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Bayesian Estimation of A one Parameter Akshaya Distribution with Progressively Type II Censord Data

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Abstract: A progressively type-II right censored sample has been examined in this paper for the inference about parameters for the one-parameter Akshaya distribution. As point estimates for the parameter, the maximum likelihood estimate (MLE), and Bayesian estimate are obtained. The asymptotic distribution of MLE is obtained. Also, the approximate confidence intervals (ACIs) and bootstraps confidence intervals for unknown parameter are obtained. Further, for symmetric loss functions such as squared error loss function, Bayesian estimates are obtained. Gibbs within Metropolis–Hasting samplers use the Monte Carlo chain (MCMC) technique to get the estimate of the unknown parameter from Bayes algorithm is used and the relevant credible interval (CRI) is obtained. Finally, the proposed methods are applied a real data set.

Keywords: Akshaya distribution, Progressively type-II censoring, Maximum likelihood estimator, Bayesian method, Markov chain Monte Carlo

1 Introduction

Statisticians have spent much time studying the failure of components and units being the most structure of operating systems in the industrial and mechanical engineering field. Their study concerns with observing the operating units till failure, registering the lifetime of those units, applying the statistical inference tools to the collected data, then estimating the reliability and the hazard functions for the entire system through the collected data. But, some experimental units are expensive and have high reliability, this example requires reducing the number of experimental units and the time of the lifetime experiment of these units. The progressive type-II censoring scheme satisfies obtaining good estimators with the lifetime experiment and keeping some experimental units from failure. The progressive type-II censoring scheme is frequently defined as follows, first, the experimenter places n independent and identical units on the measure of life. When the first failure happens, say at time $x_{(1)}$, r_1 units are randomly removed from remaining $n-1$ surviving units. When the second failure occurs at time $x_{(2)}$, r_2 units are randomly removed from remaining $n-r_1-2$ surviving units. This experiment terminates when the m th failure occurs at time x_m, and $r_m = n-m-\sum_{i=1}^{m-1} r_i$ surviving units are removed from the test. We call $R = (R_1, R_2, \dots, R_m)$, the progressive Type-II censoring scheme. Progressive Type-II right censoring, the censoring scheme R is fixed before the experiment. It are often seen that Type-II censoring may be a particular case of progressive Type-II censoring, where the scheme is $R = (0, 0, \dots, n-m)$, see [1]

Let $X_{1:m:n}, X_{2:m:n}, \dots, X_{m:m:n}; 1 \le m \le n$ be a progressively type-II censored sample observed from a lifetime test involving n units and $r_1, r_2, \dots, r_m$ being the censoring scheme. The joint PDF of a progressively type-II censored sample is given by

$$f(x_{1:m:n}, x_{2:m:n}, \dots, x_{m:m:n}) = C \prod_{i=1}^{m} f\left(x_{i:m:n}\right)\left[1-F\left(x_{i:m:n}\right)\right]^{r_i}, \tag{1}$$

where C may be a constant defined as

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$C = n\,(n-r_1-1)\cdots(n-\sum_{i=1}^{m-1}(r_i+1))$, (see [1] for details).

The focus has been on advancing progressive type II censoring for the past two to three decades. One may refer to, among others, [1,2,3], for some useful results on this censoring scheme. In reliability analysis, [4] discussed extended cosine generalized family of distributions for reliability modeling. [5] introduced reliability modelling of the COVID-19 mortality rate with a new versatile modification of the log-logistic distribution. [6] obtained fuzzy reliability model for inverse Rayleigh distribution.

In almost every field of applied science, including living science, engineering, finance and insurance, the statistical analysis and modelling of data for lifetime is important. In the statistics for modelling lifetime data, classical lifetime distributions, respectively exponential and Lindley [7,8]. But from a theoretical and applied point of view these two classic lifetime distributions do not fit. [9] performed a crucial comparative analysis of lifetime modelling with exponential and lindley distributions; it was found that there are many lifetime data for their shapes, hazard rate functions and mean lifetime characteristics, among others, do not make these typical lifetime distributions relevant. Recently, a number of one parameter lifetime distributions are introduced by [10,11,12,13] namely Akash, Shanker, Amarendra, Aradhana, Sujatha, Devya and Shambhu. While these distributions for a lifetime fit more well than the classical lindley and exponential distributions, some lifetime data still do not match those distributions. [13] proposes a new lifetime distribution, better for the modelling of lifetime data than Akash, Shanker, Amarendra, Aradhana, Sujatha, Devya and Shambhu. Also, the Akshaya distribution is a new one parametric life-time distribution which has a better flexibility in handling lifetime data as compared to exponential distribution. The random variable X has a one-parameter Akshaya distribution if its probability density function (PDF) is given by

