Optimal Degree Estimation of the Determinant of a Polynomial Matrix

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Abstract: In this paper we present a recursive formula to find the degree of the determinant of a bivariate polynomial matrix. The proposed algorithm returns the optimal estimation of the degree but has a very large computational cost. The recursive formula can be represented as an assignment problem which is solved with the Hungarian method that has a very small computational cost. The contribution of the proposed formula is to reduce the required interpolation points for the *evaluation–interpolation* technique. That is, we reduce the evaluations which are computations of determinants or inverse matrices.

Keywords: polynomial matrix, determinant, degree, assignment problem

1 Introduction

The computation of a) the determinant [1], [2], [3], b) the inverse [4], [5], [6], [7], c) the generalized inverse [8] and d) the Drazin matrix [9] of a bivariate polynomial matrix are some problems in control theory. We can solve numerically the above problems with the aim of the bivariate polynomial interpolation. In particular, we use the technique *evaluation–interpolation* [10], [11], [12]. According to the *evaluation–interpolation* technique, first, we must evaluate the values of the determinant or the inverse matrix and in sequence interpolate at the evaluate values. In the above procedure we have the critical problem to define the number of initial values which is the number of interpolation points. The number of initial values must be the minimum because the initial values are determinants of inverse matrices. The number of interpolation points depends on the degree of the interpolating polynomial (determinant or element of inverse matrix) [13], [14], [15]. The optimal degree estimation of the determinant is the aim of this work.
Let a two variable polynomial matrix $A(x,y) \in \mathbb{R}[x,y]^{m \times m}$ where

$$A(x,y) = \begin{bmatrix} a_{1,1}(x,y) & \dots & a_{1,m}(x,y) \\ \vdots & \ddots & \vdots \\ a_{m,1}(x,y) & \cdots & a_{m,m}(x,y) \end{bmatrix}$$

We define the total degree matrix $D \in \mathbb{N}^{m \times m}$ where

$$D = \begin{bmatrix} d_{1,1} & \dots & d_{1,m} \\ \vdots & \ddots & \vdots \\ d_{m,1} & \cdots & d_{m,m} \end{bmatrix}$$

with $d_{i,j} = \deg[a_{i,j}(x,y)]$ where $\deg[.]$ denotes the total degree of polynomial.
Similarly, we define the degree matrix in terms of x, $D^x \in \mathbb{N}^{m \times m}$ where

$$D^x = \begin{bmatrix} d_{1,1}^x & \dots & d_{1,m}^x \\ \vdots & \ddots & \vdots \\ d_{m,1}^x & \cdots & d_{m,m}^x \end{bmatrix}$$

with $d_{i,j}^x = \deg_x[a_{i,j}(x,y)]$ where $\deg_x[.]$ denotes the greatest degree in terms of variable z and the degree matrix in terms of y, $D^y \in \mathbb{N}^{m \times m}$ where

$$D^y = \begin{bmatrix} d_{1,1}^y & \dots & d_{1,m}^y \\ \vdots & \ddots & \vdots \\ d_{m,1}^y & \cdots & d_{m,m}^y \end{bmatrix}$$

with $d_{i,j}^y = \deg_y[a_{i,j}(x,y)]$ where $\deg_y[.]$ denotes the greatest degree in terms of variable y. For the above

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elements hold that

$$d_{i,j} \leq d_{i,j}^x + d_{i,j}^y \leq 2 \cdot d_{i,j}$$

According to [5], [8], [6], [7] the upper bound of degree of the determinant of polynomial matrix $A(x,y)$ is

$$n_1 = m \cdot d \text{ where } d = \max_{1 \leq i,j \leq m} \{d_{i,j}\}$$

In this case we need

$$N_1 = \binom{n_1 + 2}{n_1}$$

interpolation points. Additionally, the upper bound of degree of the determinant of polynomial matrix $A(x,y)$ is n_2 where

$$n_2 = \min \left\{ \sum_{i=1}^{m} \left(\max_{1 \leq j \leq m} \{\deg[a_{i,j}(x,y)]\} \right), \right.$$
$$\left. \sum_{j=1}^{m} \left(\max_{1 \leq i \leq m} \{\deg[a_{i,j}(x,y)]\} \right) \right\}$$

In this case we need

$$N_2 = \binom{n_2 + 2}{n_2}$$

interpolation points.
With the same way we define

$$n_3^z = (m \cdot d^z) \text{ where } d^z = \max_{1 \leq i,j \leq m} \{d_{i,j}^z\}$$

with $z \in \{x,y\}$ and

$$n_4^z = \min \left\{ \sum_{i=1}^{m} \left(\max_{1 \leq j \leq m} \{\deg_z[a_{i,j}(x,y)]\} \right), \right.$$
$$\left. \sum_{j=1}^{m} \left(\max_{1 \leq i \leq m} \{\deg_z[a_{i,j}(x,y)]\} \right) \right\}$$

In these cases we need

$$N_3 = (n_3^x + 1) \cdot (n_3^y + 1)$$

and

$$N_4 = (n_4^x + 1) \cdot (n_4^y + 1)$$

interpolation points respectively.

