{N} h_k(t) e^{-iwt}.$$

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2.1 Statistical properties of $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$

We study some statistical properties of the spectral estimator $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$,

(i)Bias of $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$.
The expected value of $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$ is given by

$$E\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] = \sum_{k=1}^{K}\sum_{j=1}^{L}\sum_{t_1=1}^{M}\sum_{t_2=1}^{M}\gamma_{j,k}^{-1}h_{j,k}(t_1)h_{j,k}(t_2)E\left[X^{(j)}(t_1)\times X^{(j)}(t_2)\right]e^{i(\lambda_1 t_1+\lambda_2 t_2)} = \sum_{k=1}^{K}\sum_{j=1}^{L}\sum_{t_1=1}^{M}\sum_{t_2=1}^{M}\gamma_{j,k}^{-1}h_{j,k}(t_1)h_{j,k}(t_2)C_{XX}(t_1,t_2)\times e^{i(\lambda_1 t_1+\lambda_2 t_2)}.$$

Using formula (3), then

$$E\left(\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right) = \sum_{k=1}^{K}\sum_{j=1}^{L}\sum_{\alpha_1=-\pi}^{\pi}\sum_{\alpha_2=-\pi}^{\pi}\sum_{t_1=1}^{M}\sum_{t_2=1}^{M}\gamma_{j,k}^{-1}h_{j,k}(t_1)h_{j,k}(t_2)\times f_{XX}(\alpha_1,\alpha_2)e^{i(\lambda_1+\alpha_1)t_1}e^{i(\lambda_2+\alpha_2)t_2}.$$

Set $\lambda_1-\alpha_1=w_1$, $\lambda_2-\alpha_2=w_2$, in the above equation, we get

$$E\left(\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right) = \sum_{k=1}^{K}\sum_{j=1}^{L}\gamma_{j,k}^{-1}\left(\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{j,k}(w_1)H_{j,k}^{*}(w_2)\right)\times f_{XX}(\lambda_1-w_1,\lambda_2-w_2).$$

Using Taylor expansion for $f_{XX}(\lambda_1-w_1,\lambda_2-w_2)$ about (λ_1,λ_2) implies

$$E\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] = \sum_{k=1}^{K}\sum_{j=1}^{L}\gamma_{j,k}^{-1}\left(\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{j,k}(w_1)H_{j,k}^{*}(w_2)\right)\times \left[f_{XX}(\lambda_1,\lambda_2)-\left(w_1\frac{\partial}{\partial\lambda_1}+w_2\frac{\partial}{\partial\lambda_2}\right)f_{XX}(\lambda_1,\lambda_2)+\frac{1}{2!}\left(w_1\frac{\partial}{\partial\lambda_1}+w_2\frac{\partial}{\partial\lambda_2}\right)^2 f_{XX}(\lambda_1,\lambda_2)+\cdots\right].$$

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Hence,

$$E\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] = f_{XX}(\lambda_1,\lambda_2) + \sum_{k=1}^{K}\sum_{j=1}^{L}\gamma_{j,k}^{-1}\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{j,k}(w_1) \times H_{j,k}^{*}(w_2)\left[-\left(w_1\frac{\partial}{\partial\lambda_1}+w_2\frac{\partial}{\partial\lambda_2}\right)f_{XX}(\lambda_1,\lambda_2)+ \frac{1}{2!}\left(w_1\frac{\partial}{\partial\lambda_1}+w_2\frac{\partial}{\partial\lambda_2}\right)^2 f_{XX}(\lambda_1,\lambda_2)+\cdots\right]. \tag{9}$$

Therefore, the bias of $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$ is given by

$$\text{Bias}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] = E\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] - f_{XX}(\lambda_1,\lambda_2) = \frac{1}{KL}\sum_{k=1}^{K}\sum_{j=1}^{L}\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{j,k}(w_1)H_{j,k}^{*}(w_2)\right]^{-1} \times\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{j,k}(w_1)H_{j,k}^{*}(w_2)\right]\times \left[-\left(w_1\frac{\partial}{\partial\lambda_1}+w_2\frac{\partial}{\partial\lambda_2}\right)f_{XX}(\lambda_1,\lambda_2)+ \frac{1}{2!}\left(w_1\frac{\partial}{\partial\lambda_1}+w_2\frac{\partial}{\partial\lambda_2}\right)^2 f_{XX}(\lambda_1,\lambda_2)+\cdots\right]. \tag{10}$$

We conclude that

$$\text{Bias}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] \to 0 textas \quad L\to\infty.$$

Hence, the multi-segment multi-taper spectral density estimator, $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$ is asymptotically an unbiased estimator of the spectral density function $f_{XX}(\lambda_1,\lambda_2)$.

