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A New Two-Parameter Estimator in the Linear Regression Model with Correlated Regressors

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Abstract: The inefficiency of the ordinary least square estimator for the parameter estimation of a linear regression model with multicollinearity problem has led to the development of various ridge regression estimators. These estimators are recently classified as one-parameter and two-parameter ridge-type estimators. This paper proposes a new two-parameter estimator following a newly developed one-parameter ridge estimator to handle multicollinearity in the linear regression model. Theoretical and simulation results show that, under some conditions, the proposed estimator performs better than some popular existing estimators in that it has a smaller mean square error. Furthermore, we used real-life data to illustrate the paper's findings establishes the same results from theory and simulation.
Keywords: Proposed Estimator, Ridge Estimator, Monte Carlo Simulation, Multicollinearity, Mean Square Error.

1 Introduction

Consider the general linear regression model define in matrix form as:

$$y = X\beta + \varepsilon, \tag{1}$$

where y is a $n\times 1$ vector of the response variable, X is a known $n\times p$ full-rank matrix of predictor variables, β is a $p\times 1$ vector of unknown regression parameters to be estimated, and ε is $n\times 1$ vector of random error such that E(ε) = 0 and Cov (ε) = $\sigma^2 I$. Equation (1) can be written in a canonical form as:

$$y = Z\alpha + \varepsilon \tag{2}$$

where $Z = XQ$, $\alpha = Q'\beta$ and Q is the orthogonal matrix whose columns constitute the eigenvectors of $X'X$. Then $Z'Z = Q'X'XQ = \Lambda = diag(\lambda_1,...,\lambda_p)$,

where $\lambda_1 \geq \lambda_2 \geq ...\lambda_p > 0$ are the ordered eigen values $X'X$. The ordinary least square estimator (OLSE) of β in (1) can be defined as:

$$\hat{\alpha}_{OLS} = \Lambda^{-1}X'y \tag{3}$$

where $\Lambda = X'X$ is the design matrix.
The OLSE is considered the Best Linear Unbiased Estimator (BLUE) when the assumptions of the classical linear regression model are not violated [1,2] One of the assumptions is that the explanatory variables are independent [3]. Literature has shown that the OLS will not be the best in the presence of multicollinearity. The problem of multicollinearity arises whenever two or more explanatory variables are related. Multicollinearity is a situation where there is an exact (or

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nearly exact) linear relation among two or more of the explanatory variables [4,5] Whenever multicollinearity exists, the OLSE suffers a set back by yielding regression coefficients whose regression coefficients produce wrong signs with large standard error, imprecise confidence intervals and incorrect small t-ratios [6,7].
Some biased estimators have been developed to deal with the problem of multicollinearity. These estimators include the Stein estimator [8] the principal component estimator [9] the ordinary ridge regression estimator by Hoerl and Kennard [10] , the modified ridge regression by Swindel [11] Dawoud and Kibria [12,13] The ordinary ridge regression estimator by Hoerl and Kennard [10], which is one of the most widely used among these estimators, helps overcome multicollinearity by adding a positive value, *k*, to the diagonal elements of the Z'Z matrix. This constant *k* is known as the biasing parameter. A major problem of the ridge regression parameter is the choice of *k* because the biasing parameter *k* plays a very significant role in controlling the regression's bias toward the mean of the dependent variable [14]. There are several works on the choice of biasing parameter k. Some of them are Hoerl and Kennard [10], Hoerl *et al.*[15] McDonald and Galarneau [16], Hocking *et al.*[17], Lawless and Wang [18], Nomura[19], Firinguetti [20], Kibria [21], Batach *et al.*[22], among many others.
Liu [23] proposed another estimator where *d* is the biasing parameter. This estimator combined the advantages of the Stein estimator [8] and the ordinary ridge regression estimator by Hoerl and Kennard[10]. Liu Estimator is a linear function of biasing parameter d, while the ridge regression estimator is a nonlinear function of the biasing parameter k. This makes the choice of a suitable k remains difficult and the Liu Estimator becomes a preferred choice allowing an appropriate selection of *d* [24]. The objective of this paper is to propose a new two-parameter ridge-type estimator for the regression. The performance of the proposed estimator is compared with OLSE, ridge regression, Liu estimator, Ozkale and Kanciranlar two-parameter estimator [25], Modified Ridge Type (MRT) by Lukman *et al*. [26] and Kibria and Lukman [27].

2 Some Alternative Biased Estimators and the Proposed Estimator

2.1 Some Ridge Estimators as Alternative to OLS

Ridge-type estimators have been proposed as alternative to the OLSE. The canonical form of OLSE is written in Equation (3). Following this, the ordinary ridge regression (RE) proposed by Hoerl and Kennard [10] is given as:

$$\hat{\alpha}_{RE}(k)=(\Lambda+kI)^{-1}Z'y \tag{4}$$

where k is the non-negative constant known as the biasing parameter.
The Liu estimator (LE) is defined as:

$$\hat{\alpha}_{L}(d)=[\Lambda+I]^{-1}[\Lambda+dI]\hat{\alpha}_{OLS} \tag{5}$$

where d is the biasing parameter of Liu Estimator.
The Kibria-Lukman (KL) estimator is given as:

$$\hat{\alpha}_{KL}(k)=(\Lambda+kI)^{-1}(\Lambda-kI)\hat{\alpha}_{OLS} \tag{6}$$

The two parameter estimator by Ozkale and Kaciranlar [25] is given as:

$$\hat{\alpha}_{TP}(k,d)=(\Lambda+kI)^{-1}(\Lambda+kdI)\hat{\alpha}_{OLS} \tag{7}$$

where k and d is the biasing parameter of Liu Estimator
The Modified Ridge Type Two (MRT) Parameters proposed by Lukman *et al.*[26] is given as:

$$\hat{\alpha}_{MRT}(k,d)=(\Lambda+k(1+d)I)^{-1}\Lambda\hat{\alpha}_{OLS}=R_K\hat{\alpha}_{OLS}$$

(8)

2.2 The Proposed Estimator

The proposed Two Parameter Estimator of α is obtained by minimizing $(\alpha+\hat{\alpha})'(\alpha+\hat{\alpha})$ subject to $(y-Z\alpha)'(y-Z\alpha)=c$, where c is a constant.

$$(y-Z\alpha)'(y-Z\alpha)+kd\left[(\alpha+\hat{\alpha})'(\alpha+\hat{\alpha})-c\right] \tag{9}$$

where *k* and *d* are the Langrangian multipliers

Following Kibria and Lukman (KL) [27] as defined in equation (6), the solution to (9) gives the solution to the proposed estimator as follows:

$$\hat{\alpha}_p(k,d) = (\Lambda + kdI)^{-1}(\Lambda - kdI)\hat{\alpha}_{OLS} \quad (10)$$

$$\hat{\alpha}_p(k,d) = Z_0 Z_1 \hat{\alpha}_{OLS} \quad (11)$$

where $Z_0 = (\Lambda + kdI)$ and $Z_1 = (\Lambda - kdI)$, k>0 and 0<d<1.

