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A New Class of Distributions with Applications to Complete Data and Survival Data with Long-Term Survivors

M. M. E. Abd El-Monsef[1], *M. M. El-Awady*[1,*] *and M. M. Seyam*[1,2]

[1]Mathematics Department, Faculty of Science, Tanta University, Tanta, Egypt
[2]Mathematics Department, Faculty of Science and Arts, Jouf University, Al-Jawf, KSA

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Abstract: The exponential power distribution (EP) is a lifetime model that can exhibit increasing and bathtub hazard rate function. This paper aims to propose a new class of distributions called the exponential power power series (EPPS) class of distributions. The hazard function of the proposed class can be increasing, decreasing, modified bathtub (increasing-decreasing-increasing) and bathtub shaped. Among others, the exponential power geometric (EPG) distribution is presented as a special case of the proposed family. A defective version of EPG distribution is defined to estimate the fraction of long-term survivors in a population. The moments of the defective EPG distribution are obtained. More importantly, owing to the proposed defective distribution, a cure rate regression model is proposed for modelling lifetime data contains long-term survivors with associated covariates. The maximum likelihood method and Bayesian method are used for estimating the unknown parameters. The performance of these estimation methods is examined by conducting a simulation study. The importance of the proposed family is illustrated by means of three distinctive real data sets.

Keywords: Exponential power distribution, Power series distributions, Cure rates, Long-term survivors, Defective distributions, Censored data, Regression model.

1 Introduction

The exponential power (EP) distribution with bathtub shape or increasing hazard rate is proposed by Smith and Bain [1]. Its distribution function is given by

$$F_{EP}(x) = 1 - e^{-\left(e^{\lambda x^{\alpha}} - 1\right)}, \tag{1}$$

where $\alpha > 0$ is a shape parameter and $\lambda > 0$ is a scale parameter. This distribution may be thought of as a truncated extreme-value distribution with a Weibull type parametrization rather than the usual location-scale parametrization.

An extension of EP distribution has been proposed by Barriga et al. [2], called complementary exponential power (CEP) distribution, based on the exponentiated type family of distributions. Based on modification of the EP distribution, Chen [3] proposed two parameter distribution with bathtub or increasing hazard rate function. Xei et al. [4] proposed an extension of Chen's model, known as the Weibull extension model, by adding scale parameter. It has been further extended by Pappas et al. [5] using the technique of Marshall and Olkin [6]. Another extension proposed by Chaubey and Zhang [7] using exponentiated type family of distribution.

When modelling survival data, all individuals in the study population are assumed to be susceptible to the interested event. However, such assumption may be violated because some individuals in the population may never experience the event of interest, such individuals are often called cured or immune or long-term survivors. The proportion of cured individuals are known in the literature as cure rate or the fraction of long-term survivors.

In order to model the cure rates, a strategy is needed to make the survival function tends to a value $p \in (0,1)$, representing the cure rate, as time increases. The standard mixture model is commonly used for modelling cure rates, [8]. This model is a mixture of cured and uncured individuals, that is, $\bar{F}(t) = p + (1-p)\bar{F}_0(t)$, where $p \in (0,1)$ is the cure

* Corresponding author e-mail: m.m.elawady@science.tanta.edu.eg

rate and $\bar{F}_0(t)$ is proper survival function for the uncured individuals. Common choices for $\bar{F}_0(t)$ are the exponential, Weibull, log-logistic, log-normal and Gompertz distributions.

An alternative model is due Yakovlev et al. [9] who proposed the promotion time cure model (PTCM) as a different approach for modelling survival data with long term survivors. In this type of models the cumulative hazard function is defined as $H_{pt}(x) = \lambda F_0(x)$, where $\lambda > 0$. Thus the improper survival function is then given by $\bar{F}_{pt}(x) = e^{-\lambda F_0(x)}$ and the cure fraction is $\lim_{x\to\infty} \bar{F}_{pt}(x) = e^{-\lambda}$.

Another interesting way for modelling cure rates is to use defective distributions. A defective distribution is characterized by having probability density function integrated to values less than 1, when changing the usual domain for some of its parameters. In this case, the survival function approaches a proportion $p \in (0,1)$ as time increases. Gompertz and inverse Gaussian distributions are examples of such distributions, Balka et al. [10]. Rocha et al. [11] derived a new property for Marshall-Olkin class of distributions which allows one to generate new defective distributions.

This paper aims to propose an extension to exponential power distribution by compounding the EP distribution and power series distributions. The compounding procedure is based on competing risks problems, see [12], following the technique of Marshall and Olkin. The new class called exponential power power series (EPPS) class of distributions. The EPPS contains the exponential power geometric (EPG) distribution as a special case which can be redefined as a defective distribution by changing the usual domain for some of its parameters. The defective EPG distribution can be used to model survival data with long-term survivors and censoring data in presence of covariates. Therefore, we propose a regression model that allows us to estimate cure rates based on defective EPG distribution.

The organization of the paper is as follows: Section 2 proposes the EPPS family. Section 3 presents some characteristics for the density function of the EPPS family. The quantile function and ordinary moments are also provided in this section. In section 4, special cases of the proposed family are presented. The defective EPG distribution and its regression model are defined in section 5. The ordinary moments for defective EPG distribution is also derived in this section. Estimation of the parameters using maximum likelihood and Bayesian methods are addressed in section 6. Further, in this section, detailed simulation studies are performed to examine the accuracy of the maximum likelihood and Bayesian estimators. Finally, section 7 explore applications to real datasets that illustrate the superiority of the EPPS family over other models.

2 The exponential power power series family of distributions

Consider $X_i,\ i = 1,\ldots,N$ be independent and identically distributed random variables following exponential power distribution with cumulative distribution function (cdf) in (1) with corresponding survival function and probability density function (pdf) given by

$$\bar{F}_{EP}(x) = e^{-\left(e^{\lambda x^{\alpha}}-1\right)}, \tag{2}$$

and

$$f_{EP}(x) = \lambda\alpha x^{\alpha-1} e^{-(e^{\lambda x^{\alpha}}-1)+\lambda x^{\alpha}}, \tag{3}$$

respectively. These random variables represent lifetimes of a series system of N components. Here, N is a zero truncated discrete random variable following a power series distribution with probability function

$$p_n = P\{N = n\} = \frac{a_n\theta^n}{C(\theta)}, n = 1,2,\ldots, \tag{4}$$

where the coefficient $a_n \geq 0$ depends only on n and $C(\theta) = \sum_{n=1}^{\infty} a_n\theta^n$. Table 1 shows useful quantities of some power series distributions such as Poisson, logarithmic, geometric and binomial (with m being the number of replicas) distributions.

The cdf of the exponential power power series (EPPS) class of distributions is the marginal distribution of the first order statistics $X_{(1)} = \min\{X_i : i = 1,\ldots,N\}$ which given by

$$F_{EPPS}(x) = \sum_{i=1}^{n} \left\{1 - \bar{F}_{EP}(x)^n\right\}\ P\{N = n\}. \tag{5}$$

Therefore, the cdf for the EPPS family is given by

$$F_{EPPS}(x) = 1 - \frac{C(\theta e^{-(e^{\lambda x^{\alpha}}-1)})}{C(\theta)}. \tag{6}$$

Table 1: Useful quantities of some power series distributions.

Distribution	a_n	$C(\theta)$	$C'(\theta)$	$C''(\theta)$	$C^{-1}(\theta)$	
Poisson	$n!^{-1}$	$e^{\theta}-1$	e^{θ}	e^{θ}	$\log(\theta+1)$	$\theta\in(0,\infty)$
Logarithmic	n^{-1}	$-\log(1-\theta)$	$(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$1-e^{-\theta}$	$\theta\in(0,1)$
Geometric	1	$\theta(1-\theta)^{-1}$	$(1-\theta)^{-2}$	$2(1-\theta)^{-3}$	$\theta(\theta+1)^{-1}$	$\theta\in(0,1)$
Binomial	$\binom{m}{n}$	$(\theta+1)^m-1$	$m(\theta+1)^{m-1}$	$\frac{m(m-1)}{(\theta+1)^{2-m}}$	$(\theta-1)^{1/m}-1$	$\theta\in(0,1)$

The survival function, pdf and hazard function for EPPS random variable are, respectively, given by

$$\bar{F}_{EPPS}(x)=\frac{C(\theta e^{-(e^{\lambda x^{\alpha}}-1)})}{C(\theta)}. \tag{7}$$

$$f_{EPPS}(x)=\theta\lambda\alpha x^{\alpha-1}e^{-(e^{\lambda x^{\alpha}}-1)+\lambda x^{\alpha}}\frac{C'(\theta e^{-(e^{\lambda x^{\alpha}}-1)})}{C(\theta)}. \tag{8}$$

$$h_{EPPS}(x)=\theta\lambda\alpha x^{\alpha-1}e^{-(e^{\lambda x^{\alpha}}-1)+\lambda x^{\alpha}}\frac{C'(\theta e^{-(e^{\lambda x^{\alpha}}-1)})}{C(\theta e^{-(e^{\lambda x^{\alpha}}-1)})}. \tag{9}$$

3 Properties

3.1 Density function

The following propositions discuss the limiting behaviour and some other characteristics of the EPPS family.

Proposition 3.1. For the pdf of EPPS family of distributions we have

$$\lim_{x\to 0} f_{EPPS}(x)=\begin{cases}\infty, & \text{if } 0<\alpha<1\\ 0, & \text{if } \alpha>1\\ \frac{\theta C'(\theta)}{C(\theta)}, & \text{if } \alpha=1\end{cases} \quad \text{and} \quad \lim_{x\to\infty} f_{EPPS}(x)=0.$$

Proposition 3.2. The pdf of EPPS family is monotone decreasing if $\alpha\leq 1$ and has at least a mode if $\alpha>1$.

Proof. We have that

$$\frac{d\log f_{EPPS}(x)}{dx}=0\Longrightarrow\frac{(\alpha-1)e^{u}}{\alpha u}=(1-e^{-(u-1)})+\theta e^{-(u-1)}\frac{C''(\theta e^{-(u-1)})}{C'(\theta e^{-(u-1)})} \tag{10}$$

where $u=\lambda x^{\alpha}>0$ and $C''(.),C'(.)>0$. If $\alpha\leq 1$ the above equation does not have a solution, for $u>0$. Therefore, $f_{EPPS}(x)$ is monotone decreasing. For $\alpha>1$, let $g_1(u)=\frac{(\alpha-1)e^{u}}{\alpha u}$ and $g_2(u)=(1-e^{-(u-1)})+\theta e^{-(u-1)}\frac{C''(\theta e)}{C'(\theta e)}$. It is obvious that $g_1(u)\to\infty$ and $g_2(u)\to\theta e\frac{C''(\theta e)}{C'(\theta e)}$ as $u\to 0$, $g_1(u)\to 0$ and $g_2(u)\to 1$ as $u\to\infty$. With this and Proposition 3.1, equation (10) has at least one root and hence the density $f_{EPPS}(x)$ has at least a mode. □

Proposition 3.3. The exponential power distribution is a limiting distribution of the EPPS family when $\theta\to 0^{+}$.