(ii)Consistency of $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$.

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Using formula (7), the covariance of $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$, can be obtained as follows:

$$\begin{aligned}
&\mathrm{Cov}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2),\hat{f}_{XX}^{(m,mt)}(\mu_1,\mu_2)\right]= \\
&\sum_{k=1}^{K}\sum_{n=1}^{K}\sum_{j=1}^{L}\sum_{m=1}^{L}\left(\gamma_{j,k}\gamma_{m,n}\right)^{-1}\times \\
&\mathrm{Cov}\left[J_X^{(j,k)}(\lambda_1)J_X^{*(j,k)}(\lambda_2),J_X^{(m,n)}(\mu_1)J_X^{*(m,n)}(\mu_2)\right] \\
&=\sum_{k=1}^{K}\sum_{n=1}^{K}\sum_{j=1}^{L}\sum_{m=1}^{L}\left(\gamma_{j,k}\gamma_{m,n}\right)^{-1}\times \\
&E\left[J_X^{(j,k)}(\lambda_1)J_X^{(m,n)}(\mu_1)\right]E\left[J_X^{*(j,k)}(\lambda_2)J_X^{*(m,n)}(\mu_2)\right] \\
&+\sum_{k=1}^{K}\sum_{n=1}^{K}\sum_{j=1}^{L}\sum_{m=1}^{L}\left(\gamma_{j,k}\gamma_{m,n}\right)^{-1}\times \\
&E\left[J_X^{(j,k)}(\lambda_1)J_X^{*(m,n)}(\mu_2)\right]E\left[J_X^{*(j,k)}(\lambda_2)J_X^{*(m,n)}(\mu_1)\right].
\end{aligned} \tag{11}$$

From Equation (6), we get:

$$\begin{aligned}
&E\left[J_X^{(j,k)}(\lambda_1)J_X^{(m,n)}(\mu_1)\right]= \\
&\sum_{t_1=1}^{M}\sum_{s_1=1}^{M}h_{j,k}(t_1)h_{m,n}(s_1)C_{XX}(t_1,s_1)e^{-i(\lambda_1 t_1+\mu_1 s_1)} \\
&=\sum_{\alpha_1=-\pi}^{\pi}\sum_{\beta_1=-\pi}^{\pi}\sum_{t_1=1}^{M}\sum_{s_1=1}^{M}h_{j,k}(t_1)h_{m,n}(s_1)\times \\
&e^{-i(\lambda_1-\alpha_1)t_1}e^{-i(\mu_1+\beta_1)s_1}f_{XX}(\alpha_1,\beta_1) \\
&=\sum_{\alpha_1=-\pi}^{\pi}\sum_{\beta_1=-\pi}^{\pi}H_{j,k}(\lambda_1-\alpha_1),H_{m,n}(\mu_1+\beta_1)f_{XX}(\alpha_1,\beta_1).
\end{aligned} \tag{12}$$

Similarly, we can get

$$\begin{aligned}
&E\left[J_X^{*(j,k)}(\lambda_2)J_X^{*(m,n)}(\mu_2)\right]= \\
&\sum_{\alpha_2=-\pi}^{\pi}\sum_{\beta_2=-\pi}^{\pi}H_{j,k}^{*}(\lambda_2+\alpha_2),H_{m,n}^{*}(\mu_2-\beta_2)f_{XX}(\alpha_2,\beta_2),
\end{aligned} \tag{13}$$

$$\begin{aligned}
&E\left[J_X^{(j,k)}(\lambda_1)J_X^{*(m,n)}(\mu_2)\right]= \\
&\sum_{\alpha_1=-\pi}^{\pi}\sum_{\beta_1=-\pi}^{\pi}H_{j,k}(\lambda_1-\alpha_1),H_{m,n}^{*}(\mu_2-\beta_2)f_{XX}(\alpha_1,\beta_2),
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
&E\left[J_X^{*(j,k)}(\lambda_2)J_X^{*(m,n)}(\mu_1)\right]= \\
&\sum_{\alpha_2=-\pi}^{\pi}\sum_{\beta_1=-\pi}^{\pi}H_{j,k}^{*}(\lambda_2+\alpha_2),H_{m,n}(\mu_1+\beta_1)f_{XX}(\alpha_2,\beta_1).
\end{aligned} \tag{15}$$