Some of the differences between the proposed estimator and Kibria-Lukman estimator are:

i. The KL estimator is a one-parameter estimator while the proposed estimator is a two parameter estimator. This also makes their Mean Squared Error different from each other.

ii. The KL estimator is a function of biasing parameter *k* while the proposed estimator is a function of biasing parameters *k* and *d*.

iii. The KL estimator is obtained based on the objective function $(y - Z\alpha)'(y - Z\alpha) + k\left[(\alpha + \hat{\alpha})'(\alpha + \hat{\alpha}) - c\right]$ while the objective function used in obtaining the proposed estimator is $(y - Z\alpha)'(y - Z\alpha) + kd\left[(\alpha + \hat{\alpha})'(\alpha + \hat{\alpha}) - c\right]$.

iv. The KL estimator is a special case of the proposed estimator when d=1. Hence, the proposed estimator is a general estimator.

Properties of the Proposed Estimator

$$E(\hat{\alpha}_p(k,d)) = Z_0 Z_1 \alpha \quad (12)$$

$$B(\hat{\alpha}_p(k,d)) = (Z_0 Z_1 - I)\alpha \quad (13)$$

$$D[\hat{\alpha}_p(k,d)] = \sigma^2 Z_0 Z_1 \Lambda^{-1} Z'_0 Z'_1 \quad (14)$$

$$MSEM[\hat{\alpha}_P(k,d)] = \sigma^2 Z_0 Z_1 \Lambda^{-1} Z'_0 Z'_1 + (Z_0 Z_1 - I)\alpha\alpha'(Z_0 Z_1 - I)' \quad (15)$$

The following lemmas are used to make some theoretical comparisons among estimators and to prove the statistical properties of the proposed estimator.

Lemma 1. Let n x n matrices M > 0 and N > 0 (or $N \geq 0$), Then, M > N if and only if $\lambda_i(NM^{-1}) < 1$ where $\lambda_i(NM^{-1})$ is the largest eigenvalue of matrix NM^{-1} [28].

Lemma 2. *Let M be an n x n positive definite matrix, that is, M > 0 and α be some vector, then, $M - \alpha\alpha' \geq 0$ if and only if $\alpha' M^{-1} - \alpha \leq 1$* [29].

Lemma 3. *Let $\hat{\alpha}_i = A_i y$, i = 1, 2, be two linear estimators of α. Suppose that $D = Cov(\hat{\alpha}_1) - Cov(\hat{\alpha}_2) > 0$, where $Cov(\hat{\alpha}_i), i = 1,2$ denotes the covariance matrix of $\hat{\alpha}_i$ and $b_i = Bias(\hat{\alpha}_i) = (A_i X - I)\alpha, i = 1,2$. Consequently,*

$$\Delta(\hat{\alpha}_1 - \hat{\alpha}_2) = MSEM(\hat{\alpha}_1) - MSEM(\hat{\alpha}_2) = \sigma^2 D + b_1 b_2' - b_2 b_2' > 0 \quad (16)$$

if and only if $b_2'\left[\sigma^2 D + b_1 b_1'\right]^{-1} b_2 < 1$ *where* $MSEM(\hat{\alpha}_i) = Cov(\hat{\alpha}_i) + b_i b_i'$ [30]

2.3 Comparison among the Estimators

In this section, theoretical comparison was carried out among the estimators to examine the performance of the proposed modified two-parameter estimator, $\hat{\alpha}_p(k,d)$ over other estimators; $\hat{\alpha}_{OLS}$, $\hat{\alpha}_{RE}$, $\hat{\alpha}_{LE}$, $\hat{\alpha}_{NTP}$, $\hat{\alpha}_{MRT}$, $\hat{\alpha}_{KL}$.

2.3.1 Comparison between $\hat{\alpha}_{OLS}$ and $\hat{\alpha}_p(k,d)$

The MSEM of the estimator $\hat{\alpha}_{OLS} = \Lambda^{-1} Z' y$ is as follows:

$$MSEM\left[\hat{\alpha}_{OLS}\right]=\sigma^2\Lambda^{-1} \quad (17)$$

The difference between (15) and (17)

$$MESM\left(\hat{\alpha}_{OLS}\right)-MSEM\left[\hat{\alpha}(k,d)\right]=\sigma^2\Lambda^{-1}-\sigma^2 Z_0 Z_1\Lambda^{-1}Z'_0\,Z'_1+\left(Z_0Z_1-I\right)\alpha\alpha'\left(Z_0Z_1-I\right)' \quad (18)$$

Let $k > 0$ and $0 < d < 1$. Thus, the following theorem holds.

Theorem 1: The proposed estimator $\hat{\alpha}_p(k,d)$ is superior to $\hat{\alpha}_{OLS}$ if and only if

$$\alpha'\left(Z_0Z_1-I\right)'\ \sigma^2\left[\left(\Lambda^{-1}-Z_0Z_1\Lambda^{-1}Z'_0\,Z'_1\right)\right]^{-1}\left(Z_0Z_1-I\right)\alpha<1 \quad (19)$$

Proof: $D\left(\hat{\alpha}_{OLS}\right)-D\left(\hat{\alpha}_p(k.d)\right)=\sigma^2\left(\Lambda^{-1}-Z_0Z_1\Lambda^{-1}Z'_0\,Z'_1\right)$

$$=\sigma^2 diag\left\{\frac{1}{\lambda_i}-\frac{\left(\lambda_i-kd\right)^2}{\lambda_i\left(\lambda_i+kd\right)^2}\right\}_{i=1}^{p} \quad (20)$$

$\Lambda^{-1}-Z_0Z_1\Lambda^{-1}Z'_0\,Z'_1$ will be pdf if and only if $\left(\lambda_i+kd\right)^2-\left(\lambda_i-kd\right)^2>0$. By lemma 3 the proof is completed.

2.3.2 Comparison between $\hat{\alpha}_{RE}(k)$ and $\hat{\alpha}_p(k,d)$

The bias vector, covariance matrix and MSEM of the estimator $\hat{\alpha}_{RE}(k)=\left(\Lambda+kI\right)^{-1}Z'y$ are as follows:

$$B\left[\hat{\alpha}_{RE}(k)\right]=-k\left(\Lambda+kI\right)^{-1}\alpha \quad (21)$$

$$D\left[\hat{\alpha}_{RE}(k)\right]=\sigma^2\left(\Lambda+kI\right)^{-1}\Lambda\left(\Lambda+kI\right)^{-1} \quad (22)$$

$$MSEM\left[\hat{\alpha}_{RE}(k)\right]=\sigma^2 B_0\Lambda B'_0+k^2B_0\alpha\alpha'B'_0 \quad (23)$$

where $B_0=\left(\Lambda+kI\right)^{-1}$.