Example 1. Let the polynomial matrix

$$A(x,y) = \begin{bmatrix} x^3 - x^2 y^2 & 2x + 3y & 5x + 2 \\ x^6 + y^4 - 6x^4 y^3 & x^2 + y^2 + 2 & x^3 - 2y \\ x^3 & y & 4 \end{bmatrix}$$

The polynomial matrix $A(x,y)$ has total degree matrix, degree matrix in terms of x and degree matrix in terms of y respectively,

$$D = \begin{bmatrix} 4 & 1 & 1 \\ 7 & 4 & 3 \\ 4 & 1 & 0 \end{bmatrix}$$

and

$$D^x = \begin{bmatrix} 3 & 1 & 1 \\ 6 & 2 & 3 \\ 2 & 0 & 0 \end{bmatrix}$$

and

$$D^y = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Then, $d = 7$, $d^x = 6$, $d^y = 4$ and $m = 3$ thus

$$n_1 = m \cdot d = 3 \cdot 7 = 21$$

$$n_3^x = m \cdot d^x = 18$$

and

$$n_3^y = m \cdot d^y = 12$$

Additionally, we have

$$n_2 = \min \left\{ \sum_{i=1}^{m} \left(\max_{1 \leq j \leq m} \{\deg[a_{i,j}(x)]\} \right), \right.$$
$$\left. \sum_{j=1}^{m} \left(\max_{1 \leq i \leq m} \{\deg[a_{i,j}(x)]\} \right) \right\} =$$
$$= \min\{4 + 7 + 4, 7 + 4 + 3\} = 14$$

$$n_4^x = \min \left\{ \sum_{i=1}^{m} \left(\max_{1 \leq j \leq m} \{\deg_x[a_{i,j}(x,y)]\} \right), \right.$$
$$\left. \sum_{j=1}^{m} \left(\max_{1 \leq i \leq m} \{\deg_x[a_{i,j}(x,y)]\} \right) \right\} =$$
$$= \min\{3 + 6 + 2, 6 + 2 + 3\} = 11$$

and

$$n_4^y = \min \left\{ \sum_{i=1}^{m} \left(\max_{1 \leq j \leq m} \{\deg_y[a_{i,j}(x,y)]\} \right), \right.$$
$$\left. \sum_{j=1}^{m} \left(\max_{1 \leq i \leq m} \{\deg_y[a_{i,j}(x,y)]\} \right) \right\} =$$
$$= \min\{2 + 4 + 2, 4 + 2 + 1\} = 7$$

The numbers of interpolation points are presented in Table 1.

Table 1: The number of interpolation points for all cases

N_1	N_2	N_3	N_4
253	120	247	96

2 Main propose

In the previous approaches, we compute the upper bounds of the total degree or the degrees in terms of variables x or y of the determinant respectively, as well as the number of points which are required for each case. Then, we will define a recursive formula for the computation of the degree of the determinant. The construction of the recursive formula is based on the corresponding recursive formula for the calculation of the determinant.

Let the polynomials $f(x,y)$ and $g(x,y)$. We know that

$$\deg[f(x,y) \cdot g(x,y)] = \deg[f(x,y)] + \deg[g(x,y)]$$

and

$$\deg[f(x,y) + g(x,y)] = \max\{\deg[f(x,y)], \deg[g(x,y)]\}$$

Moreover, the calculation of the determinant of the polynomial matrix $A(x,y)$ is based on operations of multiplication and addition between the polynomials. Consequently, the respective operations in degrees of polynomials is the sum of the degrees of polynomials when the polynomials multiplied and the maximum degree of polynomials when the polynomials added.

Therefore, we apply the recursive formula for the calculation of the determinant on matrix D by replacing the multiplication with the operation of addition and the adding with the maximum selection. This returns us the degree of polynomial matrix $A(x,y)$.

The above computational procedure is given by the recursive formula

$$DegDet(D) = \begin{cases} \max_{1 \le j \le m}\{d_{i,j} + DetDeg(D_{i,j})\} & m > 2 \\ \max\{d_{1,1} + d_{2,2}, d_{1,2} + d_{2,1}\} & m = 2 \end{cases} \quad (1)$$

where $D_{i,j} \in \mathbb{N}^{(m-1)\times(m-1)}$ is a sub-matrix of matrix D without the i row and the j column.