Proof. Using $C(\theta)=\sum_{n=1}^{\infty}a_n\theta^n$, we have

$$\begin{aligned}\lim_{\theta\to 0^{+}}F_{EPPS}(x)&=1-\lim_{\theta\to 0^{+}}\frac{\sum_{n=1}^{\infty}a_n(\theta e^{-(e^{\lambda x^{\alpha}}-1)})^n}{\sum_{n=1}^{\infty}a_n\theta^n}\\&=1-\lim_{\theta\to 0^{+}}\frac{a_1e^{-(e^{\lambda x^{\alpha}}-1)}+\sum_{n=2}^{\infty}na_{n-1}\theta^{n-1}e^{-n(e^{\lambda x^{\alpha}}-1)}}{a_1+\sum_{n=2}^{\infty}na_{n-1}\theta^{n-1}}\\&=1-e^{-(e^{\lambda x^{\alpha}}-1)}=F_{EP}(x).\end{aligned}$$

Table 2: Closed-form expressions for $\nu_{k,n}$.

Distribution	$\nu_{k,n}$
Poisson	$\frac{\theta^k}{(1-e^{-\theta})k!}$
Logarithmic	$\frac{-1}{\log(1-\theta)^k}\left(\frac{\theta}{1-\theta}\right)^k$
Geometric	$\frac{\theta^{k-1}}{(1-\theta)^k}$
Binomial	$\frac{\theta^k(1+\theta)^{m-k}}{(1+\theta)^m-1}\binom{m}{k}$

□

Proposition 3.4. The pdf of the EPPS family can be written as a linear combination of density of $X_{(1)} = min\{X_1, X_2, \ldots, X_n\}$.

Proof. Using $C'(\theta) = \sum_{n=1}^{\infty} n a_n \theta^{n-1}$ in (8), it follows that

$$f_{EPPS}(x) = \sum_{n=1}^{\infty} p_n f_{X_{(1)}}(x), \tag{11}$$

where $f_{X_{(1)}}(x) = n f_{EP}(x)(1 - F_{EP}(x))^{n-1}$. □

An interesting representation of the pdf of EPPS can be formulated as follows: Using binomial theorem, (11) can be written as

$$\begin{aligned} f_{EPPS}(x) &= \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}k\binom{n}{k}p_n f_{EP}(x)F_{EP}(x)^{k-1} \\ &= \sum_{n=1}^{\infty}\sum_{k=1}^{n}(-1)^{k-1}p_n\binom{n}{k}f_{CEP}(x) \end{aligned}$$

where

$$f_{CEP}(x;\lambda,\alpha,k) = k\lambda\alpha x^{\alpha-1}e^{-(e^{\lambda x^{\alpha}}-1)+\lambda x^{\alpha}}\left[1-e^{-(e^{\lambda x^{\alpha}}-1)}\right]^{k-1} \tag{12}$$

is the pdf of the complementary exponential power distribution. Changing the order of the double summation to write

$$f_{EPPS}(x) = \sum_{k=1}^{\infty}(-1)^{k-1}\nu_{k,n}f_{CEP}(x), x > 0, \tag{13}$$

where $\nu_{k,n} = \sum_{n=k}^{\infty} p_n \binom{n}{k}$. Table 2 shows closed expressions of $\nu_{k,n}$ for Poisson, logarithmic, geometric and binomial distributions. Therefore, the pdf of EPPS class of distribution can be represented as an infinite mixture of CEP density. The CEP distribution, as presented by Barriga et al. [2] with parametrization $f_{CEP}(x; 1/\alpha, \beta, \theta)$, is a flexible lifetime distribution which can handle bathtub shaped, unimodal, increasing and decreasing hazard rate functions.

Theorem 3.1. Let X follows EPPS$(\alpha, \lambda, \theta)$. Then:

(i)EPPS is closed under scale transformation, i.e., cX follows EPPS$(\alpha, \lambda/c^{\alpha}, \theta)$, $c > 0$.
(ii)EPPS is closed under power transformation, i.e., X^c follows EPPS$(\alpha/c, \lambda, \theta)$, $c > 0$.
(iii)EPPS is closed inverse power transformation , i.e., X^{-1} has the inverse EPPS distribution.

Proof. (i) Let $c > 0$, $Z = cX$. Thus, $F_Z(z) = P(Z \le z) = P(X \le \frac{z}{c}) = F_X(\frac{z}{c}; \alpha, \lambda, \theta)$. So that, $f_Z(z) = c^{-1} f_X(\frac{z}{c}; \alpha, \lambda, \theta)$ is given by

$$\begin{aligned} f_Z(z) =& c^{-1}\theta\lambda\alpha(\frac{z}{c})^{\alpha-1}e^{-(e^{\lambda(\frac{z}{c})^{\alpha}}-1)+\lambda(\frac{z}{c})^{\alpha}}\frac{C'(\theta e^{-(e^{\lambda(\frac{z}{c})^{\alpha}}-1)})}{C(\theta)}. \\ =& f_{EPPS}(z;\alpha,\lambda/c^{\alpha},\theta). \end{aligned}$$

The proofs of (ii) and (iii) are similar to the previous proof and therefore are omitted. □

3.2 Quantile function, Bowley skewness and Moors kurtosis

The quantile function of EPPS class of distribution is given by

$$Q(u) = \lambda^{-\alpha^{-1}} \log\left\{1 + \log\theta - \log\{C^{-1}((1-u)C(\theta))\}\right\}^{\alpha^{-1}} \tag{14}$$

where u has a uniform $U(0,1)$ distribution. The effect of the shape parameter θ on the skewness and kurtosis of the distribution can be studied using $Q(u)$. The Bowley skewness [13] and Moors kurtosis [14] can be utilized for such investigation.
Bowley skewness is

$$sk = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{2}{4}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}.$$

Moors kurtosis is

$$ku = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}.$$

3.3 Moments and mean deviations

The moments of a distribution is important for studying some of the most important features and characteristics of the distribution such as tending, dispersion, skewness and kurtosis. Let X be a random variable following the EPPS distribution with pdf (8). Using (13), the r_{th} moment is given by

$$E(X^r) = k\lambda\alpha \sum_{k=1}^{\infty} (-1)^{k-1} \nu_{k,n} \int_0^{\infty} x^{r+\alpha-1} e^{-(e^{\lambda x^\alpha}-1)+\lambda x^\alpha} \left[1 - e^{-(e^{\lambda x^\alpha}-1)}\right]^{k-1} dx$$

Using binomial theorem and let $y = e^{\lambda x^\alpha}$, we get

$$E(X^r) = k \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} (-1)^{i+k-1} \nu_{k,n} \binom{k-1}{i} e^{i+1} \lambda^{-\frac{r}{\alpha}} I_1$$

where $I_1 = \int_1^{\infty} (\log y)^{\frac{r}{\alpha}} e^{-(i+1)y} dy$.

For the positive integer values of $(\frac{r}{\alpha})$, the integral $I_1 = (\frac{r}{\alpha})!\, E_0^{\frac{r}{\alpha}}(i+1)$, where

$$\begin{aligned} E_s^j(z) =& \frac{(-1)^j}{j!} \frac{\partial^j \Gamma(1-s,z)}{\partial s^j} \\ =& \frac{1}{j!} \int_1^{\infty} (\log t)^j t^{-s} e^{-zt} dt, j \in \mathbb{N}_0; s, z \in \mathbb{C} \end{aligned} \tag{15}$$

is the generalized integro-exponential function, for properties and numerical tables see Milgram [15]. Note that, $E_0^r(z) = \frac{E_1^{r-1}(z)}{z}$, $E_1^0(z) = \Gamma(0,z)$, where $\Gamma(s,z) = \int_z^{\infty} t^{s-1} e^{-t} dt$ is the upper incomplete gamma function.

Using Meijer G-function, $G_{p+1,q+1}^{m,n+1}\left(zy \mid \begin{matrix} \boldsymbol{a}_p \\ \boldsymbol{b}_q \end{matrix}\right)$, where $\boldsymbol{a}_p = a_1, \ldots, a_n; a_{n+1}, \ldots, a_p$ and $\boldsymbol{b}_q = b_1, \ldots, b_m; b_{m+1}, \ldots, b_q$, the generalized integro-exponential function can be written as

$$E_s^j(z) = G_{j+1,j+2}^{j+2,0}\left(z \mid \begin{matrix} ;s,\ldots,s \\ 0, s-1, \ldots, s-1; \end{matrix}\right) \tag{16}$$

For the positive real values of $(\frac{r}{\alpha})$, the integral I_1 requires to be computed numerically. In this case, an extension of the generalized integro-exponential function, introduced by Pogany et al. [16], can also be used to calculate the integral I_1. They derived triple power series expansion of the generalized integro-exponential function $\mathbb{E}_s^{\tau}(z)$ for real $\tau > 0$. Thus, the function $\mathbb{E}_s^j(z)$ can be presented as

$$\mathbb{E}_s^{\tau}(z) = \sum_{l \geq 0} \frac{(s+2)_l}{l!} \Phi_{\mu,1}^{(0,1)}(-l, \tau+1, 1)\ {}_1F_1(s+l+2; s+2; -z),$$

where $\Phi_{\mu,\nu}^{(\rho,\sigma)}(z,s,u)=\sum_{n\geq 0}\frac{(\mu)_{\rho n}z^n}{(\nu)_{\sigma n}(n+u)^s}$ is the Lin-Srivastava generalized Hurwitz-Lerch Zeta function, [17]. Here $(s+2)_l=\frac{\Gamma(s+l+2)}{\Gamma(s+2)}$ denotes the generalized Pochhammer symbol and ${}_1F_1(a;b;x)=\sum_{n\geq 0}\frac{(a)_n x^n}{(b)_n n!}$ is the confluent hypergeometric function - Kummer's function [18].

$$\mu_r'=E(X_{EPPS}^r)=k\sum_{k=1}^{\infty}\sum_{i=0}^{\infty}(-1)^{i+k-1}\nu_{k,n}\binom{k-1}{i}e^{i+1}\lambda^{-\frac{r}{\alpha}}(\frac{r}{\alpha})!\,\mathbb{E}_0^{\frac{r}{\alpha}}(i+1). \tag{17}$$

The mean deviation about the mean and the mean deviation about the median are defined by

$$D(\mu)=\int_0^{\infty}|x-\mu|f(x)\,dx,\qquad D(M)=\int_0^{\infty}|x-M|f(x)\,dx$$

respectively, where $\mu=E(X)=\mu_1'$ and $M=Median(X)=Q(0.5)$. These measures can be expressed as

$$D(\mu)=2\mu F(\mu)-2\int_0^{\mu}xf(x)\,dx,\qquad D(M)=\mu-2\int_0^{M}xf(x)\,dx.$$

4 Special Cases

4.1 Exponential power Poisson distribution

The exponential power Poisson (EPP) distribution is a special case of EPPS distribution with $a_n=\frac{1}{n!}$ and $C(\theta)=e^{\theta}-1$. Using cdf (6) the cdf of EPP distribution is given by

$$F_{EPP}(x)=1-\frac{e^{\theta e^{-(e^{\lambda x^{\alpha}}-1)}}-1}{e^{\theta}-1} \tag{18}$$

The pdf and the hazard rate function of EPG distribution are given, respectively, by

$$f_{EPP}(x)=\theta\alpha\lambda x^{\alpha-1}e^{\lambda x^{\alpha}-(e^{\lambda x^{\alpha}}-1)}\frac{e^{\theta e^{-(e^{\lambda x^{\alpha}}-1)}}}{e^{\theta}-1} \tag{19}$$

$$h_{EPP}(x)=\theta\alpha\lambda x^{\alpha-1}e^{\lambda x^{\alpha}-(e^{\lambda x^{\alpha}}-1)}\frac{e^{\theta e^{-(e^{\lambda x^{\alpha}}-1)}}}{e^{\theta e^{-(e^{\lambda x^{\alpha}}-1)}}-1} \tag{20}$$