The difference between (15) and (23)

$$MESM\left(\hat{\alpha}_{RE}(k)\right)-MSEM\left[\hat{\alpha}_p(k,d)\right]$$

$$=\sigma^2B_0\Lambda B'_0-\sigma^2Z_0Z_1\Lambda^{-1}Z'_0\,Z'_1+k^2B_0\alpha\alpha'B'_0-\left(Z_0Z_1-I\right)\alpha\alpha'\left(Z_0Z_1-I\right)' \quad (24)$$

Let $k > 0$ and $0 < d < 1$. Thus, the following theorem holds.

Theorem 2: The proposed estimator $\hat{\alpha}_p(k,d)$ is superior to $\hat{\alpha}_{RE}(k)$ if and only if

$MSEM\left[\hat{\alpha}_{RE}(k)\right]-MSEM\left[\hat{\alpha}_p(k,d)\right]>0$ if and only if

$$\alpha'\left(Z_0Z_1-I\right)'\ \left[\sigma^2\left(B_0\Lambda B'_0-Z_0Z_1\Lambda^{-1}Z'_0\,Z'_1\right)+k^2B_0\alpha\alpha'B'_0\right]^{-1}\left(Z_0Z_1-I\right)\alpha<1$$

(25)

Proof: Considering the dispersion matrix difference between $D\left[\hat{\alpha}_{RE}(k)\right]$ and $D\left[\hat{\alpha}_p(k,d)\right]$

$$D_d=\sigma^2B_0\Lambda B'_0-\sigma^2Z_0Z_1\Lambda^{-1}Z'_0\,Z'_1$$

$$=\sigma^2\left(\Lambda+kI\right)^{-1}\Lambda\left(\Lambda+kI\right)^{-1}-\sigma^2\left(\Lambda-kdI\right)\left(\Lambda+kdI\right)^{-1}\Lambda^{-1}\left(\Lambda-kdI\right)\left(\Lambda+kdI\right)^{-1}$$

$$=\sigma^2\left(\Lambda+kI\right)^{-1}\left(\Lambda+kdI\right)^{-1}\left[\Lambda^2\left(\Lambda+kdI\right)^2-\left(\Lambda+kI\right)^2\left(\Lambda-kdI^2\right)\right]\Lambda^{-1}\left(\Lambda+kI\right)^{-1}\left(\Lambda+kdI\right)^{-1} \quad (26)$$

It is observed that D_d is positive definite. By lemma 3, the proof is completed.

2.3.3 Comparison between $\hat{\alpha}_{LE}(d)$ and $\hat{\alpha}_p(k,d)$

The bias vector, covariance matrix and MSEM of the estimator $\hat{\alpha}_{Liu}=\left(\Lambda+I\right)^{-1}\left(\Lambda+dI\right)\hat{\alpha}_{OLS}$ are as follows:

$$B\left[\hat{\alpha}_{LE}(d)\right]=\left(d-1\right)\left(\Lambda+I\right)^{-1}\alpha \quad (27)$$

$$D\left[\hat{\alpha}_{LE}\right]=\sigma^2\left(\Lambda+I\right)^{-1}\left(\Lambda+dI\right)^{1}\ \Lambda^{-1}\left(\Lambda+I\right)^{-1}\left(\Lambda+dI\right)^{1} \quad (28)$$

$$MSEM\ \left[\hat{\alpha}_{LE}\right]=\sigma^2D_0D_1\Lambda^{-1}D'_0\,D'_1+\left(d-1\right)D_0\alpha\alpha'\left(d-1\right)'D'_0{}' \quad (29)$$

Where $D_0=\left(\Lambda+I\right)^{-1}$ and $D_1=\left(\Lambda+dI\right)$.

The difference between (15) and (29) becomes:

$$MESM\left(\hat{\alpha}_{LE}(d)\right)-MSEM\left[\hat{\alpha}_p(k,d)\right]$$
$$=\sigma^2 D_0 D_1 \Lambda^{-1} D'_0\, D'_1-\sigma^2 Z_0 Z_1 \Lambda^{-1} Z'_0\, Z'_1+(d-1)D_0\alpha\alpha'(d-1)' D'_0-(Z_0Z_1-I)\alpha\alpha'(Z_0Z_1-I)' \tag{30}$$

Let k > 0 and 0 <d < 1. Thus, the theorem 3 holds.

Theorem 3: $MSEM\left[\hat{\alpha}_{Liu}(d)\right]-MSEM\left[\hat{\alpha}_p(k,d)\right]>0$ if and only if

$$\alpha'(Z_0Z_1-I)'\left[\sigma^2\left(D_0D_1\Lambda^{-1}D'_0\,D'_1-Z_0Z_1\Lambda^{-1}Z'_0\,Z'_1\right)+(d-1)D_0\alpha\alpha'(d-1)'D'_0\right]^{-1}(Z_0Z_1-I)\alpha<1 \tag{31}$$

Proof: Considering the dispersion matrix difference between $D\left[\hat{\alpha}_{LE}(d)\right]$ *and* $D\left[\hat{\alpha}_p(k,d)\right]$

$$D_d=\sigma^2 D_0 D_1 \Lambda^{-1} D'_0\, D'_1-\sigma^2 Z_0 Z_1 \Lambda^{-1} Z'_0\, Z'_1$$
$$D_d=\sigma^2(\Lambda+I)^{-1}(\Lambda+dI)^1\ \Lambda^{-1}(\Lambda+I)^{-1}(\Lambda+dI)^1-\sigma^2(\Lambda-kdI)(\Lambda+kdI)^{-1}\Lambda^{-1}(\Lambda-kdI)'(\Lambda+kdI)^{-1}$$
$$D_d=\sigma^2(\Lambda+1)^{-1}(\Lambda+kd)^{-1}\left[(\Lambda+d)^2(\Lambda+kd)^2-(\Lambda-kd)^2(\Lambda+1)^2\right]\Lambda^{-1}(\Lambda+1)^{-1}(\Lambda+kd)^{-1}$$
$$D_d=\sigma^2(\Lambda+1)^{-1}(\Lambda+kd)^{-1}\begin{bmatrix}4\Lambda^3kd+8\Lambda^2kd+2\Lambda kd+2\Lambda^3d+k^2d^4+2\Lambda k^2d^3+\Lambda^2d^2\\+2\Lambda kd^3-2\Lambda k^2d^2-k^2d^2-2\Lambda^3-\Lambda^2\end{bmatrix}\Lambda^{-1}(\Lambda+1)^{-1}(\Lambda+kd)^{-1} \tag{32}$$

It is observed that D_d is positive definite. By lemma 3, the proof is completed.