$$f(x) = \left[\frac{\theta^4(1+x)^3}{\theta^3+3\theta^2+6\theta+6}\right]e^{-\theta x}, x>0, \theta>0, \tag{2}$$

and the cumulative distribution function

$$F(x) = 1-\left[1+\frac{\theta^3x^3+3\theta^2(\theta+1)x^2+3\theta(\theta^2+2\theta+2)x}{\theta^3+3\theta^2+6\theta+6}\right]e^{-\theta x}, \tag{3}$$

the survival rate function is

$$\overline{F}(x) = \left[1+\frac{\theta^3x^3+3\theta^2(\theta+1)x^2+3\theta(\theta^2+2\theta+2)x}{\theta^3+3\theta^2+6\theta+6}\right]e^{-\theta x},\ x\geq 0 \tag{4}$$

and the hazard rate function is

$$h(x) = \frac{\theta^4(1+x)^3}{\theta^3x^3+3\theta^2(\theta+1)x^2+3\theta(\theta^2+2\theta+2)x+(\theta^3+3\theta^2+6\theta+6)},\ x\geq 0, \tag{5}$$

where θ is the shape parameter.

[13] discussed statistical properties for the Akshaya distribution. He also examined the maximum probability estimators for the uncertain parameters and assisted the complete data in their asymptotic confidence intervals. He studied the structure, time, failure rate function and mean residual function, stochastic order, mean deviations, and curves of Bonferroni and Lorenz. Besides another one parameter lifetime distribution the conditions under which the distribution Akshaya is excessively dispersed, equally dispersed and undispersed. [16] introduced generalized power Akshaya distribution and its applications.

[17] proposed the maximum product of spacing (MPS) method as an alternative to the MLE method for estimating the parameters of continuous univariate distributions. They stated that the MPS approach possesses much of the maximum likelihood properties by replacing the likelihood function with a product of spacings. This method is devoloped to estimate parameter under censored sample by different authors. For complete sample see [18,19,20,14,15]. For Type-I and Type-II censored sample see [21,22]. For Progressive Type-II see [23,24]. For adaptive progressive Type-II see [25,26,27].

The following paper is structured as follows, in Section 2, maximum likelihood and product of spacing. In Section 3, asymptotic intervals of confidence estimates of θ are estimated, based on maximum likelihood estimates of θ and the confidence interval of unknown parameter will be introduced by two parametric bootstrap procedures. In the 4 section, Bayes' estimate θ for Squared error loss function is obtained. In section 5, the real data set was analysed. Finally, in Section 6 simulation analysis is carried out to evaluate the standard of the different estimators developed in this paper.

2 Classical Estimation

We discussed the MLE and MPS for parameter estimator of the Akshaya distribution based on progressive type-II censored sample. Let $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$, $1 \le m \le n$ be a progressively type-II censored sample observed from a life test involving n units taken from a population with PDF $f(x)$ and CDF $F(x)$ given in Equations (2) and (3), with the censoring scheme $(r_1, r_2, \ldots r_m)$.

2.1 Maximum-likelihood estimation

From Equation (1) the likelihood function of is then given by

$$L(\theta \mid \underline{x}) \propto \left(\frac{\theta^4}{\theta^3+3\theta^2+6\theta+6}\right)^m \prod_{i=1}^{m}((1+x_{i:m:n})^3 e^{-\theta x_{i:m:n}}) \left(\left[1+\frac{\theta^3 x_{i:m:n}^3+3\theta^2(\theta+1)x_{i:m:n}^2+3\theta(\theta^2+2\theta+2)x_{i:m:n}}{\theta^3+3\theta^2+6\theta+6}\right] e^{-\theta x_{i:m:n}}\right)^{r_i}, \tag{6}$$

where $\underline{x} = x_{1:m:n}, x_{2:m:n}, \ldots, x_{m:m:n}$.
The corresponding log-likelihood function for the parameters θ is

$$\ell = \log L(\theta \mid \underline{x}) = m(4\log\theta - \log(\theta^3+3\theta^2+6)) - \sum_{i=1}^{m} x_{i:m:n} + \sum_{i=1}^{m} 3\log(1+x_{i:m:n}) + \sum_{i=1}^{m} r_i \log\left(\left[1+\frac{\theta^3 x_{i:m:n}^3+3\theta^2(\theta+1)x_{i:m:n}^2+3\theta(\theta^2+2\theta+2)x_{i:m:n}}{\theta^3+3\theta^2+6\theta+6}\right] e^{-\theta x_{i:m:n}}\right). \tag{7}$$