4.2 Exponential power logarithmic distribution

The exponential power logarithmic distribution (EPL) is a special case of EPPS distribution with $a_n=n^{-1}$ and $C(\theta)=-\log(1-\theta)$. Using cdf (6) the cdf of EPL distribution is given by

$$F_{EPL}(x)=1-\frac{\log(1-\theta e^{-(e^{\lambda x^{\alpha}}-1)})}{\log(1-\theta)} \tag{21}$$

The pdf and the hazard rate function of EPG distribution are given, respectively, by

$$f_{EPL}(x)=\frac{\theta\alpha\lambda x^{\alpha-1}e^{\lambda x^{\alpha}-(e^{\lambda x^{\alpha}}-1)}}{(\theta e^{-(e^{\lambda x^{\alpha}}-1)}-1)\log(1-\theta)} \tag{22}$$

$$h_{EPL}(x)=\frac{\theta\alpha\lambda x^{\alpha-1}e^{\lambda x^{\alpha}-(e^{\lambda x^{\alpha}}-1)}}{(\theta e^{-(e^{\lambda x^{\alpha}}-1)}-1)\log(1-\theta e^{-(e^{\lambda x^{\alpha}}-1)})} \tag{23}$$

4.3 Exponential power geometric distribution

The exponential power geometric distribution (EPG) is a special case of EPPS distribution with $a_n = 1$ and $C(\theta) = \theta(1-\theta)^{-1}$. Using cdf (6) the cdf of EPG distribution is given by

$$F_{EPG}(x) = 1 - \frac{(1-\theta)e^{-(e^{\lambda x^{\alpha}}-1)}}{1-\theta e^{-(e^{\lambda x^{\alpha}}-1)}}, x > 0, \tag{24}$$

and the survival function is given by

$$\bar{F}_{EPG}(x) = \frac{(1-\theta)e^{-(e^{\lambda x^{\alpha}}-1)}}{1-\theta e^{-(e^{\lambda x^{\alpha}}-1)}}, x > 0, \tag{25}$$

The pdf and the hazard rate function of EPG distribution are given, respectively, by

$$f_{EPG}(x) = \frac{(1-\theta)\alpha\lambda x^{\alpha-1}e^{\lambda x^{\alpha}-(e^{\lambda x^{\alpha}}-1)}}{(1-\theta e^{-(e^{\lambda x^{\alpha}}-1)})^2} \tag{26}$$

$$h_{EPG}(x) = \frac{\alpha\lambda x^{\alpha-1}e^{\lambda x^{\alpha}}}{1-\theta e^{-(e^{\lambda x^{\alpha}}-1)}} \tag{27}$$

where $\alpha, \lambda > 0$ and $\theta \in (0,1)$. Equation (26) is a proper density function even for $\theta \leq 0$. The hazard function for the EPG distribution has increasing, decreasing, increasing-decreasing-increasing shaped. Fig. 1 shows the possible shapes for the pdf and the hazard function of the EPG distribution.

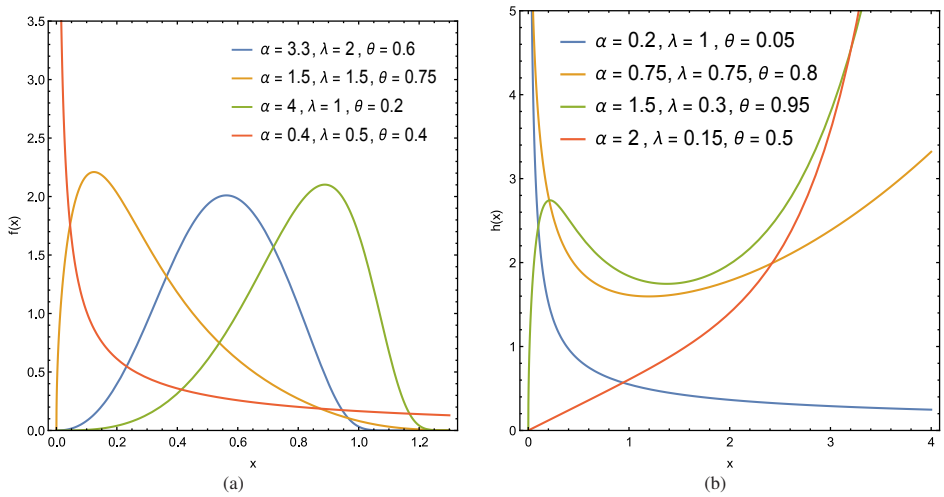


Fig. 1: (a) pdf for EPG distribution (b) hazard rate function for EPG distribution for different parameters' values

Taking $\lambda = 1$, skewness and kurtosis of the EPG distribution are plotted as functions of θ for selected choices of α. Fig. 2 shows plots of skewness and kurtosis of the EPG distribution which reveal that the shapes of the proposed distribution have strong dependence on the values of the parameter θ. Further, the EPG distribution can be used to model positive and negative skewness as well as symmetric datasets.

Table 3 shows the first four moments and some descriptive statistics of EPG distribution. These results show that, the first four moments increase while the skewness and kurtosis decrease as the parameter θ decreases.

5 Cure rate modelling

This section presents a defective version of EPG distribution in order to model survival data contains cured individuals. Further, it discusses an approach on how to include covariates information to the proposed model.

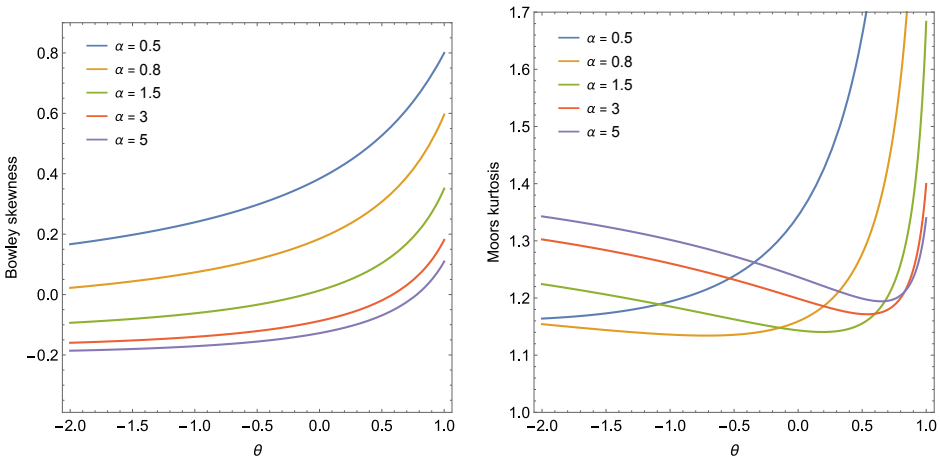


Fig. 2: Bowley skewness andMoors kurtosis measures of the EPG distribution for some parameters' values

Table 3: The first four moments and some descriptive statistics for the EPG distribution where $\lambda = 1$.

α	θ	μ_1'	μ_2'	μ_3'	μ_4'	sk	ku	$D(\mu)$	$D(M)$
0.5	0.9	0.0988	0.0913	0.1620	0.3944	5.8770	50.4194	0.1376	0.0968
	0.5	0.3403	0.4029	0.7675	1.9200	2.8271	13.5243	0.3617	0.3077
	-0.5	0.6729	0.9947	2.0823	5.4523	1.7136	6.5880	0.5604	0.5259
	-1	0.7857	1.2352	2.6671	7.1019	1.4938	5.6357	0.6098	0.5822
	-2	0.9618	1.6501	3.7352	10.2208	1.2203	4.6454	0.6735	0.6549
1.5	0.9	0.2796	0.1401	0.0988	0.0855	1.6203	5.8556	0.1876	0.1767
	0.5	0.5335	0.3899	0.3403	0.3335	0.5817	2.7252	0.2668	0.2645
	-0.5	0.7440	0.6702	0.6729	0.7275	0.0142	2.3377	0.2826	0.2826
	-1	0.8001	0.7562	0.7857	0.8717	-0.1236	2.3894	0.2803	0.2799
	-2	0.8783	0.8837	0.9618	1.1061	-0.3121	2.5577	0.2732	0.2720
2.5	0.9	0.4259	0.2347	0.1556	0.1180	0.8348	3.3934	0.1838	0.1807
	0.5	0.6520	0.4917	0.4082	0.3631	0.0441	2.3695	0.2130	0.2130
	-0.5	0.8117	0.7204	0.6774	0.6642	-0.4738	2.7372	0.2015	0.2001
	-1	0.8513	0.7833	0.7580	0.7607	-0.6095	2.9749	0.1947	0.1926
	-2	0.9048	0.8722	0.8762	0.9071	-0.8014	3.4178	0.1832	0.1802

5.1 Defective exponential power geometric

The defective version of the EPG distribution is defined by changing the usual domain of the parameters λ and θ to be $\lambda < 0$ and $\theta > 1$. The density and survival functions for the defective EPG distribution are given by

$$\bar{F}(x) = \frac{(1-\theta)e^{-(e^{\lambda x^{\alpha}}-1)}}{1-\theta e^{-(e^{\lambda x^{\alpha}}-1)}}, x > 0, \lambda < 0, \alpha > 0, \theta > 1, \tag{28}$$

and

$$f(x) = \frac{(1-\theta)\alpha\lambda x^{\alpha-1}e^{\lambda x^{\alpha}-(e^{\lambda x^{\alpha}}-1)}}{(1-\theta e^{-(e^{\lambda x^{\alpha}}-1)})^2}, x > 0, \lambda < 0, \alpha > 0, \theta > 1, \tag{29}$$

respectively. So that, modelling survival data with long-term survivors using defective EPG distribution is available with cure rate given as follows

$$p = \lim_{x\to\infty} \bar{F}(x) = \frac{\theta - 1}{\theta - e^{-1}} \in (0,1) \tag{30}$$

The density function of the defective EPG distribution in (29) is a positive density integrated to the proportion $\frac{1-e^{-1}}{\theta-e^{-1}} \in (0,1)$. The following theorem presents the ordinary moments for a defective EPG distributed random variable.

Theorem 5.1. Let X be a random variable following the defective EPG distribution with density function (29), then the r_{th} ordinary moments of the random variable X is given by

$$E(X^r) = \sum_{k,i\geq 0} (-\lambda(i+1))^{-\frac{r}{\alpha}} \frac{(1-\theta)(k+1)^{i+1}}{\theta^{k+2}e^{k+1}} \frac{\Gamma(\frac{r}{\alpha}+1)}{\Gamma(i+2)} \tag{31}$$

where $\Gamma(z) = \int_0^\infty y^{z-1}e^{-y}dy$ is gamma function and $\alpha > 0, \lambda < 0, \theta > 1$.