2.3.4 Comparison between $\hat{\alpha}_{KL}(k)$ and $\hat{\alpha}_p(k,d)$

The bias vector, covariance matrix and MSEM of the estimator $\hat{\alpha}_{KL}(k)=(\Lambda+kI)^{-1}(\Lambda-kI)\hat{\alpha}$ are as follows:

$$B\left[\hat{\alpha}_{KL}(k)\right]=\left[W(k)M(k)-I\right]\alpha \tag{33}$$
$$D\left[\hat{\alpha}_{KL}(k)\right]=\sigma^2W(k)M(k)\Lambda^{-1}M'(k)W'(k) \tag{34}$$
$$MSEM\left[\hat{\alpha}_{KL}(k)\right]=\sigma^2W(k)M(k)\Lambda^{-1}M'(k)W'(k)+\left[W(k)M(k)-I\right]\alpha\alpha'\left[W(k)M(k)-I\right]' \tag{35}$$

The difference between (15) and (35) becomes:

$$MESM\left(\hat{\alpha}_{KL}(k)\right)-MSEM\left[\hat{\alpha}_p(k,d)\right]$$
$$=\sigma^2W(k)M(k)\Lambda^{-1}M'(k)W'(k)-\sigma^2Z_0Z_1\Lambda^{-1}Z'_0\,Z'_1+\left[W(k)M(k)-I\right]\alpha\alpha'\left[W(k)M(k)-I\right]'$$
$$-(Z_0Z_1-I)\alpha\alpha'(Z_0Z_1-I)' \tag{36}$$

Let $k>0$ and $0<d<1$. Thus, the following theorem holds.

Theorem 4: The proposed estimator $\hat{\alpha}_p(k,d)$ is superior to $\hat{\alpha}_{KL}(k)$ if and only if:

$MSEM\left[\hat{\alpha}_{KL}(k)\right]-MSEM\left[\hat{\alpha}_P(k,d)\right]>0$ if and only if

$$\alpha'(Z_0Z_1-I)'\left[\sigma^2\left(W(k)M(k)\Lambda^{-1}W'(k)M'(k)-Z_0Z_1\Lambda^{-1}Z'_0\,Z'_1\right)+W(k)M(k)\alpha\alpha'W'(k)M'(k)\right]^{-1}$$
$$(Z_0Z_1-I)\alpha<1 \tag{37}$$

Proof: Considering the dispersion matrix difference between $D\left[\hat{\alpha}_{KL}(k)\right]$ and $D\left[\hat{\alpha}_p(k,d)\right]$

$$D_d=\sigma^2(\Lambda-kI)(\Lambda+kI)^{-1}\Lambda^{-1}(\Lambda-kI)(\Lambda+kI)^{-1}-\sigma^2(\Lambda-kdI)(\Lambda+kdI)^{-1}\Lambda^{-1}(\Lambda-kdI)'(\Lambda+kdI)^{-1} \tag{38}$$
$$D_d=\sigma^2(\Lambda+k)^{-1}(\Lambda+kd)^{-1}\left[(\Lambda-k)^2(\Lambda+kd)^2-(\Lambda-kd)^2(\Lambda+k)^2\right]\Lambda^{-1}(\Lambda+k)^{-1}(\Lambda+kd)^{-1} \tag{39}$$
$$D_d=\sigma^2(\Lambda+k)^{-1}(\Lambda+kd)^{-1}\left[4\Lambda^3kd+4\Lambda k^3d+4\Lambda^2k^2-4\Lambda^3k-4\Lambda^2k^2d-4\Lambda k^3d^2\right]$$
$$\Lambda^{-1}(\Lambda+k)^{-1}(\Lambda+kd)^{-1} \tag{40}$$

It is observed that D_d is positive definite. By lemma 3, the proof is completed.

2.3.5 Comparison between $\hat{\alpha}_{MRT}(k,d)$ and $\hat{\alpha}_p(k,d)$

The bias vector, covariance matrix and MSEM of the estimator $\hat{\alpha}_{MRT}(k,d)=(\Lambda+k(1+d)I)^{-1}\Lambda\hat{\alpha}_{OLS}=R_K\hat{\alpha}_{OLS}$ are as follows:

$$B[\hat{\alpha}_{MRT}(k,d)]=[R_k-I]\alpha \tag{41}$$

$$D[\hat{\alpha}_{MRT}(k,d)]=\sigma^2 R_k\Lambda^{-1}R'_k \tag{42}$$

$$MSEM[\hat{\alpha}_{MRT}(k,d)]=\sigma^2 R_k\Lambda^{-1}R'_k+[R_k-I]\alpha\alpha'[R_k-I]' \tag{43}$$

Where $R_k=\Lambda(\Lambda+k(1+d)I)^{-1}$. Let k > 0 and 0 < d < 1. Thus, the following theorem holds.

Theorem 5: The proposed estimator $\hat{\alpha}_p(k,d)$ is superior to $\hat{\alpha}_{MRT}(k,d)$ if and only if

$MSEM[\hat{\alpha}_{MRT}(k,d)]-MSEM[\hat{\alpha}_p(k,d)]>0$ if and only if

$$\alpha'(Z_0Z_1-I)'\left[\sigma^2\left(R_k\Lambda^{-1}R'_k-Z_0Z_1\Lambda^{-1}Z'_0Z'_1\right)+R_k\alpha\alpha'R'_k\right]^{-1}(Z_0Z_1-I)\alpha<1$$
(44)

Proof: Considering the dispersion matrix difference between $D[\hat{\alpha}_{MRT}(k,d)]$ and $D[\hat{\alpha}_p(k,d)]$

$$D_d=\sigma^2 R_k\Lambda^{-1}R'_k-\sigma^2 Z_0Z_1\Lambda^{-1}Z'_0Z'_1$$

$$D_d=\sigma^2(\Lambda+k(1+d)I)^{-1}\Lambda(\Lambda+k(1+d)I)^{-1}-\sigma^2(\Lambda-kdI)(\Lambda+kdI)^{-1}\Lambda^{-1}(\Lambda-kdI)'(\Lambda+kdI)^{-1}$$

$$D_d=\sigma^2\, diag\left\{\frac{\lambda_i}{(\lambda_i+k(1+d))^2}-\frac{(\lambda_i-kd)^2}{\lambda_i(\lambda_i+kd)^2}\right\}_{i=1}^{p} \tag{45}$$

will be pdf if and only if $\lambda_i^2(\lambda_i+kd)^2-(\lambda_i-kd)^2(\lambda_i+k(1+d))^2>0$. For 0 < d < 1 and k > 0, it was observed that $\lambda_i^2(\lambda_i+kd)^2-(\lambda_i-kd)^2(\lambda_i+k(1+d))^2>0$. By lemma 3, the proof is completed.