Calculating the first partial derivatives of ℓ with relation to θ and equating it to zero, we get the likelihood equations as

$$\frac{4m}{\theta} - \frac{(3\theta^2+6\theta)m}{\theta^3+3\theta^2+6} + \sum_{i=1}^{m} r_i \left(-x_{i:m:n}\log\left[1+\frac{\theta^3 x_{i:m:n}^3+3\theta^2(\theta+1)x_{i:m:n}^2+3\theta(\theta^2+2\theta+2)x_{i:m:n}}{\theta^3+3\theta^2+6\theta+6}\right]\right) + e^{-\theta x_{i:m:n}} \frac{1}{\log\left(\left[1+\frac{\theta^3 x_{i:m:n}^3+3\theta^2(\theta+1)x_{i:m:n}^2+3\theta(\theta^2+2\theta+2)x_{i:m:n}}{\theta^3+3\theta^2+6\theta+6}\right] e^{-\theta x_{i:m:n}}\right)} \frac{(3\theta^2 x_{i:m:n}^3+(9\theta^2+6\theta)x_{i:m:n}^2+(9\theta^2+12\theta+6)x_{i:m:n})(\theta^3+3\theta^2+6\theta+6)}{(\theta^3+3\theta^2+6\theta+6)^2} \frac{-(3\theta^2+6\theta+6)(\theta^3 x_{i:m:n}^3+3\theta^2(\theta+1)x_{i:m:n}^2+3\theta(\theta^2+2\theta+2)x_{i:m:n})}{(\theta^3+3\theta^2+6\theta+6)^2} = 0, \tag{8}$$

Since, Equation (8) does not has closed-form solution, the Newton–Raphson iteration method is employed to get the estimates. The algorithm is described in [28].

It is standard that under some regularity conditions, see [29], $\hat{\theta}$ is approximately distributed as multivariate normal with mean θ and covariance matrix $I^{-1}(\theta)$. Then, the $100(1-\gamma)\%$ two sided confidence interval of θ, can be given by

$$\widehat{\theta}_i \pm Z_{\frac{\gamma}{2}} \sqrt{Var\left(\hat{\theta}_i\right)}, i = 1,2,3, \tag{9}$$

where $Z_{\frac{\gamma}{2}}$ is that the percentile of the standard normal distribution with right-tail probability $\frac{\gamma}{2}$.

2.2 Maximum product of spacing method

According to [23], the MPS under progressive type-II censored sample as:

$$D_{i:m:n}(\theta) = \prod_{i=1}^{m+1} (F(x_{i:m:n},\theta) - F(x_{i-1:m:n},\theta)) \prod_{i=1}^{m} (1-F(x_{i:m:n},\theta))^{R_i}. \tag{10}$$

The MPS estimators of the Akshaya distribution based on progressive type-II censored sample can be obtained by maximizing

$$G(\theta)=\prod_{i=1}^{m+1}\left[1+\frac{\theta^3x_{i-1:m:n}^3+3\theta^2(\theta+1)x_{i-1:m:n}^2+3\theta(\theta^2+2\theta+2)x_{i-1:m:n}}{\theta^3+3\theta^2+6\theta+6}\right]e^{-\theta x_{i-1:m:n}}$$
$$-\left[1+\frac{\theta^3x_{i:m:n}^3+3\theta^2(\theta+1)x_{i:m:n}^2+3\theta(\theta^2+2\theta+2)x_{i:m:n}}{\theta^3+3\theta^2+6\theta+6}\right]e^{-\theta x_{i:m:n}} \tag{11}$$
$$\prod_{i=1}^{m}\left[1+\frac{\theta^3x_{i:m:n}^3+3\theta^2(\theta+1)x_{i:m:n}^2+3\theta(\theta^2+2\theta+2)x_{i:m:n}}{\theta^3+3\theta^2+6\theta+6}\right]^{R_i}e^{-\theta R_i x_{i:m:n}}.$$

Further, the log-MPS of the Akshaya parameter can also be obtained by solving the first partial derivatives of log-MPS with relation to θ and equating to zero, we get the MPS estimate by using the Newton–Raphson iteration method.