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Hence, the covariance of $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2), \hat{f}_{XX}^{(m,mt)}(\mu_1,\mu_2)$ takes the form

$$\begin{aligned}
&\operatorname{Cov}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2), \hat{f}_{XX}^{(m,mt)}(\mu_1,\mu_2)\right] = \\
&\frac{1}{(KL)^2}\sum_{k=1}^{K}\sum_{n=1}^{K}\sum_{j=1}^{L}\sum_{m=1}^{L}\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi} H_{j,k}(w_1)H_{j,k}^{*}(w_2)\right]^{-1} \\
&\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi} H_{m,n}(w_1)H_{m,n}^{*}(w_2)\right]^{-1} \times \\
&\sum_{\alpha_1=-\pi}^{\pi}\sum_{\alpha_2=-\pi}^{\pi}\sum_{\beta_1=-\pi}^{\pi}\sum_{\beta_2=-\pi}^{\pi} H_{j,k}(\lambda_1-\alpha_1) \\
&H_{m,n}(\lambda_1+\beta_1)H_{j,k}^{*}(\lambda_2+\alpha_2)H_{m,n}^{*}(\lambda_2-\beta_2)\times \\
&[f_{XX}(\alpha_1,\beta_1)f_{XX}(\alpha_2,\beta_2)+f_{XX}(\alpha_1,\beta_2)f_{XX}(\alpha_2,\beta_1)].
\end{aligned} \tag{16}$$

Therefore,

$$\operatorname{Cov}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2), \hat{f}_{XX}^{(m,mt)}(\mu_1,\mu_2)\right] \to 0 \quad \text{as} \quad L\to\infty.$$

That is, $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$ and $\hat{f}_{XX}^{(m,mt)}(\mu_1,\mu_2)$ are asymptotically uncorrelated.
By setting $\lambda_1=\mu_1$ and $\lambda_2=\mu_2$ in formula (16), we get the variance of the spectral estimator $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$:

$$\begin{aligned}
&\operatorname{Var}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] = \\
&\frac{1}{(KL)^2}\sum_{k=1}^{K}\sum_{n=1}^{K}\sum_{j=1}^{L}\sum_{m=1}^{L}\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi} H_{j,k}(w_1)H_{j,k}^{*}(w_2)\right]^{-1} \\
&\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi} H_{m,n}(w_1)H_{m,n}^{*}(w_2)\right]^{-1} \times \\
&\sum_{\alpha_1=-\pi}^{\pi}\sum_{\alpha_2=-\pi}^{\pi}\sum_{\beta_1=-\pi}^{\pi}\sum_{\beta_2=-\pi}^{\pi} H_{j,k}(\lambda_1-\alpha_1)\times \\
&H_{m,n}(\lambda_1+\beta_1)H_{j,k}^{*}(\lambda_2+\alpha_2)H_{m,n}^{*}(\lambda_2-\beta_2)\times \\
&[f_{XX}(\alpha_1,\beta_1)f_{XX}(\alpha_2,\beta_2)+f_{XX}(\alpha_1,\beta_2)f_{XX}(\alpha_2,\beta_1)].
\end{aligned} \tag{17}$$

It is obvious that

$$\operatorname{Var}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] \to 0 \quad \text{as} \quad L\to\infty.$$

Using Equations (10) and (17), the mean-squared error (MSE) of the estimator $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$ can be obtained by the expression

$$\begin{aligned}
&\operatorname{MSE}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] = \\
&\left|\operatorname{Bias}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right]\right|^2 + \operatorname{Var}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right].
\end{aligned} \tag{18}$$

We stated previously that Bias $\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] \to 0$ and Var $\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] \to 0$ as $L\to\infty$.
This implies that MSE $\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] \to 0$ as $L\to\infty$.
This means that, the spectral estimator $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$ is consistent.
Notice that, we can select K which balanced the bias and variance of $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$ for obtaining a best estimator of the spectral density $f_{XX}(\lambda_1,\lambda_2)$.