Proof. By definition, the r_{th} ordinary moments of the random variable X is given by

$$E(X^r) = \int_0^\infty (1-\theta)\alpha\lambda x^{r+\alpha-1} e^{\lambda x^\alpha - (e^{\lambda x^\alpha}-1)} \{1-\theta e^{-(e^{\lambda x^\alpha}-1)}\}^{-2} dx$$

by expanding the binomial and the exponential terms it follows

$$E(X^r) = \alpha\lambda \sum_{k,i\geq 0} \frac{(1-\theta)(k+1)^{i+1}}{\theta^{k+2}e^{k+1}i!} \int_0^\infty x^{r+\alpha-1} e^{(i+1)\lambda x^\alpha} dx$$

The result is obtained by setting $y = -(i+1)\lambda x^\alpha$. □

5.2 Model with covariates

Here, a regression model is discussed in order to include covariates information to the defective EPG distribution. Following the work of de Castro et al [19] and Leão et al. [20], the proposed cure rate regression model considers reparameterizing the EPG distribution in terms of the cure rate (30) such that

$$\theta = \frac{pe^{-1} - 1}{p - 1} \tag{32}$$

where $p \in (0,1)$ is the cure rate parameter. Consequently, the covariates can be included directly through the cure rate imposing that the cure rate $p(\boldsymbol{z})$ satisfies the functional relation

$$g(p(\boldsymbol{z})) = \boldsymbol{\beta}^T \boldsymbol{z},$$

where $\boldsymbol{z} = (1, z_1, z_2, \ldots, z_k)^T$ be a vector of covariates and $\boldsymbol{\beta} = (\beta_0, \beta_1, \ldots, \beta_k)^T$ be a vector of regression coefficients. Different choices for the link function $g(.)$ cen be considered. Due to the direct interpretation of the parameters in terms of odds, in this paper we consider the logit link. Thus

$$g(p(\boldsymbol{z})) = \log\left(\frac{p(\boldsymbol{z})}{1-p(\boldsymbol{z})}\right).$$

Under this parametrization, the cure rate is given by

$$p(\boldsymbol{z}) = \frac{e^{\boldsymbol{\beta}^T \boldsymbol{z}}}{1 + e^{\boldsymbol{\beta}^T \boldsymbol{z}}} \tag{33}$$

This approach is attractive because the cure rate is easily calculated through the logit function. So that, the cure rate depends directly on the regression coefficients, making it very easy to interpret. If $\boldsymbol{\beta}^T \boldsymbol{z}$ increases, so does the cure rate towards 1. If $\boldsymbol{\beta}^T \boldsymbol{z}$ decreases, so does the cure rate towards 0.

The defective EPG cure rate regression mode is given by the survival function

$$\bar{F}(x|\boldsymbol{z}) = \frac{p(\boldsymbol{z})(1-e^{-1})e^{-(e^{\lambda x^{\alpha}}-1)}}{(1-e^{-1}p(\boldsymbol{z}))e^{-(e^{\lambda x^{\alpha}}-1)}-(1-p(\boldsymbol{z}))},$$

and the pdf

$$f(x|\boldsymbol{z}) = \frac{\left(1-e^{-1}\right)\alpha\lambda(p(\boldsymbol{z})-1)p(\boldsymbol{z})x^{\alpha-1}e^{\lambda x^{\alpha}-\left(e^{\lambda x^{\alpha}}-1\right)}}{\left((p(\boldsymbol{z})-1)-(p(\boldsymbol{z})e^{-1}-1)\,e^{1-e^{\lambda x^{\alpha}}}\right)^{2}},$$

where $\lambda<0, \alpha>0$.

6 Estimation and simulation

6.1 Maximum likelihood method

In order to estimate the parameters of the EPPS class, we utilize the maximum likelihood (MLE) method. Assume that the lifetimes are independently distributed and independent from the censoring mechanism. Consider a sample of size n and the observed time is $X_i = min(T_i, C_i)$, where C_i denote the censored time and T_i the time to event of interest, that is, if the censoring indicator $\delta_i = 1$ then $X_i = T_i$ and $\delta_i = 0$ otherwise, $i = 1,\ldots,n$. The MLE $\hat{\boldsymbol{\Theta}} = (\hat{\lambda}, \hat{\alpha}, \hat{\theta})$ for the vector parameter $\boldsymbol{\Theta} = (\lambda, \alpha, \theta)$ is obtained from maximizing the likelihood function, or equivalently maximizing the log-likelihood function, corresponding to the EPPS class. The log-likelihood function is given by

$$\begin{aligned}\ell(\boldsymbol{\Theta}) =& \sum_{i=1}^{n}\delta_i\log(\lambda\alpha\theta) - n\log(C(\theta)) + (\alpha-1)\sum_{i=1}^{n}\delta_i\log x_i + \sum_{i=1}^{n}\delta_i\left\{\lambda x_i^{\alpha} - (e^{\lambda x_i^{\alpha}}-1)\right\}\\ &+\sum_{i=1}^{n}\delta_i\log C'(\theta e^{-(e^{\lambda x_i^{\alpha}}-1)}) + \sum_{i=0}^{n}(1-\delta_i)\log C(\theta e^{-(e^{\lambda x_i^{\alpha}}-1)}).\end{aligned} \tag{34}$$

Assume that for each i, an explanatory variable vector $\boldsymbol{z_i} = (1, z_{i1}, z_{i2}, \ldots, z_{ik})^T$ independent of x_i. Let $\boldsymbol{\eta} = (\lambda, \alpha, \boldsymbol{\beta})$ corresponds to the regression model, proposed in section 5.2, be the vector of (k+3) unknown parameters to be estimated using the method of maximum likelihood. The log-likelihood function of $\boldsymbol{\eta}$ is then given by

$$\begin{aligned}\ell(\boldsymbol{\eta}) =& n\log(1-e^{-1}) + \sum_{i=1}^{n}\log(p_i) + \sum_{i=1}^{n}\delta_i\log(\alpha\lambda) + \sum_{i=1}^{n}\delta_i(p_i-1) + (\alpha-1)\sum_{i=1}^{n}\delta_i\log(x_i)\\ &+\lambda\sum_{i=1}^{n}\delta_i x_i^{\alpha} - \sum_{i=1}^{n}(e^{\lambda x_i}-1) - \sum_{i=1}^{n}(1+\delta_i)\log\left((p_i-1)-(p_i e^{-1}-1)e^{e^{\lambda x_i}{}^{\alpha}-1}\right)\end{aligned} \tag{35}$$

The log-likelihood functions (34) and (35) can be maximized directly using a mathematical software, i.e. Mathematica (NMaximize and FindMaximum functions), R (optim and MaxLik functions).

Confidence intervals for the parameters were based on asymptotic normality. If $\hat{\boldsymbol{\eta}}$ is MLE for $\boldsymbol{\eta}$, then its well known that the distribution of $\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}$ can be approximated by a $(k+3)-$variate normal distribution with zero means and covariance matrix $I^{-1}(\hat{\boldsymbol{\eta}})$, where $I(\boldsymbol{\eta})$ is the observed information matrix defined by

$$I(\boldsymbol{\eta}) = -\left[\frac{\partial\ell(\boldsymbol{\eta})}{\partial\boldsymbol{\eta}\partial\boldsymbol{\eta}^T}\right]$$

The asymptotic $100(1-\gamma)\%$ confidence interval for η_j is

$$\hat{v}_j \pm \mathscr{Z}_{\frac{\gamma}{2}} se(\hat{\eta}_j),$$

where $\mathscr{Z}_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}$ percentile of standard normal distribution and $se(\hat{\eta}_j)$ is the asymptotic standard error of $\hat{\eta}_j$. Note that $se(\hat{\eta}_j)$ is the square root of the j_{th} diagonal element of the matrix $I^{-1}(\hat{\boldsymbol{\eta}})$.

The estimated cure fraction $\hat{p}$ is calculated as a function of the estimated parameters. The delta method with first order Taylor's approximation can be used to estimate the variance of the estimated $\hat{p}$, [21]. Let $\hat{p} = g(\hat{\boldsymbol{\eta}})$, this function can be expanded as

$$g(\hat{\boldsymbol{\eta}}) = g(\boldsymbol{\eta}) + g'(\boldsymbol{\eta})(\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}).$$

Since $\hat{\boldsymbol{\eta}} - \boldsymbol{\eta}$ is approximated by a $(k+3)-$variate normal distribution with zero means and covariance matrix $I^{-1}(\hat{\boldsymbol{\eta}})$, then $g(\hat{\boldsymbol{\eta}})$ can be approximated by a univariate normal distribution with mean $g(\hat{\boldsymbol{\eta}})$ and variance $\boldsymbol{D}(\hat{\boldsymbol{\eta}})^T I^{-1}(\hat{\boldsymbol{\eta}})\boldsymbol{D}(\hat{\boldsymbol{\eta}})$, where the column vector $\boldsymbol{D}(\hat{\boldsymbol{\eta}})$, with $(k+3)$ component, is the first order partial derivative $\frac{\partial g(\boldsymbol{\eta})}{\partial \boldsymbol{\eta}}|_{\eta=\hat{\eta}}$.

6.2 MLE simulation

Here, the performance of the MLEs of the parameters of the EPPS family are examined. We considered EPG distribution and generated random samples using the inverse transformation method for different parameter combinations. The simulation study is repeated 1000 times each with sample sizes $n = 20, 50, 80, 100, 150$ and 200 for different values of the parameters λ, α and θ, where the shape of the density function (26) varies between symmetric, left-skewed and right-skewed. As a result of this simulation study, the mean square errors (MSEs) and the average biases (ABs) are obtained and presented in Table 4. We noticed from these results that: (i) the MSEs and ABs decrease toward zero as the sample size increases, (ii) the average widths (AWs) of the 95% asymptotic confidence intervals of the parameters decreases as the sample size increases, and (iii) the coverage probability (CPs) are quite close to 0.95. Similar results are obtained when generating random samples from EPP or EPL distributions and therefore are omitted.

Table 4: MSEs, ABs, AWs and CPs for the parameters of EPG distribution.

	$\lambda = 1.5$				$\alpha = 1.5$				$\theta = 0.5$			
n	MSE	AB	AW	CP	MSE	AB	AW	CP	MSE	AB	AW	CP
20	0.2190	0.1155	2.0046	0.964	0.2199	0.1094	2.6646	0.989	0.4770	0.2205	4.2579	0.933
50	0.0703	0.0500	1.2413	0.955	0.1102	0.0172	1.7041	0.98	0.3181	0.1886	2.8004	0.941
80	0.0521	0.0405	0.9918	0.952	0.0819	0.0141	1.3747	0.967	0.2324	0.1594	2.1792	0.932
100	0.0425	0.0411	0.8907	0.943	0.0694	0.0105	1.2166	0.959	0.2310	0.1535	1.9069	0.937
150	0.0298	0.0276	0.7386	0.951	0.0531	0.0022	1.0057	0.965	0.1283	0.1064	1.4461	0.952
200	0.0240	0.0363	0.6397	0.934	0.0394	0.0102	0.8676	0.972	0.1028	0.1001	1.2214	0.958
	$\lambda = 0.5$				$\alpha = 2.5$				$\theta = -0.5$			
n	MSE	AB	AW	CP	MSE	AB	AW	CP	MSE	AB	AW	CP
20	0.0455	0.0930	1.4901	0.897	1.7799	0.7163	6.4763	0.972	0.8982	0.2139	8.8592	0.862
50	0.0311	0.0590	1.0627	0.908	0.6822	0.3673	4.0732	0.971	0.7873	0.0969	6.4663	0.875
80	0.0237	0.0415	0.8948	0.925	0.4329	0.2580	3.2374	0.974	0.7044	0.0518	5.5708	0.882
100	0.0222	0.0257	0.8507	0.922	0.3787	0.1920	2.9546	0.961	0.7138	0.0311	5.4221	0.893
150	0.0176	0.0220	0.7177	0.934	0.2639	0.1479	2.4554	0.964	0.5588	0.0223	4.4541	0.906
200	0.0150	0.0084	0.6419	0.94	0.1931	0.0910	2.1434	0.969	0.5473	0.0535	4.0246	0.916
	$\lambda = 1$				$\alpha = 0.5$				$\theta = -0.75$			
20	0.0611	0.0745	1.9754	0.968	0.0472	0.1149	1.4166	0.996	0.8121	0.2169	9.748	0.906
50	0.0421	0.0641	1.3817	0.958	0.0261	0.0676	0.8979	0.972	0.7782	0.1332	7.8380	0.898
80	0.0336	0.0640	1.1042	0.958	0.0169	0.0552	0.7015	0.978	0.7426	0.1436	6.296	0.896
100	0.0279	0.0539	0.9924	0.964	0.0135	0.0484	0.6280	0.978	0.7038	0.0855	5.8343	0.908
150	0.0252	0.0214	0.8287	0.956	0.0102	0.0211	0.5023	0.97	0.6612	0.0743	5.0881	0.912
200	0.0209	0.0175	0.7380	0.952	0.0086	0.0152	0.4483	0.96	0.6102	0.0194	4.2179	0.924

