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New Generalization Families of Higher Order Changhee Numbers and Polynomials

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Abstract: In this paper, we present a new definition for the generalization of first and second kinds of higher-order Changhee numbers and polynomials. Furthermore, some new results are derived for these generalizations. Moreover, some interesting special cases are deduced of these Changhee numbers and polynomials.

Keywords: Changhee numbers and polynomials, Changhee numbers of higher order, Changhee polynomials of higher order

1 Introduction

The first kind usual and singles Stirling numbers, $s(m,r)$ and $s_1(m,r)$, respectively, are defined by

$$(x)_m = \sum_{r=0}^{m} s(m,r)x^r, \quad s(m,0) = \delta_{m,0}, \text{ and } s(m,r) = 0, \text{ for } r > m, \tag{1}$$

$$\langle x\rangle_m = \sum_{r=0}^{m} s_1(m,r)x^r, \quad s_1(m,0) = \delta_{m,0}, \text{ and } s_1(m,r) = 0, \text{ for } r > m, \tag{2}$$

where, $(x)_m = \prod_{i=0}^{m-1}(x-i)$, $\langle x\rangle_m = \prod_{i=0}^{m-1}(x+i)$ and $m \in \mathbb{Z} \geq 0$.

The numbers $s(m,r)$ satisfy

$$s(m+1,r) = s(m,r-1) - ms(m,r). \tag{3}$$

The generalized first and second kinds of Comtet numbers, $s_{\bar{\boldsymbol{\alpha}}}(m,\ell;\bar{\boldsymbol{r}})$ and $S_{\bar{\boldsymbol{\alpha}}}(m,\ell;\bar{\boldsymbol{r}})$, respectively, are defined by (see [1]),

$$(x;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}})_m = \sum_{\ell=0}^{m} s_{\bar{\boldsymbol{\alpha}}}(m,\ell;\bar{\boldsymbol{r}})x^\ell, \tag{4}$$

and

$$x^m = \sum_{\ell=0}^{m} S_{\bar{\boldsymbol{\alpha}}}(m,\ell;\bar{\boldsymbol{r}})(x;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}})_\ell, \tag{5}$$

where $(x;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}})_m = \prod_{\ell=0}^{m-1}(x-\alpha_\ell)^{r_\ell}$, $\bar{\boldsymbol{\alpha}} = (\alpha_0,\alpha_1,\cdots,\alpha_{m-1})$ and $\bar{\boldsymbol{r}} = (r_0,r_1,\cdots,r_{m-1})$.

By the generating function, the Changhee polynomials can be defined as follows, [2,3,4,5,6,7,8,9],

$$\left(\frac{2}{t+2}\right)(1+t)^x = \sum_{m=0}^{\infty} Ch_m(x)\frac{t^m}{m!}. \tag{6}$$

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Setting $x=0$, into (6), the Changhee numbers, Ch_m, are obtained. Also,

$$\int_{\mathbb{Z}_p}(x)_m d\mu_{-1}(x)=Ch_m. \tag{7}$$

For $k\in\mathbb{N}$ and $m\in\mathbb{Z}\geq 0$, the first kind Changhee numbers of higher order, $Ch_m^{(k)}$, are defined by Kim [2], as follows

$$Ch_m^{(k)}=\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+x_2+\cdots+x_k)_m d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k). \tag{8}$$

The generating function of $Ch_m^{(k)}$ are given by

$$\sum_{m=0}^{\infty}Ch_m^{(k)}\frac{t^m}{m!}=\left(\frac{2}{2+t}\right)^k. \tag{9}$$

The first kind of Changhee polynomials of order k, $Ch_m^{(k)}(x)$, are defined as

$$Ch_m^{(k)}(x)=\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1+x_2+\cdots+x_k+x)_m d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k). \tag{10}$$

The Euler polynomials, $E_m^{(k)}(x)$, of or order k are defined as, [2,9,10,11],

$$\left(\frac{2}{e^t+1}\right)^k e^{xt}=\sum_{m=0}^{\infty}E_m^{(k)}(x)\frac{t^m}{m!}. \tag{11}$$

Setting $x=0$ into (11), the Euler numbers, $E_m^{(k)}$, of order k can be obtained.