*Example 2.*Let the total degree matrix of the Example 1

$$D = \begin{bmatrix} 4 & 1 & 1 \\ 7 & 4 & 3 \\ 4 & 1 & 0 \end{bmatrix}$$

Then

$$DegDet(D) = \max\{d_{1,1} + DegDet(D_{1,1}),$$
$$d_{1,2} + DegDet(D_{1,2}), d_{1,3} + DegDet(D_{1,3})\}$$

where

$$DegDet(D_{1,1}) = \max\{d_{2,2} + d_{3,3}, d_{3,2} + d_{2,3}\} = 4$$
$$DegDet(D_{1,2}) = \max\{d_{2,1} + d_{3,3}, d_{2,3} + d_{3,1}\} = 7$$
$$DegDet(D_{1,3}) = \max\{d_{2,1} + d_{3,2}, d_{2,2} + d_{3,1}\} = 8$$

Thus

$$DegDet(D) = \max\{4 + 4, 1 + 7, 1 + 8\} = 9$$

instead to $n_1 = 21$ and $n_2 = 14$.

2.1 Optimal computation of the degree

The disadvantage of the recursive formula (1) is its computational complexity which is of $O(n!)$. For the computation of *DegDet* we need $n!$ combinations of n elements from each column and each row. These elements must be from different rows and different columns. We get the sums S_k where $k = 1, ..., n!$ of these combinations and we compute the degree

$$DegDet = \max_{k=1,...,n!}\{S_k\}$$

Therefore, this computation can be written as an assignment problem in linear programming. In particular, the objective function is

$$\max \quad z = \sum_{i=1}^{m}\sum_{j=1}^{m}(d_{i,j} \cdot x_{ij})$$

subject to

$$\sum_{j=1}^{m} x_{ij} = 1 \quad i = 1, 2, \ldots, m$$

and

$$\sum_{i=1}^{m} x_{ij} = 1 \quad j = 1, 2, \ldots, m$$

with $x_{ij} = \{0, 1\}$ for all i, j.

The above assignment problem can be solved with the Hungarian method or algorithm [16] which has computational complexity of $O(n^3)$. [17] show that the computational complexity is reduced to $O(n^2 \log n)$. The matrix D of the Example 2 with the Hungarian method will be

$$\begin{bmatrix} 4 & 1 & 1 \\ 7 & 4 & 3 \\ 4 & 1 & 0 \end{bmatrix} \xrightarrow[\text{method}]{\text{Hungarian}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

In the last matrix 2 combinations of 3 independent zeros exists which means the solutions of the problem is the elements $(3,1), (2,2), (1,3)$ or $(2,1), (3,2), (1,3)$ and the objective function is equal to

$$d_{3,1} + d_{2,2} + d_{1,3} = 9$$

or

$$d_{2,1} + d_{3,2} + d_{1,3} = 9$$

Table 2: The degree of the determinant and the number of interpolation points for all cases

N_1	N_2	N_3	N_4	N_5	N_6
253	120	247	96	55	48

*Example 3.*Let the polynomial matrix $A(x,y)$ of the Example 1 which has total degree matrix, degree matrix in terms of x and degree matrix in terms of y respectively,

$$D = \begin{bmatrix} 4 & 1 & 1 \\ 7 & 4 & 3 \\ 4 & 1 & 0 \end{bmatrix}; D^x = \begin{bmatrix} 3 & 1 & 1 \\ 6 & 2 & 3 \\ 2 & 0 & 0 \end{bmatrix}; D^y = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

We have

$$n_5 = DegDet(D) = 9$$

$$n_5^x = DegDet(D^x) = 7$$

instead to $n_3^x = 18$ and $n_4^x = 11$ and

$$n_5^y = DegDet(D^y) = 5$$

instead to $n_3^y = 12$ and $n_4^y = 7$. The numbers of interpolation points are presented in Table 2.

From the Table 2 we conclude that the reduction of the required interpolation points is significantly large.

3 Conclusions

The proposed recursive formula (1) reduces a lot the interpolation points which are required for the calculation of the determinant. This is the advantage of this formula for the technique *evaluation–interpolation*, because the interpolation points are determinants or adjoint matrices of evaluating matrices which have a very large computational cost. Additionally, the recursive formula (1) is represented as an assignment problem and the computational complexity is of $O(n^2 \log n)$. Finally, this formula can be used in one variable polynomial *evaluation–interpolation* problems such as the computation of the determinant, inverse, Smith form of a polynomial matrix.

Acknowledgements

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