In order to conduct a simulation study to examine the performance of the maximum likelihood estimates and the asymptotic confidence intervals of the parameters of the proposed regression model, we consider

$$logit(p(\boldsymbol{z})) = \beta_0 + \beta_1 z_i, \quad i = 1, \ldots, n.$$

Random samples of size n are generated using the following algorithm:

–Determine the parameters' values, $\lambda < 0, \alpha > 0$, β_0 and $\beta_1 > 0$ and then evaluate the cure rates p_0 and p_1 using equation (33).
–Generate a covariate $z_i \sim Bernoulli(0.5)$ for $i = 1, \ldots, n$.
–For $z_i = 0$, generate a random variable $M_i \sim Bernoulli(1 - p_0)$.
–Take $t_i^{'} = F_{EPG}^{-1}(u)$ where

$$F_{EPG}^{-1}(u) = \left\{ \frac{1}{\lambda} \log(1 + \log(\frac{1-\theta u}{1-u})) \right\}^{\frac{1}{\alpha}}, u \sim Uniform(0, 1 - p_0)$$

–Repeat steps 3 and 4, for $z_i = 1$, replacing p_0 with p_1.
–Set $\delta_i = 1$ if $M_i = 1$ and $t_i^{'} = min(t_i^{'}, u_i^{'})$; Otherwise set $\delta_i = 0$ if $M_i = 0$ or $u_i^{'} = min(t_i^{'}, u_i^{'})$.

We generated 1000 random samples each of sizes $n = 50, 100, 150, 200, 400, 600$ and parameter vector $\boldsymbol{\eta} = (-1, 1.5, 1, -1.5)$. The cure rates for each group is $p_0 = 0.731$ and $p_1 = 0.378$. Table 5 shows the MSEs and the absolute value of ABs for the model parameters and the cure rates. Table 6 shows the AWs and CPs for the asymptotic confidence intervals. From the simulation, we can conclude that: (i) the MSEs decrease and approach zero as the sample size increases; (ii) the MLEs of the model's parameters are asymptotically not biased since the ABs approach zero as the sample size increases; (iii) the MSEs and ABs of the estimated cure rates is generally much less than for the other parameters; (iv) the coverage probability for all parameters are very close to 0.95; (v) the AW of the confidence intervals decreases as the sample size increases; (vi) the AW is very small for the cure rates.

Table 5: MSEs and ABs for the parameters of univariate defective EPG regression model and for cure rates for each group.

	$\lambda = -1$		$\alpha = 1.5$		$\beta_0 = 1$		$\beta_1 = -1.5$		$p_0 = 0.731$		$p_1 = 0.378$	
n	MSEs	ABs	MSEs	ABs	MSEs	ABs	MSEs	ABs	MSEs	ABs	MSEs	ABs
50	1.8369	-0.4955	0.1814	0.1517	1.7951	0.5273	4.0551	1.2143	0.0336	0.0956	0.0219	0.0147
100	0.6133	-0.2695	0.0797	0.0974	0.1941	0.1521	3.0487	0.5661	0.0071	0.0355	0.0086	0.0057
150	0.2196	-0.1262	0.0477	0.0645	0.1725	0.1084	2.6188	0.3478	0.0058	0.0243	0.0058	0.0036
200	0.1741	-0.0936	0.0332	0.0459	0.0686	0.0615	2.1331	0.3980	0.0030	0.0149	0.0034	0.0019
400	0.0497	-0.0461	0.0135	0.0184	0.0338	0.0121	1.3196	0.3025	0.0013	0.0038	0.0014	0.0003
600	0.0242	-0.0111	0.0082	0.0081	0.0212	0.0031	0.1458	0.2282	0.0008	0.0003	0.0010	0.0001

Table 6: AWs and CPs for the parameters of univariate defective EPG regression model and for cure rates for each group.

	$\lambda = -1$		$\alpha = 1.5$		$\beta_0 = 1$		$\beta_1 = -1.5$		$p_0 = 0.731$		$p_1 = 0.378$	
n	AWs	CPs	AWs	CPs	AWs	CPs	AWs	CPs	AWs	CPs	AWs	CPs
50	5.0301	0.883	1.5608	0.928	4.1151	0.973	4.7433	0.833	0.8091	0.930	0.6887	0.908
100	2.5816	0.893	1.0175	0.910	1.6686	0.958	1.8965	0.893	0.3281	0.935	0.3540	0.910
150	2.1537	0.899	0.9193	0.925	1.4502	0.960	1.5820	0.880	0.2851	0.955	0.3170	0.913
200	1.4816	0.908	0.6814	0.933	1.1131	0.960	1.2918	0.898	0.2188	0.948	0.2384	0.945
400	0.9100	0.925	0.4552	0.945	0.7530	0.960	0.8932	0.913	0.1480	0.955	0.1611	0.965
600	0.6856	0.943	0.3619	0.950	0.6031	0.963	0.7228	0.918	0.1186	0.965	0.1284	0.958

6.3 Bayesian estimation method

This section concerns with the Bayesian estimates (BSE) for the parameter of the proposed regression model. For this purpose, we consider reparametrizing the proposed regression model in terms of $\lambda = -\lambda^*$ and $\theta = e^{\theta^*}$, where $\lambda^* and \theta^* >$

0. Thus, independent gamma prior distributions are assumed for the parameters α, λ^* and θ^* as

$$\begin{aligned}
\pi_1(\lambda^*) &\propto \lambda^{*a_1-1}e^{-b_1\lambda^*}, \quad \lambda^* > 0, \quad a_1 > 0, \quad b_1 > 0,\\
\pi_2(\alpha) &\propto \alpha^{a_2-1}e^{-b_2\alpha}, \quad \alpha > 0, \quad a_2 > 0, \quad b_2 > 0,\\
\pi_3(\theta^*) &\propto \theta^{*a_3-1}e^{-b_3\theta^*}, \quad \theta^* > 0, \quad a_3 > 0, \quad b_3 > 0,
\end{aligned} \tag{36}$$

where a_1, a_2, a_3, b_1, b_2 and b_3 are positive known hyperparameters. Whereas, independent normal prior distributions are considered for the covariate coefficients $\beta_j, j = 0, 1, \ldots, k$, that is $\beta_j \sim N(c_j, d_j)$, where c_j and $d_j, j = 0, 1, \ldots, k$, are known hyperparameters. The hyperparameters are chosen to reflect prior knowledge about the parameters $\lambda^*, \alpha, \theta^*, \beta_j, j = 0, \ldots, k$.

In Bayesian statistics the joint posterior distribution, $\pi(\boldsymbol{\eta}|\boldsymbol{x}, \delta)$, contains all relevant information on the unknown parameters given data. It is defined by combining the joint prior distribution with the likelihood function for the parameters $\lambda^*, \alpha, \theta^*, \beta_j, j = 0, \ldots, k$. In this work, we consider Markov chain Monte Carlo (MCMC) method to generate samples from the posterior distributions of the parameters of interest and then compute the Bayes estimators and construct the corresponding credible intervals (CI).

Bayesian analysis of complex statistical models using the MCMC method with a Gibbs sampling algorithm is available in the OpenBUGS software. OpenBUGS just requires the specification of the likelihood function and the prior distributions for the parameters in the model, Lunn et al. [22].

6.4 BSE simulation

To compute the posterior summaries, we generated sample of sizes $n = 50$, 150, 200, 600 assuming parameter vector $\boldsymbol{\eta} = (-1, 1.5, 1, -1.5)$. The samples were generated using the algorithm proposed in the previous section. Further, two different types of priors are considered, i.e. informative prior (IP) and non-informative prior (WIP). We used the method of moments to estimate the values of the hyperparameters, which yield mean approximately equal to the nominal value of parameter with variance 0.5 for the IP case and variance 2 for the WIP case. We generated 100,000 MCMC samples, discarding the first 5000 values as burn-in and taking every 10_{th} variate as iid observations. The resulting study is tabulated in table 7.

From table 7, it is noted that the posteriors means are closer to the parameters' initial values for larger sample sizes. The means are closer to parameters' initials for the case of IP than of WIP. The posteriors standard deviations (sd) and MCMC error decrease as the sample size increases and IP has the least sd as compared to the WIP cases. Further, credible intervals for the IP are narrower than the WIP cases.

7 Application

A practical use for the proposed family is presented by applications to three real datasets one of which contains cured individuals and censored data in presence of covariates informations.

7.1 Turbocharger data

The first dataset considered, represents the time to failure of turbocharger of a certain type of engine reported in [23]. This dataset is approximately symmetric with skewness -0.1603. The EPG and EPP distributions are fitted to this dataset. For comparison, other alternative models are fitted to the datasets, such as exponential power distribution (EP), complementary exponential power distribution (CEP), modified Weibull distribution (MW) and Gomperts distribution (G). In order to compare the fitted distributions the following statistics are considered: Kolmogorov-Smirnov (KS) distances between the empirical distribution function and the fitted distribution function (and its p-value), Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), Anderson-Darling statistic (A^*) and Cramer-von Mises statistic (W^*).
Table 8 summarizes the fitting results including MLEs (standard errors in parentheses) and the statistics used for comparison. Furthermore, visual comparison using the empirical and fitted densities along side Kaplan-Meier and fitted survival curves presented in fig. 3. These results show that EPG and EPP distributions provides the best fit to the dataset which proves the superiority of the proposed family over the other models.