2.3.6. Comparison between $\hat{\alpha}_{TP}(k,d)$ and $\hat{\alpha}_p(k,d)$

The bias vector, covariance matrix and MSEM of the estimator $\hat{\alpha}_{NTP}(k,d)=(\Lambda+kI)^{-1}(\Lambda+kdI)\hat{\alpha}_{OLS}$ are as follows:

$$B[\hat{\alpha}_{TP}(k,d)]=[R_0R_1-I]\alpha \tag{46}$$

$$D[\hat{\alpha}_{NTP}(k,d)]=\sigma^2 R_0R_1\Lambda^{-1}R'_0R'_1 \tag{47}$$

$$MSEM[\hat{\alpha}_{TP}(k,d)]=\sigma^2 R_0R_1\Lambda^{-1}R'_0R'_1+[R_0R_1-I]\alpha\alpha'[R_0R_1-I]' \tag{48}$$

Where $R_0=(\Lambda+kI)^{-1}$ and $R_1=(\Lambda+kdI)$. Let k > 0 and 0 < d < 1. Thus the following theorem holds.

Theorem 6: The proposed estimator $\hat{\alpha}_p(k,d)$ is superior to $\hat{\alpha}_{TP}(k,d)$ if and only if

$MSEM[\hat{\alpha}_{TP}(k,d)]-MSEM[\hat{\alpha}_p(k,d)]>0$. That is, if and only if,

$$\alpha'(Z_0Z_1-I)'\left[\sigma^2\left(R_0R_1\Lambda^{-1}R'_0R'_1-Z_0Z_1\Lambda^{-1}Z'_0Z'_1\right)+(R_0R_1-I)\alpha\alpha'(R_0R_1-I)'\right]^{-1}(Z_0Z_1-I)\alpha<1 \tag{49}$$

Proof: Considering the dispersion matrix difference between $D[\hat{\alpha}_{NTP}(k,d)]$ and $D[\hat{\alpha}_p(k,d)]$

$$D_d=\sigma^2 R_0R_1\Lambda^{-1}R'_0R'_1-\sigma^2 Z_0Z_1\Lambda^{-1}Z'_0Z'_1$$

$$D_d=\sigma^2(\Lambda+kdI)(\Lambda+kI)^{-1}\Lambda^{-1}(\Lambda+kdI)(\Lambda+kI)^{-1}-\sigma^2(\Lambda-kdI)(\Lambda+kdI)^{-1}\Lambda^{-1}(\Lambda-kdI)'(\Lambda+kdI)^{-1}$$

$$D_d = \sigma^2 \, diag\left\{ \frac{(\lambda_i + kd)^2}{\lambda(\lambda_i + k)^2} - \frac{(\lambda_i - kd)^2}{\lambda_i(\lambda_i + kd)^2} \right\}_{i=1}^{p} \tag{50}$$

will be pdf if and only if $\lambda_i^2(\lambda_i + kd)^4 - \lambda_i(\lambda_i - kd)^{2^2}(\lambda_i + k))^2 > 0$. For 0 < d < 1 and k > 0, it was observed that $\lambda_i^2(\lambda_i + kd)^4 - \lambda_i(\lambda_i - kd)^{2^2}(\lambda_i + k))^2 > 0$. By lemma 3, the proof is completed.

2.4 Determination of Biasing Parameters k and d

There is a need to find an appropriate parameter for practical purpose. Following different authors such as Dorugade [31], Saleh *et al.* [32], Lukman et al. [7], Aslam and Ahmad [24] among others, the optimal values of *k* and *d* is determined for the new estimator. In determining the optimal value of *k*, *d* is fixed. The optimal value of the *k* can be considered to be those *k* that minimize

$$MSEM[\hat{\alpha}_P(k,d)] = \sigma^2 Z_0 Z_1 \Lambda^{-1} Z'_0 Z'_1 + (Z_0 Z_1 - I)\alpha\alpha'(Z_0 Z_1 - I)'$$
$$s(k,d) = MSEM[\hat{\alpha}(k,d)] = tr[MSEM(\hat{\alpha}(k,d))]$$

$$s(k,d) = \sigma^2 \sum_i^p \frac{(\lambda - kd)^2}{\lambda(\lambda + kd)^2} + 4k^2 d^2 \sum_i^p \frac{\alpha_i^2}{(\lambda + kd)^2} \tag{51}$$

Differentiating $s(k,d)$ with respect to k gives

$$\frac{\partial s(k,d)}{\partial k} = -2\sigma^2 d \sum_i^p \frac{(\lambda - kd)^2}{\lambda(\lambda + kd)^3} - 2\sigma^2 d \sum_i^p \frac{(\lambda - kd)}{\lambda(\lambda + kd)^2} - 8k^2 d^3 \sum_i^p \frac{\alpha_i^2}{(\lambda + kd)^3} + 8kd^2 \sum_i^p \frac{\alpha_i^2}{(\lambda + kd)^2}$$

Let $\frac{\partial s(k,d)}{\partial k} = 0$;

$$k = \frac{\sigma^2}{d\left(2\alpha^2 + \sigma^2/\lambda_i\right)} \tag{52}$$

For practical purpose, σ^2 and α_i^2 are replaced with $\hat{\sigma}^2$ and $\hat{\alpha}_i^2$, respectively. Consequently, (52) becomes

$$\hat{k} = \frac{\hat{\sigma}^2}{d\left(2\hat{\alpha}_i^2 + \sigma^2/\lambda_i\right)} \tag{53}$$

Following Hoerl *et al.,* the harmonic-mean version of (53) is defined as

$$\hat{k}_{HM} = \frac{p\hat{\sigma}^2}{\sum d\left(2\hat{\alpha}_i^2 + \sigma^2/\lambda_i\right)}$$

(54)

According to Ozkale and Kaçiranlar [25], the minimum version of (54) is defined as

$$\hat{k}_{\min} = \min\left[\frac{\hat{\sigma}^2}{d\left(2\hat{\alpha}_i^2 + \sigma^2/\lambda_i\right)}\right] \tag{55}$$

Likewise, the optimal value for *d* can be derived by differentiating $s(k,d)$ with respect to *d* for a fixed *k*.