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3 Autocovariance Estimator

Based on the multi-segment multi-taper spectral density estimator $\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$, we can estimate the autocovariance function $C_{XX}(t_1,t_2) = E[X(t_1)X(t_2)]$ of the harmonizable process $X(t)$ by

$$\hat{C}_{XX}(t_1,t_2) = \sum_{\lambda_1=-\pi}^{\pi}\sum_{\lambda_2=-\pi}^{\pi}\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)e^{i(\lambda_1 t_1-\lambda_2 t_2)}, \qquad 1 \le t_1,t_2 \le N. \tag{19}$$

From Equations (13), (18) and (2), the expected value of the estimator $\hat{C}_{XX}(t_1,t_2)$ is

$$\begin{aligned}
&E\left[\hat{C}_{XX}(t_1,t_2)\right] = \\
&\sum_{\lambda_1=-\pi}^{\pi}\sum_{\lambda_2=-\pi}^{\pi}E\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right]e^{i(\lambda_1 t_1-\lambda_2 t_2)} \\
&= C_{XX}(t_1,t_2)+ \\
&\sum_{\lambda_1=-\pi}^{\pi}\sum_{\lambda_2=-\pi}^{\pi}\left\{\sum_{k=1}^{K}\sum_{j=1}^{L}\gamma_{j,k}^{-1}\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{j,k}(w_1)\right. \\
&\times H_{j,k}^{*}(w_2)\left[-\left(w_1\frac{\partial}{\partial\lambda_1}+w_2\frac{\partial}{\partial\lambda_2}\right)f_{XX}(\lambda_1,\lambda_2)+\right. \\
&\left.\left.\frac{1}{2!}\left(w_1\frac{\partial}{\partial\lambda_1}+w_2\frac{\partial}{\partial\lambda_2}\right)^2 f_{XX}(\lambda_1,\lambda_2)+\cdots\right]\right\}x \\
&\qquad\qquad \times e^{[(\lambda_1 t_1-\lambda_2 t_2)]}.
\end{aligned}$$

Hence, the bias of $\hat{C}_{XX}(t_1,t_2)$ takes the form

$$\begin{aligned}
&\text{Bias}\left[\hat{C}_{XX}(t_1,t_2)\right] = \\
&\frac{1}{KL}\sum_{\lambda_1=-\pi}^{\pi}\sum_{\lambda_2=-\pi}^{\pi}\left\{\sum_{k=1}^{K}\sum_{j=1}^{L}\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{j,k}(w_1)^{-1}\right. \\
&\times H_{j,k}^{*}(w_2)^{-1}\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{j,k}(w_1)H_{j,k}^{*}(w_2)\right]\times \\
&\left[-\left(w_1\frac{\partial}{\partial\lambda_1}+w_2\frac{\partial}{\partial\lambda_2}\right)f_{XX}(\lambda_1,\lambda_2)+\right. \\
&\left.\left.\frac{1}{2!}\left(w_1\frac{\partial}{\partial\lambda_1}+w_2\frac{\partial}{\partial\lambda_2}\right)^2 f_{XX}(\lambda_1,\lambda_2)+\cdots\right]\right\}\times \\
&e^{[(\lambda_1 t_1-\lambda_2 t_2)]}.
\end{aligned} \tag{20}$$

This implies

$$\text{Bias}\left[\hat{C}_{XX}(t_1,t_2)\right] \to 0 \quad \text{as} \quad L \to \infty$$

Using Equation (16) the covariance of the estimator $\hat{C}_{XX}(t_1,t_2)$ and $\hat{C}_{XX}(s_1,s_2)$ is obtained as following