Table 7: Posterior distributions summarise for defective EPG regression model.

n	parameters	Mean		sd		MC error		95% CrI LL		95% CrI UL	
		IP	WIP	IP	WIP	IP	WIP	IP	WIP	IP	WIP
50	λ	-0.8311	0.6574	0.4000	0.4915	0.0029	0.0055	-1.7690	-1.8460	0.2365	0.0376
	α	1.4370	1.3520	0.2725	0.3053	0.0018	0.0031	0.9370	0.8173	2.0050	1.9940
	β_0	1.0520	0.9015	0.3602	0.8299	0.0024	0.0123	0.3362	-1.0330	1.7490	2.3580
	β_1	-1.5160	-1.7090	0.3820	0.6929	0.0027	0.0066	-2.2680	-3.1350	-0.7737	-0.4094
	p_0	0.7354	0.6909	0.0691	0.1632	0.0005	0.0025	0.5833	0.2626	0.8518	0.9135
	p_1	0.3905	0.3330	0.0923	0.1398	0.0007	0.0017	0.2142	0.0540	0.5728	0.5888
150	λ	-0.9521	0.8966	0.3141	0.3608	0.0023	0.0031	-1.6460	-1.6780	-0.4170	-0.2701
	α	1.6610	1.6510	0.1990	0.2158	0.0014	0.0017	1.2810	1.2340	2.0590	2.0790
	β_0	0.9116	0.8821	0.2521	0.3412	0.0018	0.0034	0.4104	0.1777	1.4020	1.5130
	β_1	-1.6410	-1.7100	0.2977	0.3759	0.0022	0.0029	-2.2250	-2.4580	-1.0590	-0.9860
	p_0	0.7106	0.7027	0.0514	0.0713	0.0004	0.0007	0.6012	0.5443	0.8025	0.8195
	p_1	0.3287	0.3100	0.0643	0.0755	0.0005	0.0007	0.2075	0.1606	0.4579	0.4578
400	λ	-1.0340	1.0380	0.1620	0.1663	0.0011	0.0011	-1.3680	-1.3760	-0.7342	-0.7271
	α	1.4170	1.4180	0.0947	0.0966	0.0006	0.0007	1.2310	1.2300	1.6030	1.6090
	β_0	1.2020	1.2390	0.1669	0.1866	0.0011	0.0012	0.8816	0.8839	1.5360	1.6120
	β_1	-1.6060	-1.6500	0.2021	0.2289	0.0013	0.0015	-2.0050	-2.1070	-1.2120	-1.2110
	p_0	0.7675	0.7738	0.0296	0.0324	0.0002	0.0002	0.7072	0.7076	0.8228	0.8337
	p_1	0.4008	0.3993	0.0359	0.0373	0.0002	0.0002	0.3310	0.3275	0.4721	0.4733
600	λ	-1.0490	0.9178	0.1729	0.2003	0.0012	0.0013	-1.4030	-1.3290	-0.7244	-0.5417
	α	1.4920	1.4680	0.0877	0.1279	0.0006	0.0009	1.3200	1.2180	1.6650	1.7200
	β_0	0.9736	0.9286	0.1371	0.1590	0.0009	0.0011	0.7056	0.6087	1.2430	1.2350
	β_1	-1.5860	-1.5100	0.1679	0.2411	0.0011	0.0016	-1.9190	-1.9860	-1.2600	-1.0370
	p_0	0.7250	0.7157	0.0273	0.0324	0.0002	0.0002	0.6694	0.6476	0.7761	0.7746
	p_1	0.3522	0.3603	0.0314	0.0515	0.0002	0.0003	0.2915	0.2620	0.4144	0.4631

Table 8: MLEs (std. errors) and discriminant criterion for turbocharger dataset.

Model	MLEs (std. errors)			K-S (p-value)	AIC	BIC	CAIC	W^*	A^*
EPG(λ, α, θ)	1.4767 (0.3823)	0.4801 (0.2144)	-130.227 (242.14)	0.0396 (0.9999)	104.869	111.571	105.238	0.0143	0.1247
EPP(λ, α, θ)	0.7247 (0.1393)	1.0589 (0.2127)	-4.7955 (1.6616)	0.0416 (0.9998)	105.599	112.301	105.968	0.0162	0.1605
CEP(α, β, θ)	1.8521 (0.1775)	1.8258 (0.5166)	1.3087 (0.5471)	0.0799 (0.7701)	111.645	118.347	112.014	0.0684	0.5288
EP (λ, α)	1.9542 (0.0669)	2.1723 (0.2107)		0.0884 (0.6536)	112.055	118.757	112.424	0.0912	0.6707
MW(λ, α, θ)	0.0322 (0.0185)	0.016 (0.0532)	0.4368 (0.2116)	0.0607 (0.9607)	107.027	113.729	107.39	0.0394	0.3164
G(a, b)	0.0842 (0.0269)	1.8805 (0.2045)		0.0811 (0.7535)	107.937	112.406	108.119	.0932	0.6767

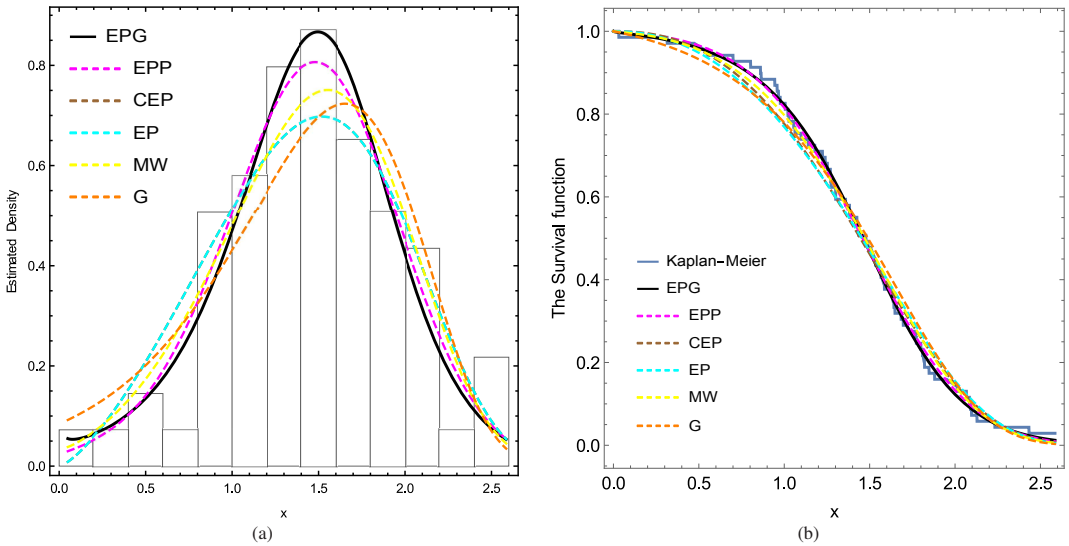


Fig. 3: Turbocharger dataset: (a) empirical and fitted densities (b) Kaplan-Meier and fitted survival curves

Table 9: MLEs (std. errors) and discriminant criterion for glass fibres dataset.

Model	MLEs (std. errors)			K-S (p-value)	AIC	BIC	CAIC	W^*	A^*
EPG(λ, α, θ)	0.956 (0.3706)	1.1789 (0.4705)	-54.477 (82.642)	0.0965 (0.6)	30.4202	36.8496	30.827	0.0733	0.4605
EPP(λ, α, θ)	0.0102 (0.0083)	5.9601 (0.6208)	5.417 (3.8365)	0.1511 (0.1123)	36.0694	42.4985	36.4758	0.2117	1.2546
CEP(α, β, θ)	1.6932 (0.09614)	2.7908 (0.6706)	1.7899 (0.6946)	0.1485 (0.1238)	36.0037	42.433	36.4105	0.2113	1.1684
EP (λ, α)	0.0987 (0.0248)	3.8778 (0.3642)		0.149 (0.122)	38.4221	44.8515	38.829	0.2578	1.3911
MW(λ, α, β)	2.8815 (2.5626)	5.6146 (1.0551)	8.3201 (39.6695)	0.152 (0.1086)	36.3769	42.806	36.7837	0.2172	1.2402
G(a, b)	0.0088 (0.0047)	3.6474 (0.3457)		0.1267 (0.2635)	33.6162	37.9025	33.8162	0.1629	0.9176

7.2 Glass fibres data

This dataset represents the strength of 1.5-cm glass fibres reported in [24]. This dataset is left-skewed with skewness -0.9. The MLEs(standard errors in parentheses), K-S statistic (its p-value), AIC, BIC, CAIC, A^* and W^* are listed in Table 9. These values verified that the EPG distribution is better than the CEP, EP, MW and Gompertz distributions in terms of fitting to this data. Plots of the empirical and fitted densities alongside Kaplan-Meier and fitted survival curves presented in fig. 4. These plots suggest that the EPG distribution is superior to the other fitted distributions. The EPP distribution provides a competitive results in fitting this dataset.

7.3 Colon cancer data

This dataset arises from one of the first successful trials of adjuvant chemotherapy for colon cancer. The event of interest here is the recurrence or death for the individual under the proposed treatment. There are 1858 observed times, of which 938 were censored (50.58 percent). A non-parametric estimate of the cure rate p is $\hat{F}(x_{(n)})$, where $\hat{F}(x_{(n)})$ is the Kaplan-Meier estimate at the largest observed time. For this data $\hat{F}(x_{(n)}) = 0.4651$. Details of this dataset can be found in [25].

In order to compare the performance of the defective EPG distribution, the following cure rate models are fitted to this dataset: Gompertz distribution G(a,b), inverse Gaussian distribution IG(a,b), The Marshal-Olkin cure rate models

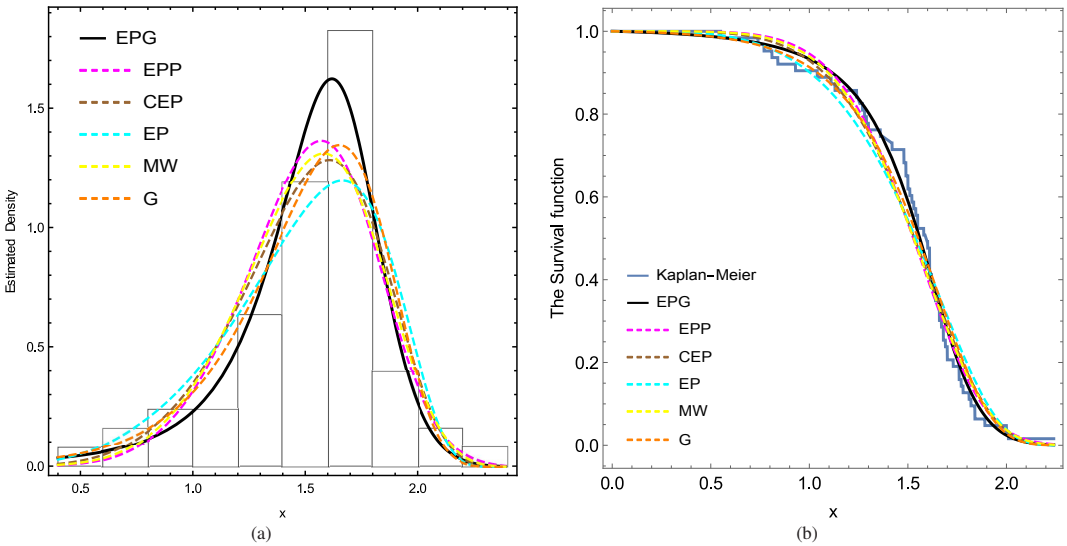


Fig. 4: Glass fibres dataset: (a) empirical and fitted densities (b) Kaplan-Meier and fitted survival curves

Table 10: MLEs, estimated cure rate (std. errors), AIC and BIC for colon cancer dataset.