The following relations are proved by Kim [4],

$$Ch_m^{(k)}(x)=\sum_{i=0}^{n}s(m,i)E_i^{(k)}(x), \tag{12}$$

and

$$E_m^{(k)}(x)=\sum_{i=0}^{m}S(m,i)Ch_i^{(k)}(x). \tag{13}$$

The numbers $Ch_m^{(k)}$ satisfy the following explicit formula

$$Ch_m^{(k)}=\left(-\frac{1}{2}\right)^m\sum_{i=0}^{m}s(m,i)(k+m-1)^i. \tag{14}$$

2 Multiparameter Changhee Numbers of the First Kind

In this section, the new definitions for the first kind of multiparameter Changhee numbers with order k, $\check{C}h_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}^{(k)}$, are introduced. Some new results are derived; also some special cases are established as follows.

Definition 2.1. For $m\in\mathbb{Z}$ and $k\in\mathbb{N}$, $\check{C}h_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}^{(k)}$, are defined by

$$\check{C}h_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}^{(k)}=\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{\ell=0}^{m-1}(x_1x_2\cdots x_k-\alpha_i)^{r_\ell}d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k), \tag{15}$$

where $\bar{\boldsymbol{\alpha}}=(\alpha_0,\alpha_1,\cdots,\alpha_{m-1})$, $\bar{\boldsymbol{r}}=(r_0,r_1,\cdots,r_{m-1})$.

Theorem 2.1. For $m \in \mathbb{Z}$ and $k \in \mathbb{N}$, the numbers $\check{C}h^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$ satisfy

$$\check{C}h^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{n=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(m,n;\bar{\boldsymbol{r}}) \sum_{\ell_1=0}^{n} \cdots \sum_{\ell_k=0}^{n} \prod_{i=0}^{k} S(n,\ell_i) Ch_{\ell_i}. \tag{16}$$

Proof. From Eq. (15), we have

$$\begin{aligned}
\check{C}h^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \sum_{n=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(m,n;\bar{\boldsymbol{r}})(x_1x_2\cdots x_k)^n \, d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k) \\
&= \sum_{n=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(m,n;\bar{\boldsymbol{r}}) \int_{\mathbb{Z}_p} (x_1)^n \, d\mu_{-1}(x_1)\cdots \int_{\mathbb{Z}_p} (x_k)^n d\mu_{-1}(x_k) \\
&= \sum_{n=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(m,n;\bar{\boldsymbol{r}}) \left[\sum_{\ell_1=0}^{n} S(n,\ell_1) \int_{\mathbb{Z}_p} (x_1)_{\ell_1} d\mu_{-1}(x_1)\cdots \sum_{\ell_k=0}^{n} S(n,\ell_k) \int_{\mathbb{Z}_p} (x_k)_{\ell_k} d\mu_{-1}(x_k) \right] \\
&= \sum_{n=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(m,n;\bar{\boldsymbol{r}}) \left[\sum_{\ell_1=0}^{n} S(n,\ell_1) Ch_{\ell_1} \cdots \sum_{\ell_k=0}^{n} S(n,\ell_k) Ch_{\ell_k} \right] \\
&= \sum_{n=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(m,n;\bar{\boldsymbol{r}}) \left[\sum_{\ell_1=0}^{n}\sum_{\ell_2=0}^{n} \cdots \sum_{\ell_k=0}^{n} S(n,\ell_1)S(n,\ell_2)\cdots S(n,\ell_k) Ch_{\ell_1}\cdots Ch_{\ell_k} \right],
\end{aligned} \tag{17}$$

then Eq. (16) is obtained.

The relationship of $\check{C}h^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$ in terms of the second kind multiparameter of non-central Stirling numbers and Stirling numbers is given in following theorem, see [5,6,7,12].

Theorem 2.2. The numbers $\check{C}h^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$ satisfy the relation

$$\check{C}h^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{n=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(m,n;\bar{\boldsymbol{r}}) \sum_{\ell_1=0}^{n} \cdots \sum_{\ell_k=0}^{n} \prod_{i=0}^{k} \frac{(-1)^{\ell_i}\ell_i! S(n,\ell_i)}{\ell_i+1}, \tag{18}$$

Proof. Substituting from Eq. (15) into (16), we obtain (18).

Remark 2.1.

$$\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \prod_{i=0}^{m-1} (x_1x_2\cdots x_k - \alpha_i)^{r_i} \, d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k) = \sum_{n=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(m,n;\bar{\boldsymbol{r}}) \sum_{\ell_1=0}^{n} \cdots \sum_{\ell_k=0}^{n} \prod_{i=0}^{k} \frac{(-1)^{\ell_i}\ell_i! S(n,\ell_i)}{\ell_i+1}. \tag{19}$$

Definition 2.2. The first kind of multiparameter Changhee polynomials, $\check{C}h^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x)$ of order k are defined by

$$\check{C}h^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x) = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \prod_{i=0}^{m-1} (x_1x_2\cdots x_k x - \alpha_i)^{r_i} \, d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k). \tag{20}$$

2.1 Some special cases

In this subsection, some special cases of the first kind of multiparameter Changhee polynomials and numbers are obtained from new generalized families.