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$$
\begin{aligned}
&\operatorname{Cov}\left[\hat{C}_{XX}(t_1,t_2),\hat{C}_{XX}(s_s,s_2)\right] = \\
&\sum_{\lambda_1=-\pi}^{\pi}\sum_{\lambda_2=-\pi}^{\pi}\sum_{\mu_1=-\pi}^{\pi}\sum_{\mu_2=-\pi}^{\pi} \\
&\operatorname{Cov}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2),\hat{f}_{XX}^{(m,mt)}(\mu_1,\mu_2)\right]e^{i[(\lambda_1 t_1-\lambda_2 t_2)]} \\
&\times e^{i[(\mu_1 s_1-\mu_2 s_2)]} \\
&= \frac{1}{(KL)^2}\sum_{\lambda_1=-\pi}^{\pi}\sum_{\lambda_2=-\pi}^{\pi}\sum_{\mu_1=-\pi}^{\pi}\sum_{\mu_2=-\pi}^{\pi}\left\{\sum_{k=1}^{K}\sum_{n=1}^{K}\sum_{j=1}^{L}\sum_{m=1}^{L}\right. \\
&\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{j,k}(w_1)H^*_{j,k}(w_2)\right]^{-1} \\
&\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{m,n}(w_1)H^*_{m,n}(w_2)\right]^{-1}\times \\
&\sum_{\alpha_1=-\pi}^{\pi}\sum_{\alpha_2=-\pi}^{\pi}\sum_{\beta_1=-\pi}^{\pi}\sum_{\beta_2=-\pi}^{\pi}H_{j,k}(\lambda_1-\alpha_1)\times \\
&H_{m,n}(\mu_1+\beta_1)H^*_{j,k}(\lambda_2+\alpha_2)H^*_{m,n}(\mu_2-\beta_2)\times \\
&\left.[f_{XX}(\alpha_1,\beta_1)f_{XX}(\alpha_2,\beta_2)+f_{XX}(\alpha_1,\beta_2)f_{XX}(\alpha_2,\beta_1)]\right\}\times \\
&e^{i[(\lambda_1 t_1-\lambda_2 t_2)]}e^{i[(\mu_1 s_1-\mu_2 s_2)]}.
\end{aligned}
$$

Hence,

$$\operatorname{Cov}\left[\hat{C}_{XX}(t_1,t_2),\hat{C}_{XX}(s_s,s_2)\right]\to 0 \quad \text{as} \quad L\to\infty.$$

That is, $\hat{C}_{XX}(t_1,t_2)$ and $\hat{C}_{XX}(s_1,s_2)$ are asymptotically uncorrelated.
Setting $t_1 = s_1$ and $t_2 = s_2$ in the previous expression implies.

$$
\begin{aligned}
&\operatorname{Var}\left[\hat{C}_{XX}(t_1,t_2)\right] = \\
&\frac{1}{(KL)^2}\sum_{\lambda_1=-\pi}^{\pi}\sum_{\lambda_2=-\pi}^{\pi}\sum_{\mu_1=-\pi}^{\pi}\sum_{\mu_2=-\pi}^{\pi} \\
&\left\{\sum_{k=1}^{K}\sum_{n=1}^{K}\sum_{j=1}^{L}\sum_{m=1}^{L}\left[\sum_{w=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{j,k}(w_1)H^*_{j,k}(w_2)\right]^{-1}\right. \\
&\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{m,n}(w_1)H^*_{m,n}(w_2)\right]^{-1}\times \\
&\left[H_{j,k}(w_1)H^*_{j,k}(w_2)\right]^{-1}\left[H_{m,n}(w_1)H^*_{m,n}(w_2)\right]^{-1}\times \\
&\sum_{\alpha_1=-\pi}^{\pi}\sum_{\alpha_2=-\pi}^{\pi}\sum_{\beta_1=-\pi}^{\pi}\sum_{\beta_2=-\pi}^{\pi}H_{j,k}(\lambda_1-\alpha_1) \\
&H_{m,n}(\mu_1+\beta_1)H^*_{j,k}(\lambda_2+\alpha_2)H^*_{m,n}(\mu_2-\beta_2)\times \\
&\left.[f_{XX}(\alpha_1,\beta_1)f_{XX}(\alpha_2,\beta_2)+f_{XX}(\alpha_1,\beta_2)f_{XX}(\alpha_2,\beta_1)]\right\}\times \\
&e^{i[(\lambda_1+\mu_1)t_1]}e^{-i[(\lambda_2+\mu_2)t_2]}.
\end{aligned}
\tag{21}
$$

It is clear that,

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$$\mathrm{Var}\left[\hat{C}_{XX}(t_1,t_2)\right] \to 0 \quad \text{as} \quad L \to \infty.$$

Making use of Equations (20) and (21), the mean-squared error of $\hat{C}_{XX}(t_1,t_2)$ tends to 0 as L tends to infinity. That is the estimator $\hat{C}_{XX}(t_1,t_2)$ is consistent.

Moreover, we can choose K, as we steed in the previous section, that makes $\hat{C}_{XX}(t_1,t_2)$ a best estimator of the auto-covariance $C_{XX}(t_1,t_2)$.

Moreover, careful choice of K implies $\hat{C}_{XX}(t_1,t_2)$ to be a best estimator of $C_{XX}(t_1,t_2)$.