$$\frac{\partial s(k,d)}{\partial d} = -2\sigma^2 k\sum_i^p \frac{(\lambda - kd)^2}{\lambda(\lambda + kd)^3} - 2\sigma^2 k\sum_i^p \frac{(\lambda - kd)}{\lambda(\lambda + kd)^2} - 8k^3 d^2 \sum_i^p \frac{\alpha_i^2}{(\lambda + kd)^3} + 8k^2 d\sum \frac{\alpha_i^2}{(\lambda + kd)^2}$$

Let $\frac{\partial s(k,d)}{\partial d} = 0$;

$$d = \frac{\sigma^2}{k\left(2\alpha^2 + \sigma^2/\lambda_i\right)} \tag{56}$$

For practical purpose, σ^2 and α_i^2 are replaced with $\hat{\sigma}^2$ and $\hat{\alpha}_i^2$, respectively. Consequently, (56) becomes

$$\hat{d}_p = \frac{\hat{\sigma}^2}{k\left(2\hat{\alpha}_i^2 + \sigma^2/\lambda_i\right)} \tag{57}$$

The selection of the estimators of the parameters *d* and *k* in $\hat{\alpha}_{MKL}(k,d)$ is obtained iteratively as follows:

Step 1: Obtain an initial estimate of d using $\hat{d} = \min\left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\right)$

Step 2: Obtain $k_{\min}$ from (53) using $\hat{d}$ in step 1.

Step 3: Estimate $\hat{d}_p$ in (55) by using $k_{\min}$ in step 2.

Step 4: Incase $\hat{d}_p$ is not between 0 and 1 use $\hat{d}_p = \hat{d}$.

3 Results and Discussion

3.1 Simulation Technique

The simulation procedure used by McDonald and Galarneau [16], Wichern and Churchill [33], Gibbons [34], Kibria [21], Lukman and Ayinde [6], Lukman *et al.* [1, 2, 3] was used to generate the explanatory variables in this study: This is given as:

$$x_{ij} = \left(1-\rho^2\right)^{1/2} z_{ij} + \rho z_{i,p+1},\ i = 1,2,...,n,\ j = 1,2,...,p. \tag{58}$$

where z_{ij} is an independent standard normal distribution with mean zero and unit variance, ρ is the correlation between any two explanatory variables and p is the number of explanatory variables. For this study, we considered the values of ρ to be 0.8, 0.9, 0.95 and 0.99. Also, explanatory variables (p) were taken to be three (3) and seven (7) for the simulation study. The error terms, u_t, were generated following Firinguetti [20] such that $u_t \sim N(0,\sigma^2 I)$. The values of $\beta'\beta = 1$ (Newhouse and Oman [35]). The standard deviations in this simulation study were σ = 3, 5 and 10.

Table 1: Estimated MSE when n=50 and 100, p=3.

N	σ	ρ	OLS	RIDGE	LIU	K-L	MRT	TP	Prop
50	3	0.8	1.0528	0.7644	1.0470	0.5537	0.6162	1.0317	**0.4029**
		0.9	1.8795	1.2325	1.8610	0.7772	0.9289	1.8312	**0.5470**
		0.95	3.5664	2.1649	3.5124	1.2412	1.5600	3.4730	**0.6962**
		0.99	17.1271	9.5370	16.7288	5.0822	6.5795	16.8416	**1.9589**
	5	0.8	2.9244	2.2014	2.9098	1.6517	1.8033	2.8692	**1.0682**
		0.9	5.2210	3.7336	5.1809	2.6324	2.9531	5.1253	**1.4745**
		0.95	9.9066	6.8429	9.8003	4.6473	5.2952	9.7490	**2.0049**
		0.99	47.5752	31.7442	46.8546	20.9350	24.0636	47.1657	**13.4065**

	10	0.8	11.6976	9.6859	11.6598	8.0159	8.3859	11.5845	**4.1898**
		0.9	20.8839	16.9539	20.7858	13.7504	14.4754	20.7057	**6.2594**
		0.95	39.6264	31.7708	39.3798	25.4601	26.8886	39.3576	**11.0181**
		0.99	190.3009	150.7865	188.7337	119.6239	126.5438	189.6824	**86.3245**
100	3	0.8	0.5704	0.4470	0.5693	0.3450	0.3702	0.5652	**0.2301**
		0.9	1.0379	0.7391	1.0339	0.5053	0.5742	1.0233	**0.3530**
		0.95	1.9852	1.2962	1.9718	0.7980	0.9600	1.9509	**0.5631**
		0.99	9.5857	5.5742	9.4573	3.1060	3.9285	9.4638	**0.3726**
	5	0.8	1.5843	1.2481	1.5812	0.9737	1.0419	1.5674	**0.6721**
		0.9	2.8830	2.1481	2.8734	1.5710	1.7290	2.8476	**1.0048**
		0.95	5.5145	3.9414	5.4857	2.7615	3.0998	5.4465	**1.3080**
		0.99	26.6269	18.1832	26.3776	12.3070	13.9893	26.4281	**2.8950**
	10	0.8	6.3374	5.3203	6.3280	4.4602	4.6409	6.2908	**2.5367**
		0.9	11.5321	9.4841	11.5067	7.7854	8.1578	11.4544	**3.7661**
		0.95	22.0581	17.8991	21.9880	14.5129	15.2666	21.9325	**5.6125**
		0.99	106.5074	85.3099	105.9113	68.4960	72.1983	106.1641	**35.0003**

NOTE: Minimum MSE value is bolded in each row.

Table 2: Estimated MSE when n=50 and 100, p=7.