2.3 Bootstrap confidence intervals

A bootstrap is an empirical approach for understanding the distributional properties of a test statistic. Also, it uses as a method of estimating statistics and their standard errors. There are three kinds of resampling plans, parametric, semi-parametric and non-parametric. Bootstrap methods depend on these three resampling plans. The bootstrap is called parametric when the probability distribution $f(x;\theta)$, from which the bootstrap data are going to be generated. Also, the parametric θ is specified, for an example, the MLE of the parameter from a real sample $\underline{X}=X_1,X_2,...,X_n$ are often computed. During this case, the parameter θ within the distribution $f(x;\theta)$ are going to be replaced with its MLE $\hat{\theta}$ and therefore the **B** bootstrap samples $\underline{X}^*=X_1^*,X_2^*,...,X_n^*$ are going to be generated from the estimated distribution $f(x;\hat{\theta})$. Applications of bootstrap methods are identified for problems in real life engineering in many fields, including radar and signal processing, geophysics, biomedical and imaging engineering, pattern, machine vision identity and image processing. Bootstrap methods can estimate the distribution of an estimator or some of its characteristics in almost all of these fields. The first is that the bootstrap P interval of trust based on the [30] idea. The second is the confidence interval bootstrap-t (Boot-t), as suggested by [31]. Boot-t established supports a ”pivot” and requires an MLE and MPS variance estimator of θ. To get the bootstrap samples for two methods, follow the algorithm:

1. From the original data $\underline{X}=X_{1:m:n},X_{2:m:n},...,X_{n:m:n}$ compute the ML estimates of the parameter θ, say $\hat{\theta}$.
2. Draw a sample of size n values, with replacement from $F_{\hat{\theta}}$. We might obtain $\underline{X}^*=X_{1:m:n}^*,X_{2:m:n}^*,...,X_{n:m:n}^*$.
3. Compute the bootstrap sample estimates of θ say $\hat{\theta}^*$.
4. Repeat Steps 2 and 3 to obtain **B** times, and obtain $\theta=\theta_1^*,\theta_2^*,...,\theta_B^*$.
5. To search out an approximate distribution of $\hat{\theta}$, sort the bootstrap estimates to get $\hat{\theta}_{(1)}^*\leq\hat{\theta}_{(2)}^*\leq...\leq\hat{\theta}_{(B)}^*$.

2.3.1 Bootstrap-p confidence interval

Let $\Phi(z)=P(\hat{\Omega}^*\leq z)$ be cumulative distribution function of $\hat{\Omega}^*$. Define $\hat{\Omega}_{Boot}^*=\Phi^{-1}(z)$ for given z. The approximation bootstrap-p $100(1-\zeta)\%$ confidence interval of $\hat{\Omega}_k^*$ is given by

$$\left(\hat{\Omega}_{Boot}^*(\zeta/2),\hat{\Omega}_{Boot}^*(1-\zeta/2)\right). \tag{12}$$

2.3.2 Bootstrap-t confidence interval

Consider the order statistics $\mu_k^{*[1]}<\mu_k^{*[2]}<...<\mu_k^{*[B]}$ where

$$\mu^{*[j]}=\frac{\sqrt{B}(\Omega^{*[j]}-\hat{\Omega})}{\sqrt{Var(\Omega^{*[j]})}}\ ,\ j=1,2,...,B, \tag{13}$$

where $Var(\Omega^{*[j]})$ represent the asymptotic variances of maximum likelihood estimates which can be calculated using the inverse of Fisher information matrix. Let $W(z) = P(\mu^* < z)$ be the cumulative distribution function of μ^*.
For a given z, define

$$\hat{\Omega}^*_{Boot-t} = \hat{\Omega} + B^{-1/2}\sqrt{Var(\Omega^*)W^{-1}(\zeta)}. \tag{14}$$

Thus, the approximation bootstrap-t $100(1-\zeta)\%$ confidence interval of $\hat{\Omega}^*$ is given by

$$\left(\hat{\Omega}^*_{Boot-t}(\zeta/2), \hat{\Omega}^*_{Boot-t}(1-\zeta/2)\right). \tag{15}$$