Case 1: (i) Setting $r_i = r$ and $\alpha_i = i$ in Eq. (20), we have

$$\begin{aligned}
\check{C}h^{(k)}_{m;i,\bar{\boldsymbol{r}}}(x) &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \prod_{i=0}^{m-1} (x_1x_2\cdots x_k x - i)^{r} \, d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k) \\
&= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} (x_1x_2\cdots x_k x)_{mr} \, d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k).
\end{aligned}$$

The higher-order Changhee polynomials can be obtained by replacing mr by m, which defined by Kim, see [4].

(ii) Setting $r_i = r$ and $\alpha_i = i$ in Eq. (15), we obtain

$$\check{C}h^{(k)}_{m;i,\bar{r}} = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_k)_{mr}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k).$$

Replacing mr by m, the higher-order Changhee numbers are obtained.

Case 2: (i) Setting $r_i = r$ and $\alpha_i = \alpha$ in Eq. (20), we obtain

$$\begin{aligned}\check{C}h^{(k)}_{m;\alpha,r}(x) &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_kx-\alpha)^{mr}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)\\ &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{\ell=0}^{mr}S(mr,\ell)(x_1x_2\cdots x_kx-\alpha)_\ell\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)\\ &= \sum_{\ell=0}^{mr}S(mr,\ell)\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_kx-\alpha)_\ell\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)\\ &= \sum_{\ell=0}^{mr}S(mr,\ell)\check{C}h^{(k)}_{\ell,\alpha}(x).\end{aligned} \tag{21}$$

(ii) Setting $r_i = r$ and $\alpha_i = \alpha$ in Eq. (15), we obtain

$$\begin{aligned}\check{C}h^{(k)}_{m;\alpha,r} &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_k-\alpha)^{mr}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)\\ &= \sum_{\ell=0}^{mr}S(mr,\ell)\check{C}h^{(k)}_{\ell,\alpha}.\end{aligned} \tag{22}$$

At $\alpha_i = 0$ in Eq. (21), we obtain

$$\check{C}h^{(k)}_{m;0,r}(x) = \sum_{\ell=0}^{mr}S(mr,\ell)\check{C}h^{(k)}_{\ell}(x).$$

At $\alpha_i = 0$ in Eq. (22), we obtain

$$\check{C}h^{(k)}_{m;0,r} = \sum_{\ell=0}^{mr}S(mr,\ell)\check{C}h^{(k)}_{\ell}.$$

Case 3: (i) Setting $r_i = 1$ and $\alpha_i = \alpha$ in Eq. (20), we obtain

$$\check{C}h^{(k)}_{m;\alpha,1}(x) = \sum_{\ell=0}^{m}S(m,\ell)\check{C}h^{(k)}_{\ell,\alpha}(x).$$

(ii) Setting $r_i = 1$ and $\alpha_i = \alpha$ in Eq. (15), we obtain

$$\check{C}h^{(k)}_{m;\alpha,1} = \sum_{\ell=0}^{m}S(m,\ell)\check{C}h^{(k)}_{\ell,\alpha}.$$

(iii) Setting $r_i = 1$ and $\alpha_i = 1$ in Eq. (20), we obtain

$$\begin{aligned}\check{C}h^{(k)}_{m;1,1}(x) &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_kx-1)^{m}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)\\ &= \sum_{\ell=0}^{m}S(m,\ell)\check{C}h^{(k)}_{\ell,1}(x).\end{aligned}$$

(iv) Setting $r_i = 1$ and $\alpha_i = 1$ in Eq. (15), we obtain

$$\begin{aligned}\check{C}h^{(k)}_{m;1,1} &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_k-1)^{m}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)\\ &= \sum_{\ell=0}^{m}S(m,\ell)\check{C}h^{(k)}_{\ell,1}.\end{aligned}$$

Case 4: (i) Setting $r_i = 1$ and $\alpha_i = 0$ in Eq. (20), we obtain

$$\check{C}h^{(k)}_{m;0,1}(x) = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_kx)^m\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \sum_{\ell=0}^{m} S(m,\ell)Ch^{(k)}_{\ell}(x).$$