n	σ	ρ	OLS	RIDGE	LIU	K-L	MRT	TP	Prop
50	3	0.8	2.2424	1.4519	2.2295	0.8785	1.0703	2.1930	**0.3147**
		0.9	4.1854	2.5279	4.1491	1.3895	1.7866	4.0960	**0.4942**
		0.95	8.1092	4.6698	8.0153	2.4199	3.2154	7.9659	**0.7282**
		0.99	39.5775	21.8209	38.9229	10.8728	14.7011	39.1638	**4.2039**
	5	0.8	6.2289	4.4384	6.2017	3.0570	3.4578	6.1384	**1.0425**
		0.9	11.6262	8.0473	11.5570	5.3516	6.1471	11.4860	**1.5664**
		0.95	22.5255	15.3191	22.3528	9.9867	11.5669	22.3191	**2.3369**
		0.99	109.9375	73.6814	108.7922	47.3930	55.1157	109.3904	**29.2026**
	10	0.8	24.9156	20.3003	24.8520	16.3816	17.2232	24.7688	**5.4781**
		0.9	46.5048	37.5459	46.3502	30.0007	31.6241	46.2968	**9.4971**
		0.95	90.1019	72.3905	89.7209	57.5421	60.7254	89.8081	**21.0711**
		0.99	439.7499	351.9615	437.4033	278.7677	294.2755	439.0564	**209.8385**
100	3	0.8	1.2201	0.8559	1.2172	0.5689	0.6532	1.2040	**0.1643**
		0.9	2.2888	1.4785	2.2792	0.8808	1.0751	2.2531	**0.2681**
		0.95	4.4426	2.6944	4.4126	1.4861	1.8983	4.3726	**0.4809**
		0.99	21.7118	12.2811	21.4355	6.3423	8.3875	21.4765	**0.6029**
	5	0.8	3.3891	2.5067	3.3818	1.7967	1.9882	3.3516	**0.5826**
		0.9	6.3577	4.5312	6.3364	3.1115	3.5138	6.2902	**0.9727**
		0.95	12.3405	8.5848	12.2803	5.7481	6.5684	12.2275	**1.3925**
		0.99	60.3107	40.9795	59.7948	26.8679	30.9547	59.9790	**5.8783**
	10	0.8	13.5563	11.1643	13.5374	9.1104	9.5369	13.4848	**2.8074**
		0.9	25.4307	20.7223	25.3797	16.7266	17.5697	25.3191	**4.3164**

		0.95	49.3619	39.9786	49.2247	32.0813	33.7531	49.1911	**6.7165**
		0.99	241.2427	194.3242	240.1605	155.1759	163.3945	240.8127	**79.1807**

NOTE: Minimum MSE value is bolded in each row.

From Table 1 and 2, the simulation results show that the proposed estimator outperforms other estimators used in this study. The proposed estimator performs best at the two different sample sizes (n = 50 and 100), three sigma levels (σ = 3, 5 and 10) and four different levels of multicollinearity levels (ρ = 0.8. 0.9, 0.95 and 0.99). It provides smaller MSE when compared with other estimators in the study when the number of parameters is three and seven. The OLS estimator is the least performed estimator as expected. The following observations were also deduced from the result:

i. An increase in the numbers of level of correlation results in an increase in the MSE for all the estimators.
ii. The MSE increases for each estimator as the level of error variances increases.
iii. Increase in the sample size, n, leads to decrease in the MSE for all the estimators.

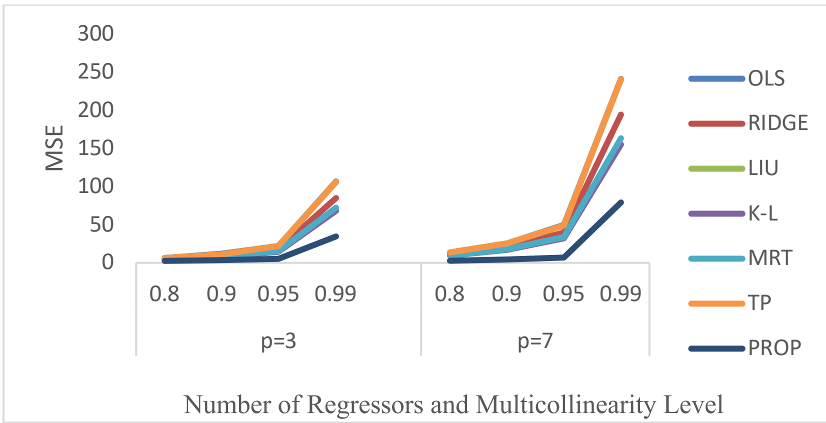

Fig. 1: The Estimators and their Estimated MSE when n=100 and σ =10.

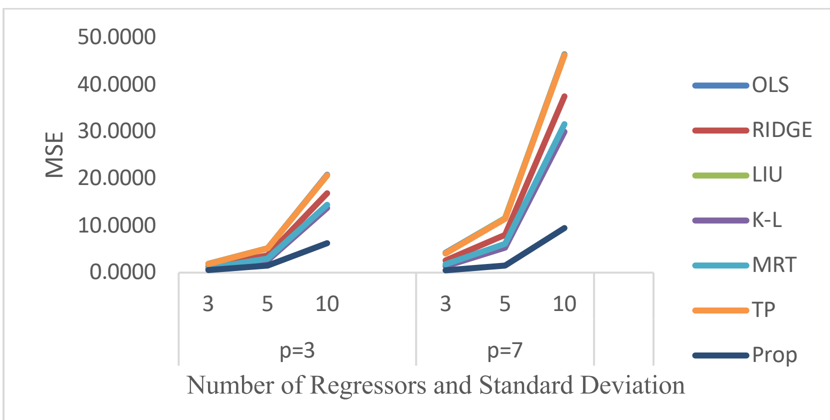

Fig.2: The Estimators and their Estimated MSE when n=50 and ρ =0.9.

From Figure 1 where n=100, σ =10 and p = 3 and 7, it appears that MSE increases as ρ increases. For the proposed estimator, it has the least MSE among all other existing estimators. Figure 2 shows the graph of n=50, ρ = 0.9 and p = 3 and 7. It reveals that MSE increases as σ increases. The proposed estimator in figure 2 also has the least MSE among all the existing estimators. Figure 3 depicts the graph of n=100, ρ = 0.8 and p = 3 and 7, which shows that MSE increases as the level of ρ increases. Also looking at Figure 3, the proposed estimator has the least MSE among all the other six estimators it is being compared with. It appears from Figure 4 where σ = 5, ρ = 0.95 and p = 3 and 7 that MSE decreases as sample size increases. For the proposed estimator, it has the least MSE among all the existing estimators. Just as in Figure 4, Figure 5 also shows that MSE decreases as sample size increases. For σ = 3, ρ = 0.99 and p = 3 and 7, the proposed estimator has the least MSE among all estimators.

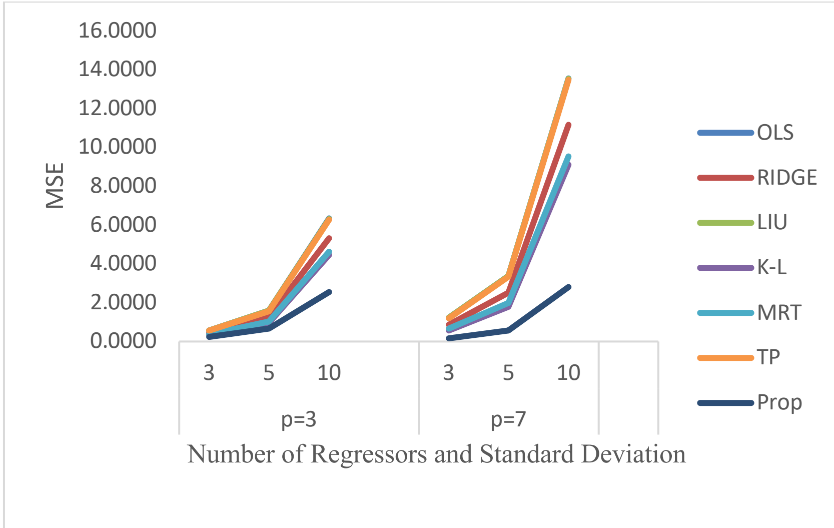

Fig. 3: The Estimators and their Estimated MSE when n=50 and ρ =0.8.