3 Bayesian Estimation

The Bayesian approach addresses the parameters randomly and uncertainties about the parameters are represented with a joint prior distribution, established before the failed data are collected. The flexibility to incorporate prior knowledge into the analyses makes the Bayesian approach very valuable in assessing reliability since the limited availability of data is one of the main challenges in terms of reliability analysis. It is assumed that the θ parameter is independent and the gamma prior distribution behave as follows,

$$\pi(\theta) \propto \theta^{a-1}e^{-b\,\theta} \qquad , \ \theta > 0, a > 0, b > 0. \tag{16}$$

The posterior distribution of the parameter θ denoted by $\pi^*(\theta \mid \underset{\sim}{x})$ are often obtained by combining the likelihood function (7) with the priors (16) and it can be written as

$$\pi^*(\theta \mid \underset{\sim}{x}) = \frac{\pi(\theta)\ L(\theta \mid \underset{\sim}{x})}{\int_0^\infty \pi(\theta)\ L(\theta \mid \underset{\sim}{x})\, d\theta}. \tag{17}$$

The SEL, which is a symmetric loss function that assigns equal loss to over estimates and underestimations, is a common function for losses. The Square Error Loss function is defined if θ is the parameter calculated with a $\hat{\theta}$ estimator, See [32]

$$L\left(\theta, \hat{\theta}\right) = \left(\hat{\theta} - \theta\right)^2. \tag{18}$$

Therefore, the Bayes estimate of function of θ, say $g(\theta)$ under the SEL function are often obtained as

$$\hat{g}_{BS}(\theta \mid \underset{\sim}{x}) = E_{\theta|\underset{\sim}{x}}(g(\theta)), \tag{19}$$

where

$$E_{\theta|\underset{\sim}{x}}(g(\theta)) = \frac{\int_0^\infty g(\theta)\ \pi(\theta)\ L(\theta \mid \underset{\sim}{x})d\theta}{\int_0^\infty \pi(\theta)\ L(\theta \mid \underset{\sim}{x})\, d\theta}. \tag{20}$$

It is noted that, the calculation of the multiple integral in (20) can not be solved analytically. Thus, the MCMC technique is used to generate samples from the joint posterior density function in (17). To implement the MCMC technique, we consider the Gibbs within Metropolis–Hasting samplers procedure. From (17), the joint posterior distribution are often written as

$$\pi^*(\theta \mid \underset{\sim}{x})L(\theta \mid \underset{\sim}{x}) \propto \left(\frac{\theta^4}{\theta^3+3\theta^2+6\theta+6}\right)^m \prod_{i=1}^{m}\left((1+x_{i:m:n})^3 e^{-\theta x_{i:m:n}}\right)$$
$$\left(\left[1+\frac{\theta^3 x^3_{i:m:n}+3\theta^2(\theta+1)x^2_{i:m:n}+3\theta(\theta^2+2\theta+2)x_{i:m:n}}{\theta^3+3\theta^2+6\theta+6}\right]e^{-\theta x_{i:m:n}}\right)^{r_i} \times \theta^{a-1}e^{-b\,\theta}. \tag{21}$$

It can be easily seen that the joint posterior of θ in Equation (21) do not present standard forms, so Gibbs sampling is not a straightforward option, the use of the Metropolis–Hasting sampler is required for the implementations MCMC technique. The algorithm of Metropolis–Hastings within Gibbs sampling is follows as:

(1) Start with initial guess $\theta^{(0)}$.
(2) Set $j=1$.
(3) Using the following M-H algorithm, generate $\theta^{(j)}$ from $\pi^*\left(\theta^{(j-1)} \mid \underline{x}\right)$ with the normal proposal distribution $N\left(\theta^{(j-1)}, var(\theta)\right)$.
(4) Generate a proposal θ^* from $N\left(\theta^{(j-1)}, var(\theta)\right)$.
(i) Evaluate the acceptance probabilities $\eta_\theta = \min\left[1, \frac{\pi_1^*(\theta^*|\underline{x})}{\pi^*\left(\theta^{(j-1)}|\underline{x}\right)}\right]$.
(ii) Generate a u_1 from a Uniform $(0,1)$ distribution.
(iii) If $u_1 < \eta_\theta$, accept the proposal and set $\theta^{(j)} = \theta^*$, else set $\theta^{(j)} = \theta^{(j-1)}$.
(5) Set $j = j+1$.
(6) Repeat Steps $(3)-(5)$, N times and obtain $\theta, i = 1,2,...N$.
(7) To compute the CRs of θ, $\theta^{(i)}$,

as $\theta^{(1)} < \theta^{(2)} ... < \theta^{(N)}$, then the $100(1-\vartheta)\%$ CRIs of θ_k is

$$\left(\theta\left(N\ \vartheta/2\right), \theta\left(N\ \left(1-\vartheta/2\right)\right)\right).$$