Model	Estimates (Std. errors)				$\hat{p}$ (Std. error)	AIC	BIC
EPG(λ,α,θ)	-2.6571 (0.3311)	1.4294 (0.0476)	1.5452 (0.0322)		0.4631 (0.0147)	1456.29	1472.87
G(a,b)	2.0018 (0.0707)	-2.3375 (0.0901)			0.4247 (0.0396)	1518.02	1529.08
IG(a,b)	-1.6688 (0.077)	7.3406 (0.1407)			0.3654 (0.2319)	1597.47	1608.53
MOE(r,ν)	-0.5871 (0.1820)	-1.2272 (0.0703)			0.3699 (0.0279)	1531.10	1542.15
MOW(r,ν,a)	-0.8805 (0.0495)	-3.6391 (0.3797)	1.3671 (0.0429)		0.4682 (0.0689)	1462.36	1478.94
MOWE(r,ν,a,b)	-0.8783 (0.0495)	-8.6310 (4.8814)	11.452 (16.4843)	1.3604 (0.0431)	0.4676 (0.0140)	1464.69	1486.8
MxW(ν,a,p)	4.9543 (0.3559)	1.2038 (0.0372)	0.4783 (0.0128)		0.4783 (0.0128)	1481.29	1497.87
PTW(ν,a,θ)	4.2819 (0.3653)	1.2791 (0.0141)	0.7477 (0.0281)		0.4734 (0.0133)	1471.2	1487.78

with exponential, Weibull and Weibull extension, respectively denoted as MOE(r,ν), MOW(r,ν,a) and MOWE(r,ν,a,b), [11], and Mixture cure rate and promotion time cure rate models with Weibull distribution as a baseline distribution, MxW(a,b,p) and PTW(a,b,λ).

Table 10 summarizes the fitting results for the dataset, including the MLEs, estimated cure rates (standard errors in parentheses), AIC and BIC. The delta method is used to calculate the standard errors for the estimated cure rates $\hat{p}$. The lowest AIC and BIC are given by the defective EPG distribution, which indicates that it is the best alternative to modelling colon cancer dataset. Furthermore, the cure rate estimated by the proposed model is 0.4631 which is closer to the one estimated by Kaplan-Meier than for the other cure rate models. Kaplan-Meier and fitted survival curves are presented in fig. 5.

Table 11 listed the results of Bayesian estimation for colon cancer dataset. These results are very similar to MLE results. The hyperparameters are estimated using the method of moments as described in the simulation study section,

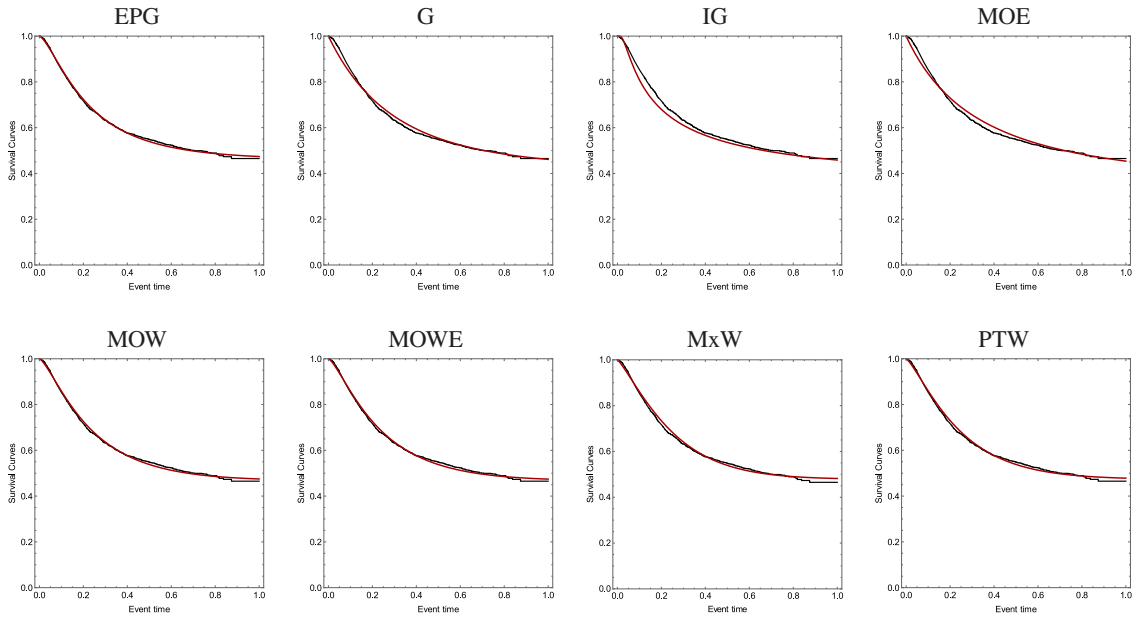


Fig. 5: Fitted survival curves and Kaplan-Meier curve for colon cancer data

Table 11: Posterior summaries for colon cancer dataset.

parameters	Mean	sd	MC error	95% CrI LL	95% CrI UL
λ	-2.6380	0.2944	0.0030	-3.2330	-2.0820
α	1.4270	0.0443	0.0004	1.3390	1.5130
θ	1.5425	0.0202	0.0002	1.4823	1.6046
p	0.4616	0.0143	0.0001	0.4328	0.4889

where the MLEs are considered as initial values. The marginal posterior density estimates of the parameters and the cure rate are and their histogram are shown in fig. 6 using the Gaussian kernel which provide evident the marginal posterior distributions are almost symmetrical. The trace plots of the iteration number against the value of the draw of the parameters and the cure rate at each iteration are provided in fig. 7.

Colon cancer data contains covariates information that would be expected to relate to survival experience. Two covariates are considered in this analysis, the adherence to the surrounding organs (adhere) and the extension of local spread (Extent). The covariate adhere assumes the value 0 if the tumer doesn't adhere to the surrounding organs (e.g. bladder) and 1 if it does. The covariate Extent is 1, 2, 3 or 4 if the tumer reach the cells of submucosa, muscle, serosa or contiguous structures, respectively.

Taking one covariate at a time, the results of fitting the univariate regression models to the data are summarized in table 12. Fig. 8 shows that the estimated survival curves capture the those estimated by the Kaplan-Meier very well for both covariates. For covariate adhere the model provide cured fraction estimates of $p_0 = 0.477$ and $p_1 = 0.383$. This means that the patients we observe adherence in the nearby organs have shorter survival times. For covariate Extent the model provide cured fraction estimates of $p_1 = 0.744$, $p_2 = 0.604$, $p_3 = 0.445$ and $p_4 = 0.296$. The patients we observe extension of the tumer in the contiguous structures have the shortest survival times. Similar results are obtained from MCMC algorithm. These results are summarised in table 13.

Table 14 shows the result of fitting multiple regression models with both covariates Adhere and Extent. Fig. 9 presents Kaplan-Meier curves for the possible scenarios from the covariate set. From these results, we can conclude that the proposed model fit very well to the colon cancer data with covariates Adhere and Extent. We used the MLEs as initial values to rune the MCMC algorithm to compute Bayesian estimates for the parameters and construct the corresponding credible intervals. The hyperparameters are estimated using the method of moments. The result of the MCMC algorithm are listed in table 15. These results are very similar to MLEs. The cure rates for the possible scenarios from the covariate set are presented in table 16.

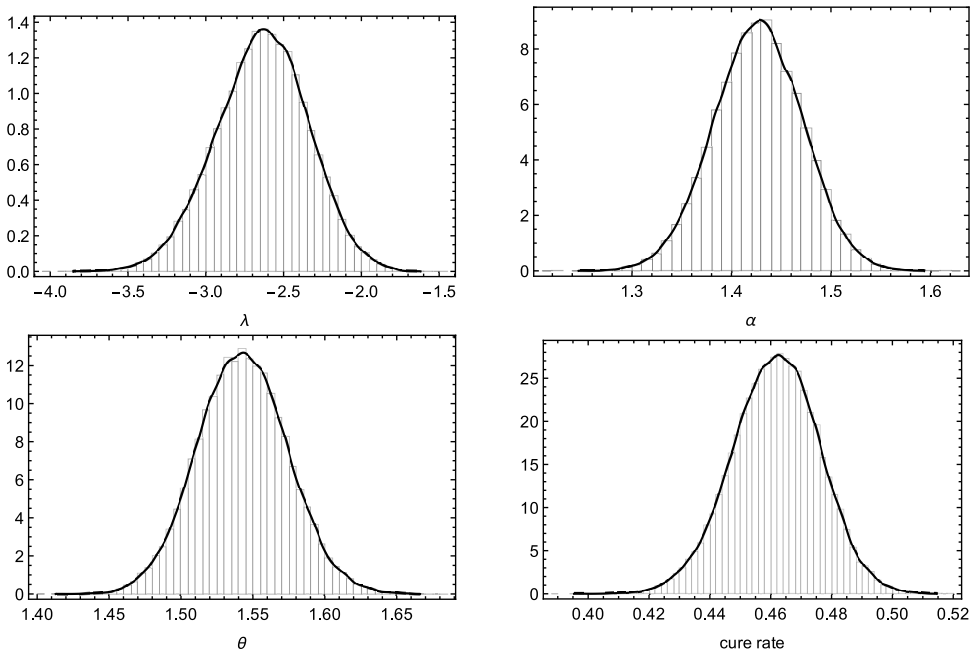


Fig. 6: Histogram and kernel density estimates of the parameters and the cure rate from colon cancer dataset

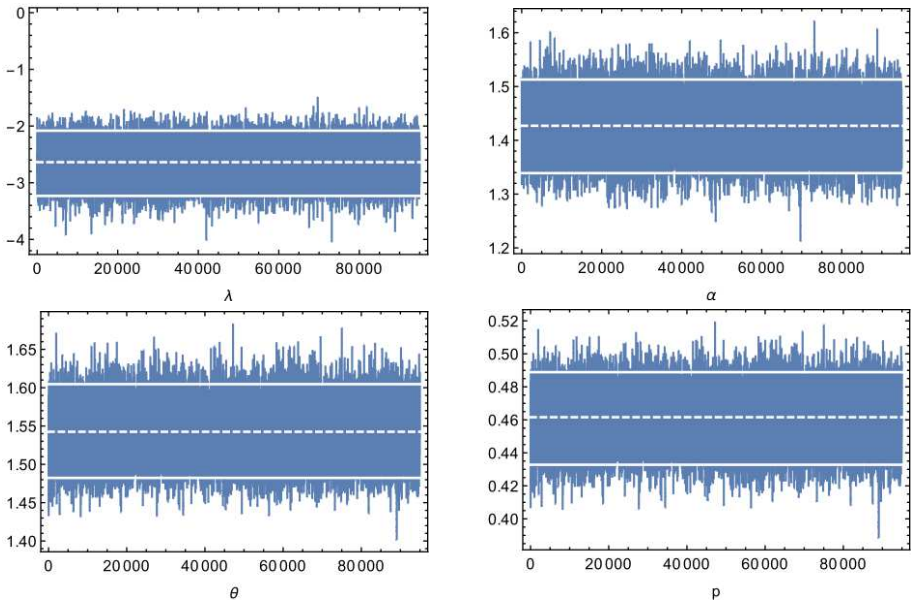


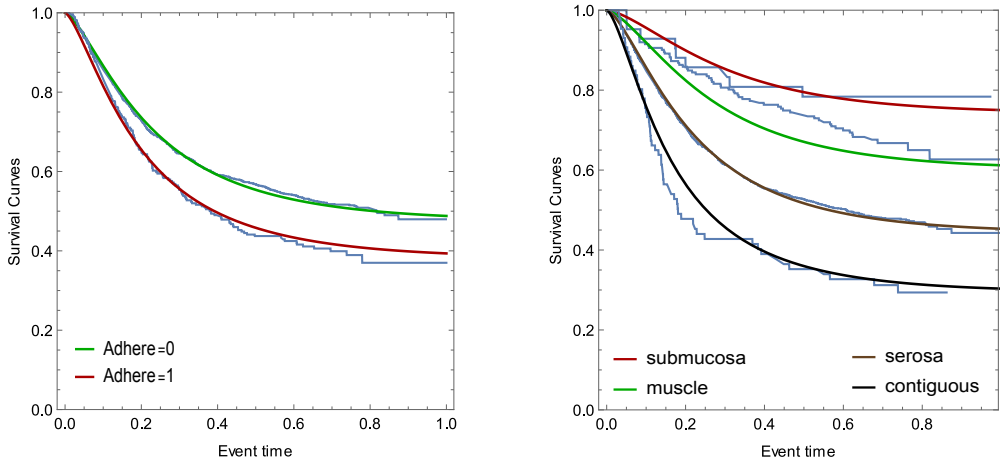
Fig. 7: Trace plots for the parameters and the cure rate: Dashed lines represent the posterior means and solid lines represent lower, and upper bounds 95% CrI interval from colon cancer dataset

Table 12: MLEs, cure rates and standard errors for the univariate regression model and (taking one covariate at a time), AIC and BIC for colon cancer dataset.