4 Spectral Distribution Estimator

If the harmonizable process $X(t)$, $t = 0,\pm1,\pm2,\cdots$, has the spectral density function $f_{XX}(\lambda_1,\lambda_2)$, then its spectral distribution function $F_{XX}(\lambda_1,\lambda_2)$, is defined by

$$F_{XX}(\lambda_1,\lambda_2) = \sum_{\alpha_1=-\pi}^{\lambda_1}\sum_{\alpha_2=-\pi}^{\lambda_2} f_{XX}(\alpha_1,\alpha_2), -\pi \leq \lambda_1,\lambda_2 \leq pi. \tag{22}$$

Therefore, we can estimate the spectral distribution function by

$$\hat{F}_{XX}(\lambda_1,\lambda_2) = \sum_{\alpha_1=-\pi}^{\lambda_1}\sum_{\alpha_2=-\pi}^{\lambda_2} \hat{f}_{XX}^{(m,mt)}(\alpha_1,\alpha_2), \tag{23}$$

where $\hat{f}_{XX}^{(m,mt)}(\alpha_1,\alpha_2)$ are multi-segment multi-taper spectral density estimator given by formula (7).

Using equations (9), (22) and (23), the expected value of the spectral distribution estimator $\hat{F}_{XX}(\lambda_1,\lambda_2)$ is given by

$$\begin{aligned} &E\left[\hat{F}_{XX}(\lambda_1,\lambda_2)\right] = \\ &F_{XX}(\lambda_1,\lambda_2) + \sum_{\alpha_1=-\pi}^{\lambda_1}\sum_{\alpha_2=-\pi}^{\lambda_2}\left\{\sum_{k=1}^{K}\sum_{j=1}^{L}\gamma_{j,k}^{-1}\right. \\ &\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi} H_{j,k}(w_1)H_{j,k}^{*}(w_2)\right]\times \\ &\qquad\left[-\left(w_1\frac{\partial}{\partial\alpha_1}+w_2\frac{\partial}{\partial\alpha_2}\right)f_{XX}(\alpha_1,\alpha_2)+\right. \\ &\qquad\left.\left.\frac{1}{2!}\left(w_1\frac{\partial}{\partial\alpha_1}+w_2\frac{\partial}{\partial\alpha_2}\right)^2 f_{XX}(\alpha_1,\alpha_2)+\cdots\right]\right\} \end{aligned}$$

Thus, the bias of the spectral estimator $\hat{F}_{XX}(\lambda_1,\lambda_2)$ takes the form

$$\begin{aligned} &\mathrm{Bias}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right] = \\ &\frac{1}{KL}\sum_{\alpha_1=-\pi}^{\lambda_1}\sum_{\alpha_2=-\pi}^{\lambda_2}\left\{\sum_{k=1}^{K}\sum_{j=1}^{L}\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi} H_{j,k}(w_1)^{-1}\right. \\ &\times H_{j,k}^{*}(w_2)^{-1}\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi} H_{j,k}(w_1)H_{j,k}^{*}(w_2)\right]\times \\ &\left[-\left(w_1\frac{\partial}{\partial\alpha_1}+w_2\frac{\partial}{\partial\alpha_2}\right)f_{XX}(\alpha_1,\alpha_2)+\right. \\ &\left.\left.\frac{1}{2!}\left(w_1\frac{\partial}{\partial\alpha_1}+w_2\frac{\partial}{\partial\alpha_2}\right)^2 f_{XX}(\alpha_1,\alpha_2)+\cdots\right]\right\}. \end{aligned} \tag{24}$$

Hence,

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$$\text{Bias}\left[\hat{f}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)\right]\to 0 \quad \text{as} \quad L\to\infty.$$

That is, the estimator $\hat{F}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$ is asymptotically unbiased.
By using formula (16), the covariance of estimator $\hat{F}_{XX}^{(m,mt)}(\lambda_1,\lambda_2)$ is