(ii) Setting $r_i = 1$ and $\alpha_i = 0$ in Eq. (15), we obtain

$$\check{C}h^{(k)}_{m;0,1} = \sum_{\ell=0}^{m} S(m,\ell)Ch^{(k)}_{\ell}.$$

Case 5: (i) Setting $r_i = 1$ and $\alpha_i = i$, in Eq. (20), we get

$$\begin{aligned}\check{C}h^{(k)}_{m;i,1}(x) &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{m-1}(x_1x_2\cdots x_kx - i)\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)\\ &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_kx)_m\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)\\ &= Ch^{(k)}_m(x),\end{aligned} \tag{23}$$

$Ch^{(k)}_m(x)$, which defined by Kim see [4], is obtained.

(ii) Setting $r_i = 1$ and $\alpha_i = i$, in Eq. (15),

$$\begin{aligned}\check{C}h^{(k)}_{m;i,1} &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{m-1}(x_1x_2\cdots x_k - i)\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)\\ &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_k)_m\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)\\ &= Ch^{(k)}_m,\end{aligned} \tag{24}$$

the numbers $Ch^{(k)}_m$, is obtained, see [4].

Case 6: Setting $x_1x_2\cdots x_k = x$ in Eq. (15), we have

$$\check{C}h_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_{\mathbb{Z}_p}(x-\alpha_0)^{r_0}(x-\alpha_1)^{r_1}\cdots(x-\alpha_{n-1})^{r_{m-1}}\,d\mu_{-1}(x). \tag{25}$$

Corollary 2.1. For $m \in \mathbb{Z}$, $\check{C}h_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$ satisfy the following relation

$$\check{C}h_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{i=0}^{|r|} S(m,i;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}})Ch_i. \tag{26}$$

Proof. Eq. (26) can be obtained, when setting $x_1x_2\cdots x_k = x$ in Eq. (16).

Corollary 2.2. For $m \in \mathbb{Z}$,

$$\check{C}h_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{\ell=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(m,\ell;\bar{\boldsymbol{r}})E_\ell. \tag{27}$$

Proof. From Eq. (12) and Eq. (26), we obtain Eq. (27).

Case 7: Setting $r_i = 1$ in Eq. (25), we obtain

$$\check{C}h_{m;\bar{\boldsymbol{\alpha}}} = \int_{\mathbb{Z}_p}(x-\alpha_0)(x-\alpha_1)\cdots(x-\alpha_{m-1})\,d\mu_{-1}(x), \tag{28}$$

which define $\check{C}h_{m;\bar{\boldsymbol{\alpha}}}$ by generalized Changhee numbers of the first kind.

Corollary 2.3. The numbers $\check{C}h_{m;\bar{\boldsymbol{\alpha}}}$ satisfy the following relation

$$\check{C}h_{m;\bar{\boldsymbol{\alpha}}} = \sum_{i=0}^{m} S(m,i;\bar{\boldsymbol{\alpha}})Ch_i. \tag{29}$$

Proof. Let $r_i = 1$ in Eq. (26), we obtain Eq. (29).

Case 8: Setting $\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} = \int_0^{\ell_1}\cdots\int_0^{\ell_k}$ in (15), the first kind of the multiparameter Poly-Cauchy numbers is obtained.

$$C^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_0^{\ell_1}\cdots\int_0^{\ell_k}\prod_{i=0}^{m-1}(x_1x_2\cdots x_k-\alpha_i)^{r_i}\,dx_1\cdots dx_k. \tag{30}$$

3 Multiparameter Chaghee Numbers of the Second Kind

In this section, new definition of the second kind of multiparameter Chaghee numbers, $\widehat{\check{C}h}^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$, are introduced. Furthermore, some special cases are established.

Definition 3.1. For $m\in\mathbb{Z}$ and $k\in\mathbb{N}$, $\widehat{\check{C}h}^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$, are defined as

$$\widehat{\check{C}h}^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{m-1}(-x_1x_2\cdots x_k-\alpha_i)^{r_i}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k). \tag{31}$$