Parameter	Adhere		Extent	
	Estimate	Std. error	Estimate	Std. error
λ	-2.641	0.3307	-2.8887	0.3517
α	1.4312	0.0477	1.4939	0.0484
β_0	-0.0922	0.062	1.7090	0.2878
β_1	-0.384	0.0154	-0.6436	0.0973
p_0	0.477	0.0155	–	–
p_1	0.383	0.0275	0.7437	0.0368
p_2	–	–	0.6039	0.0251
p_3	–	–	0.4448	0.0145
p_4	–	–	0.2962	0.0254
AIC	1447.79	–	1396.79	–
BIC	1457.61	–	1406.61	–

Table 13: Posterior summaries for the univariate regression model for colon cancer dataset.

Parameters	Adhere					Extent				
	Mean	sd	MC error	95% CI LL	95% CI UL	Mean	sd	MC error	95% CrI LL	95% CrI UL
λ	-2.6040	0.2866	0.0116	-3.200	-2.0660	-2.690	0.2875	0.0119	-3.2620	-2.1420
α	1.4270	0.0434	0.0015	1.3410	1.5140	1.4540	0.0437	0.0019	1.3720	1.5400
β_0	-0.1018	0.0590	0.0023	-0.2178	0.0145	1.9490	0.2502	0.0077	1.4510	2.4180
β_1	-0.3785	0.1104	0.0043	-0.5946	-0.1599	-0.7267	0.0854	0.0026	-0.8897	-0.5558
p_0	0.4746	0.0147	0.0006	0.4458	0.5036					
p_1	0.3825	0.0254	0.0010	0.3334	0.4325	0.7712	0.0297	0.0009	0.7091	0.8235
p_2						0.6212	0.0215	0.0007	0.5775	0.6629
p_3						0.4425	0.0139	0.0005	0.4154	0.4690
p_4						0.2734	0.0245	0.0008	0.2264	0.3233



(a) (b)

Fig. 8: The estimated survival curves for univariate regression models: (a) with covariate Adhere (b) with covariate Extent

Table 14: MLEs and standard errors for multiple regression with covariates Adhere and Extent, AIC and BIC for colon cancer dataset.

Parameter	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}_0$	$\hat{\beta}_1$	$\hat{\beta}_2$	AIC	BIC
Estimate	−2.6025	1.4464	2.0005	−0.2539	−0.7327	1391.62	1408.16
Std. error	0.3313	0.0481	0.2957	0.1193	0.1003	–	–

Table 15: Posterior summaries for multivariate regression model for colon cancer dataset.

parameters	Mean	sd	MC error	95% CrI LL	95% CrI UL
λ	-2.5800	0.2983	0.0124	-3.1990	-2.0070
α	1.4440	0.0451	0.0017	1.3520	1.5280
β_0	2.0050	0.2486	0.0103	1.5350	2.5040
β_1	-0.2604	0.1114	0.0051	-0.4776	-0.0428
β_1	-0.7368	0.0845	0.0035	-0.9050	-0.5782

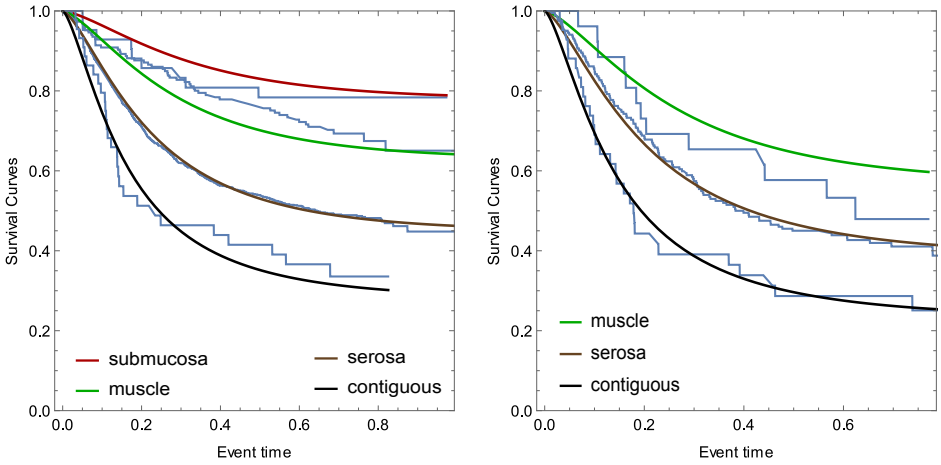


Fig. 9: The estimated survival curves for multiple regression models with covariates Adhere and Extent (a)Adhere=0, (b)Adhere=1

Table 16: Estimated cure rates for each configuration of covariates.

Cure rate	p_{01}	p_{02}	p_{03}	p_{04}	p_{12}	p_{13}	p_{14}
Estimate	0.7804	0.6306	0.4508	0.2829	0.57	0.389	0.2343
Std. error	0.034	0.0252	0.0159	0.0262	0.0387	0.028	0.0271

8 Conclusion

This paper proposed a new class of distributions, called exponential power power series (EPPS) class, by compounding the exponential power (EP) distribution and power series distributions. The hazard function of EPPS class showed more flexibility than the hazard function of EP distribution. Some characteristics of the density function and some basic properties of EPPS family are discussed. The EPPS family has the exponential power geometric (EPG) distribution as a special case which can be redefined as a defective distribution when changing the usual domains of some of its parameters. The moments for the defective EPG distribution is derived. More importantly, owing to the proposed defective distribution, a cure rate regression model is proposed for modeling lifetime data contain long-term survivors with associated covariates. The maximum likelihood method and Bayesian method are used for estimating the unknown parameters. The performance of these estimation methods is examined by conducting simulation studies. Three applications to real datasets are given to prove the superiority of the EPPS class. The proposed cure rate regression model is used to analyze the survival times for a group of patients who were subject to adjuvant chemotherapy after being diagnosed with colon cancer.

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Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

References

[1] R.M. Smith and L.J. Bain, An exponential power life-testing distribution, *Communications in Statistics*, **4**(5), 469-481 (1975).
[2] G.D.C. Barriga, F. Louzada-Neto and V.G. Cancho, The complementary exponential power lifetime model, *Computational Statistics & Data Analysis*, **55**(3), 1250-1259 (2011).
[3] Z. Chen, A new two-parameter lifetime distribution with bathtub shape or increasing failure rate function, *Statistics & Probability Letters*, **49**(2), 155-161 (2000).
[4] M. Xie, Y. Tang, and T.N. Goh, A modified weibull extension with bathtub-shaped failure rate function, *Reliability Engineering and System Safety*, **76**(3), 279-285 (2002).
[5] V. Pappas, K. Adamidis and S. Loukas, A family of lifetime distributions, *International Journal of Quality, Statistics, and Reliability*, **2012** (3), 1-6 (2012).
[6] A.W. Marshall and I. Olkin, A new method for adding a parameter to a family of distributions with application to the exponential and weibull families, *Biometrika*, **84**(3), 641-652 (1997).
[7] Y.P. Chaubey and R. Zhang, An extension of chen's family of survival distributions with bathtub shape or increasing hazard rate function, *Communications in Statistics - Theory and Methods*, **44**(19), 4049-4064 (2015).
[8] V.T. Farewell, The use of mixture models for the analysis of survival data with long-term survivors, *Biometrics*, **38**(4), 1041-1046 (1982).
[9] A. Yu Yakovlev, B. Asselain, A.Y. Yakovlev, B. Asselain, *Stochastic models of tumor latency and their biostatistical applications*, World Scientific, (1996).
[10] J. Balka, A.F. Desmond and P.D. McNicholas. Review and implementation of cure models based on first hitting times for wiener processes, *Lifetime data analysis*, **15** (2), 147-176 (2009).
[11] R. Rocha, S. Nadarajah, V. Tomazella and F. Louzada, A new class of defective models based on the marshall–olkin family of distributions for cure rate modeling, *Computational Statistics & Data Analysis*, **107**, 48-63 (2017).
[12] A.P. Basu and J.P. Klein, Some recent results in competing risks theory, *Lecture Notes-Monograph Series*, **2**, 216-229 (1982).
[13] J.F. Kenney and K.S. Keeping, *Mathematics of statistics*, Princeton, New Jersey, (1962).
[14] J.J.A. Moors, A quantile alternative for kurtosis, *Journal of the Royal Statistical Society, Series D (The Statistician)*, **37**(1), 25-32 (1988).
[15] M.S. Milgram, The generalized integro-exponential function, *Mathematics of computation*, **44**(170), 443–458 (1985).
[16] T. Pogány, G.M. Cordeiro, M.H. Tahir and H.M. Srivastava, Extension of generalized integro-exponential function and its application in study of chen distribution, *Applicable Analysis and Discrete Mathematics*, **11**(2), 434-450 (2017).
[17] H.M. Srivastava, R.K. Saxena, T.K. Pogány, and R. Saxena, Integral and computational representations of the extended hurwitz–lerch zeta function, *Integral Transforms and Special Functions*, **22**(7), 487-506 (2011).

[18] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Gezondheidszorg, (2006).
[19] M. de Castro, V.G. Cancho and J. Rodrigues, A bayesian long-term survival model parametrized in the cured fraction, *Biometrical Journal*, **51**(3), 443-455 (2009).
[20] J. Leão, V. Leiva, H. Saulo and V. Tomazella, Incorporation of frailties into a cure rate regression model and its diagnostics and application to melanoma data, *Statistics in medicine*, **37**(29), 4421-4440 (2018).
[21] G.W. Oehlert, A note on the delta method, *The American Statistician*, **46**(1), 27-29 (1992).
[22] D.J. Lunn, A. Thomas, N. Best and D. Spiegelhalter, Winbugs-a bayesian modelling framework: concepts, structure, and extensibility, *Statistics and computing*, **10**(4), 325-337 (2000).
[23] K. Xu, M. Xie, L.C. Tang and S.L. Ho, Application of neural networks in forecasting engine systems reliability, *Applied Soft Computing*, **2**(4), 255-268 (2003).
[24] R.L. Smith and J.C. Naylor, A comparison of maximum likelihood and bayesian estimators for the three-parameter weibull distribution, *Applied Statistics*, **36**(3), 358-369 (1987).
[25] J.A. Laurie, C.G. Moertel, T.R. Fleming, H.S. Wieand, J.E. Leigh, J. Rubin, G.W. McCormack, J.B. Gerstner, J.E. Krook and J. Malliard. Surgical adjuvant therapy of large-bowel carcinoma: an evaluation of levamisole and the combination of levamisole and fluorouracil, the north central cancer treatment group and the mayo clinic, *Journal of Clinical Oncology*, **7**(10), 1447-1456 (1989).