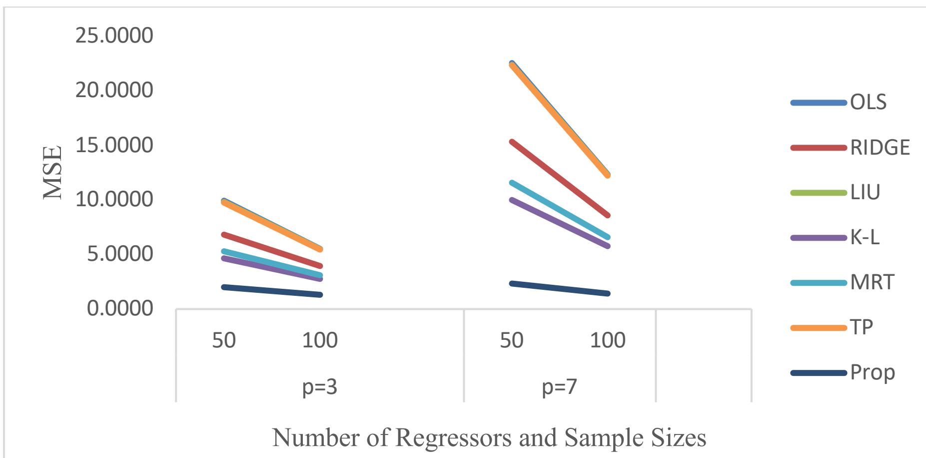

Fig. 4: The Estimators and their Estimated MSE when σ =5 and ρ =0.95.

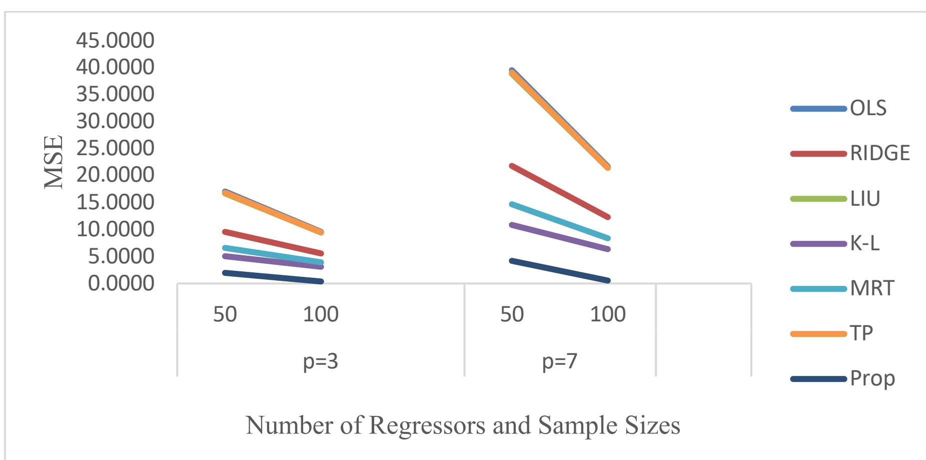

Fig. 5: The Estimators and their Estimated MSE when σ =3 and ρ =0.99

3.2 Numerical Example

In this section, Portland cement data was used to demonstrate the performance of the proposed estimator. The Portland cement data originally adopted by Woods *et al.* [36] and was later adopted by Li and Yang [37], Ayinde *et al.* [38]. The data set is widely known as the Portland cement dataset. The regression model for these data is defined as:

$$y_i = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \varepsilon_i$$

where y_i = heat evolved after 180 days of curing measured in calories per gram of cement, X_1 = tricalcium aluminate, X_2 = tricalcium silicate, X_3 = tetracalcium aluminoferrite, and X_4 = β-dicalcium silicate. The variance inflation factors are VIF1 = 38.50, VIF2 = 254.42, VIF3 = 46.87, and VIF4 = 282.51. Eigenvalues of $X'X$ matrix are λ1 = 44676.206, λ2 = 5965.422, λ3 = 809.952, and λ4 = 105.419, and the condition number of $X'X$ is approximately 424. The VIFs, the eigenvalues, and the condition number all indicate that severe multicollinearity exists. The estimated parameters and the MSE values of the estimators are presented in Table 3. The proposed estimator performs best among other estimators as it gives the smallest MSE value.

Table 3: The results of regression coefficients and the corresponding MSE values.

	OLS	RIDGE	LIU	KL	MRT	TP	PROP
$\hat{\alpha}_0$	62.40537	8.587048	27.649	-45.2313	32.37233	6.229118	27.60677
$\hat{\alpha}_1$	1.551103	2.104613	1.900972	2.658123	1.859986	2.128821	1.909055
$\hat{\alpha}_2$	0.510168	1.06485	0.870142	1.619532	0.819705	1.089162	0.868809
$\hat{\alpha}_3$	0.101909	0.668088	0.462094	1.234267	0.417863	0.692863	0.468037
$\hat{\alpha}_4$	-0.14406	0.399594	0.208183	0.943248	0.159323	0.423419	0.207454
K	-	0.008	-	0.008	0.008	0.008	0.306
d			0.44	-	0.44	0.44	0.0015
MSE	4912.09	2989.829	2170.963	14180.4	2237.804	2222.368	**2170.96**

4 Conclusions

In this paper, a new two-parameter estimator was proposed to solve the problem of multicollinearity for the linear regression models. The proposed estimator was theoretically compared with six other existing estimators. A simulation study was then conducted to compare the performance of the proposed estimator and the six existing estimators [OLS, Liu estimator [23], Ridge estimator [10], KL estimator [27], Modified Ridge Type estimator [26], Two-parameter estimator by Ozkale and Kaciranlar [25]. It is obvious from the theoretical comparison that the proposed estimator performs best among the existing estimators considered in this research work.
Simulation study also supports the theoretical study as the proposed estimator performs best among all the existing estimators. Finally, application of real-life data further established the superiority of proposed estimator as it gives the best result among the existing estimators using the Mean Square Error criterion. The proposed estimator is hereby recommended for use of researchers in different fields.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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