Theorem 3.1. For $m\in\mathbb{Z}$ and $k\in\mathbb{N}$,

$$\widehat{\check{C}h}^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{n=0}^{|r|} s_{\bar{\boldsymbol{\alpha}}}(m,n;\bar{\boldsymbol{r}})\sum_{\ell=0}^{n}L(n,m)\sum_{\ell_1=0}^{n}\cdots\sum_{\ell_k=0}^{n}\prod_{i=0}^{k}S(n,\ell_i)Ch_{\ell_i}, \tag{32}$$

where $L(n,m)$ is the Lah numbers, see [13].
Proof. From Eq. (31) we have

$$\begin{aligned}\widehat{\check{C}h}^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{|r|}s_{\bar{\boldsymbol{\alpha}}}(m,n;\bar{\boldsymbol{r}})(-x_1x_2\cdots x_k)_n\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)\\ &= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{n=0}^{|r|}s_{\bar{\boldsymbol{\alpha}}}(m,n;\bar{\boldsymbol{r}})\sum_{\ell=0}^{n}L(n,m)(x_1x_2\cdots x_k)_\ell\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k).\end{aligned} \tag{33}$$

Substituting from Eq. (16) into (33), we obtain (32).

Definition 3.2. The multiparameter higher order Changhee polynomials, $\widehat{\check{C}h}^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$, are defined by

$$\widehat{\check{C}h}^{(k)}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}(x) = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{m-1}(-x_1x_2\cdots x_kx-\alpha_i)^{r_i}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k). \tag{34}$$

3.1 Some special cases

Some special cases of the second kind of multiparameter Changhee polynomials and numbers can be obtained from new generalized families.

Case 1: (i) Setting $r_i = r$ and $\alpha_i = i$ in Eq. (34), we have

$$\widehat{\check{C}h}^{(k)}_{m;i,\bar{r}}(x) = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{m-1}(-x_1x_2\cdots x_kx-i)^r\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_kx)_{mr}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k).$$

The Changhee polynomials of order k which defined by Kim [4] can be obtained by replacing mr by m in above relation.

(ii) Setting $r_i = r$ and $\alpha_i = i$ in Eq. (31), we obtain

$$\widehat{\check{C}h}^{(k)}_{m;i,\bar{r}} = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_k)_{mr}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k).$$

Replacing mr by m, the higher-order Changhee numbers are obtained.

Case 2: (i) Setting $r_i = r$ and $\alpha_i = \alpha$ in Eq. (34), we obtain

$$\widehat{\check{C}h}^{(k)}_{m;\alpha,r}(x) = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_kx-\alpha)^{mr}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\sum_{\ell=0}^{mr}S(mr,\ell)(-x_1x_2\cdots x_kx-\alpha)_\ell\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \sum_{\ell=0}^{mr}S(mr,\ell)\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_kx-\alpha)_\ell\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \sum_{\ell=0}^{mr}S(mr,\ell)\widehat{\check{C}h}^{(k)}_{\ell,\alpha}(x). \tag{35}$$

(ii) Setting $r_i = r$ and $\alpha_i = \alpha$ in Eq. (31), we obtain

$$\widehat{\check{C}h}^{(k)}_{m;\alpha,r} = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_k-\alpha)^{mr}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \sum_{\ell=0}^{mr}S(mr,\ell)\widehat{\check{C}h}^{(k)}_{\ell,\alpha}. \tag{36}$$

At $\alpha_i = 0, r_i = r$ in Eq. (35), we obtain

$$\widehat{\check{C}h}^{(k)}_{m;0,r}(x) = \sum_{\ell=0}^{mr}S(mr,\ell)\widehat{\check{C}h}^{(k)}_{\ell}(x).$$

At $\alpha_i = 0, r_i = r$ in Eq. (36), we obtain

$$\widehat{\check{C}h}^{(k)}_{m;0,r} = \sum_{\ell=0}^{mr}S(mr,\ell)\widehat{\check{C}h}^{(k)}_{\ell}.$$

Case 3: (i) Setting $r_i = 1$ and $\alpha_i = \alpha$ in Eq. (35), we obtain

$$\widehat{\check{C}h}^{(k)}_{m;\alpha,1}(x) = \sum_{\ell=0}^{m}S(m,\ell)\widehat{\check{C}h}^{(k)}_{\ell,\alpha}(x).$$

(ii) Setting $r_i = 1$ and $\alpha_i = \alpha$ in Eq. (36), we obtain

$$\widehat{\check{C}h}^{(k)}_{m;\alpha,1} = \sum_{\ell=0}^{m}S(m,\ell)\widehat{\check{C}h}^{(k)}_{\ell,\alpha}.$$

(iii) Setting $r_i = 1$ and $\alpha_i = 1$ in Eq. (35), we obtain

$$\widehat{\check{C}h}^{(k)}_{m;1,1}(x) = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_kx-1)^m\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \sum_{\ell=0}^{m}S(m,\ell)\widehat{\check{C}h}^{(k)}_{\ell,1}(x).$$