$$\begin{aligned}
&\text{Cov}\left[\hat{F}_{XX}(\lambda_1,\lambda_2),\hat{F}_{XX}(\mu_1,\mu_2)\right]= \\
&\frac{1}{(KL)^2}\sum_{u_1=-\pi}^{\lambda_1}\sum_{u_2=-\pi}^{\lambda_2}\sum_{v_1=-\pi}^{\mu_1}\sum_{v_2=-\pi}^{\mu_2}\sum_{k=1}^{K}\sum_{n=1}^{K}\sum_{j=1}^{L}\sum_{m=1}^{L} \\
&\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{j,k}(w_1)H^*_{j,k}(w_2)\right]^{-1} \\
&\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{m,n}(w_1)H^*_{m,n}(w_2)\right]^{-1}\times \\
&\sum_{\alpha_1=-\pi}^{\pi}\sum_{\alpha_2=-\pi}^{\pi}\sum_{\beta_1=-\pi}^{\pi}\sum_{\beta_2=-\pi}^{\pi}H_{j,k}(u_1-\alpha_1) \\
&H_{m,n}(v_1+\beta_1)H^*_{j,k}(u_2+\alpha_2)H^*_{m,n}(v_2-\beta_2)\times \\
&\left[f_{XX}(\alpha_1,\beta_1)f_{XX}(\alpha_2,\beta_2)+f_{XX}(\alpha_1,\beta_2)f_{XX}(\alpha_2,\beta_1)\right].
\end{aligned}$$

$\hat{F}_{XX}(\lambda_1,\lambda_2)$ and $\hat{F}_{XX}(\mu_1,\mu_2)$ are asymptotically uncorrelated, since

$$\text{Cov}\left[\hat{F}_{XX}(\lambda_1,\lambda_2),\hat{F}_{XX}(\mu_1,\mu_2)\right]\to 0 \quad \text{as} \quad L\to\infty.$$

From the previous expression, we get the variance of $\hat{F}_{XX}(\lambda_1,\lambda_2$

$$\begin{aligned}
&\text{Var}\left[\hat{f}_{XX}(\lambda_1,\lambda_2)\right]= \\
&\frac{1}{(KL)^2}\sum_{u_1=-\pi}^{\lambda_1}\sum_{u_2=-\pi}^{\lambda_2}\sum_{v_1=-\pi}^{\lambda_1}\sum_{v_2=-\pi}^{\lambda_2} \\
&\left\{\sum_{k=1}^{K}\sum_{n=1}^{K}\sum_{j=1}^{L}\sum_{m=1}^{L}\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{j,k}(w_1)H^*_{j,k}(w_2)\right]^{-1}\right. \\
&\left[\sum_{w_1=-\pi}^{\pi}\sum_{w_2=-\pi}^{\pi}H_{m,n}(w_1)H^*_{m,n}(w_2)\right]^{-1}\times \\
&\sum_{\alpha_1=-\pi}^{\pi}\sum_{\alpha_2=-\pi}^{\pi}\sum_{\beta_1=-\pi}^{\pi}\sum_{\beta_2=-\pi}^{\pi}H_{j,k}(u_1-\alpha_1) \\
&H_{m,n}(v_1+\beta_1)H^*_{j,k}(u_2+\alpha_2)H^*_{m,n}(v-\beta_2)\times \\
&\left.\left[f_{XX}(\alpha_1,\beta_1)f_{XX}(\alpha_2,\beta_2)+f_{XX}(\alpha_1,\beta_2)f_{XX}(\alpha_2,\beta_1)\right]\right\}.
\end{aligned} \tag{25}$$

From the two previous equations we get

$$\begin{aligned}
&\text{MSE}\left[\hat{F}_{XX}(\lambda_1,\lambda_2)\right]= \\
&\left|\text{Bias}\left[\hat{F}_{XX}(\lambda_1,\lambda_2)\right]\right|^2+\text{Var}\left[\hat{F}_{XX}(\lambda_1,\lambda_2)\right]\to 0 \\
&\quad \text{as} \quad L\to\infty.
\end{aligned}$$

Since, $\text{Var}\left[\hat{F}_{XX}(\lambda_1,\lambda_2)\right]\to 0$. This proves the consistency of the spectral distribution $\hat{F}_{XX}(\lambda_1,\lambda_2)$. Careful choice of K implies that $\hat{F}_{XX}(\lambda_1,\lambda_2)$ is a best estimator of the spectral distribution $F_{XX}(\lambda_1,\lambda_2)$.

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5 Conclusion

We obtain the estimator for the spectral density of this process by using tapered Fourier transform. Also, we obtain the estimator of the autocovariance and the spectral distribution functions, based on the spectral density estimator. Statistical properties of these estimators are investigated, including the asymptotic behaviour of bias and covariance.

Conflicts of Interest

There are no conflicts of interest declared by the authors for the publication of this paper..

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