(iv) Setting $r_i = 1$ and $\alpha_i = 1$ in Eq. (36), we obtain

$$\widehat{\check{C}h}^{(k)}_{m;1,1} = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_k-1)^m d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \sum_{\ell=0}^{m} S(m,\ell)\widehat{\check{C}h}^{(k)}_{\ell,1}.$$

Case 4: (i) Setting $r_i = 1$ and $\alpha_i = 0$ in Eq. (35), we obtain

$$\widehat{\check{C}h}^{(k)}_{m;0,1}(x) = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(x_1x_2\cdots x_kx)^m d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \sum_{\ell=0}^{m} S(m,\ell)\widehat{\check{C}h}^{(k)}_{\ell}(x).$$

(ii) Setting $r_i = 1$ and $\alpha_i = 0$ in Eq. (36), we get

$$\widehat{\check{C}h}^{(k)}_{m;0,1} = \sum_{\ell=0}^{m} S(m,\ell)\widehat{\check{C}h}^{(k)}_{\ell}.$$

Case 5: (i) Setting $r_i = 1$ and $\alpha_i = i$ in Eq. (35), we have

$$\widehat{\check{C}h}^{(k)}_{m;i,1}(x) = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{\ell=0}^{m-1}(-x_1x_2\cdots x_kx-i)\, d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_kx)_m\, d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \widehat{\check{C}h}^{(k)}_{m}(x), \tag{37}$$

the Changhee polynomials which defined by Kim [4] are obtained.

(ii) Setting $r_i = 1$ and $\alpha_i = i$ in Eq. (31), we have

$$\widehat{\check{C}h}^{(k)}_{m;i,1} = \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\prod_{i=0}^{m-1}(-x_1x_2\cdots x_k-i)\, d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(-x_1x_2\cdots x_k)_m\, d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_k)$$
$$= \widehat{\check{C}h}^{(k)}_{m},$$

the second kind of Changhee numbers with order k are obtained, see [4].

Case 6: Setting $-x_1x_2\cdots x_k = -x$, in Eq. (31),

$$\widehat{\check{C}h}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \int_{\mathbb{Z}_p}(-x-\alpha_0)^{r_0}(-x-\alpha_1)^{r_1}\cdots(-x-\alpha_{m-1})^{r_{m-1}}\, d\mu_{-1}(x). \tag{38}$$

Corollary 3.1. For $m \in \mathbb{Z}$, $\widehat{\check{C}h}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}}$ satisfy that

$$\widehat{\check{C}h}_{m;\bar{\boldsymbol{\alpha}},\bar{\boldsymbol{r}}} = \sum_{\ell=0}^{|r|}(-1)^{\ell} S_{\bar{\boldsymbol{\alpha}}}(m,\ell;\bar{\boldsymbol{r}})Ch_{\ell}. \tag{39}$$

Case 7: Setting $r_i = 1$ in Eq. (38), we obtain

$$\widehat{\check{C}h}_{m;\bar{\boldsymbol{\alpha}}} = \int_{\mathbb{Z}_p}(-x-\alpha_0)(-x-\alpha_1)\cdots(-x-\alpha_{m-1})\, d\mu_{-1}(x).$$

Corollary 3.2. For $m \in \mathbb{Z}$, $\widehat{\check{C}h}_{m;\bar{\boldsymbol{\alpha}}}$, then

$$\widehat{\check{C}h}_{m;\bar{\boldsymbol{\alpha}}} = \sum_{i=0}^{m}(-1)^{i} S_{\bar{\boldsymbol{\alpha}}}(m,i)Ch_{i}. \tag{40}$$

Case 8: Setting $\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} = \int_0^{\ell_1}\int_0^{\ell_2}\cdots\int_0^{\ell_k}$ in (31), the second kind of the multiparameter Poly-Cauchy numbers are obtained as

$$\widehat{C}^{(k)}_{m;\bar{\alpha},\bar{r}} = \int_0^{\ell_1}\int_0^{\ell_2}\cdots\int_0^{\ell_k}\prod_{i=0}^{m-1}(-x_1x_2\cdots x_k-\alpha_i)^{r_i}\,dx_1dx_2\cdots dx_k, \tag{41}$$

4 Conclusion

In this article, new definition for the first and second kinds of generalization multiparameters Changhee numbers and polynomials are investigated. Some new results are derived for these families of polynomials and numbers. A connection between these families and other numbers and polynomials are given. Furthermore, Some important cases were addressed in this study.

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Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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