In order to ensure the convergence and to remove the affection of the selection of initial values, the first M simulated varieties are discarded. Then the chosen samples are $\theta^{(j)}$, $j = M+1,...N$, for sufficiently large N.
Based on SEL function, the approximate Bayes estimates of θ is given by

$$\hat{\theta} = \frac{1}{N-M}\sum_{j=M+1}^{N}\theta^{(j)}. \tag{22}$$

4 Simulation Study

In this part, Monte Carlo simulations are provided using progressive type II censored samples to compare between MLE, MPS and Bayesian estimates of the Akshaya parameter. The simulation results are performed in order to explore and output in terms of bias, mean square error and confidence interval.For many individual parameters, we produce ten thousand random samples from the Akshaya distribution. $n = 50$, 100 and 200 for different sample sizes, various failure numbers m sample ratio $ratio = \frac{m}{n}$ and schemes as various as
scheme 1: $R = (0^{(*m-1)},\ n-m)$.
Scheme 2: $R = (n-m,\ 0^{(*m-1)})$.
Scheme 3: $R = (n-2m,\ 0^{(*m-2)},\ n-2m)$.

The most easy approach is often considered to be the estimate form that minimises bias, MSE and L.CI of estimates. The results of the simulation including MSE, L.CI, B-p, B-t and MLE are described in Tables 1, 2, 5, 6, 7, 8 for different parameters. Tables 1, 2, 5, 6, 7, 8 summarise the simulation results.
Tables 1-6 are also summarised as follows in the subsequent observations.

1. The MSE, bias and L.CI decrease as the sample size increases.

2. The bias, MSE, L.CI decrease as the number of stages (m) increases.

3. The MPS estimates are efficient than anther methods for most studied cases of the Akshaya distribution under progressive type-II censored samples.

4. The B-t confidence intervals are more efficient than the B-p confidence intervals for most studied cases.

5 Application on Real Data

[14] discussed the following data 0.77, 1.74, 0.81, 1.20, 1.95, 1.20, 0.47, 1.43, 3.37, 2.20, 3.00, 3.09, 1.51, 2.10, 0.52, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.90, 2.05, which shows precipitation levels in

Table 1: The MLE, MPS and Bayesian estimation and interval estimation of parameter of Akshaya distribution based on progressive type-II censoring with different ratio when $\theta = 2.2$ and $n = 50$.

ratio	scheme	n=50					
			Bias	MSE	L.CI	B-t	B-p
0.60	1	MLE	0.01507	0.04389	0.81948	0.02518	0.02659
		MPS	0.00456	0.04291	0.81221	0.02565	0.02720
		Bayes	0.01031	0.00563	0.29143	0.00920	0.00904
	2	MLE	0.02784	0.05532	0.91597	0.02890	0.02992
		MPS	-0.02186	0.05174	0.88800	0.02842	0.02764
		Bayes	0.01619	0.00752	0.33414	0.01069	0.01073
	3	MLE	0.03062	0.04959	0.86506	0.02793	0.02687
		MPS	0.01336	0.04762	0.85420	0.02620	0.02772
		Bayes	0.01618	0.00658	0.31165	0.00966	0.00969
0.75	1	MLE	0.02239	0.03526	0.73117	0.02388	0.02391
		MPS	0.00831	0.03410	0.72351	0.02375	0.02371
		Bayes	0.01217	0.00453	0.25969	0.00799	0.00789
	2	MLE	0.01983	0.04081	0.78851	0.02479	0.02536
		MPS	-0.02363	0.03864	0.76534	0.02379	0.02554
		Bayes	0.01191	0.00518	0.27842	0.00892	0.00873
	3	MLE	0.02294	0.03851	0.76437	0.02324	0.02415
		MPS	0.00253	0.03689	0.75326	0.02339	0.02538
		Bayes	0.01325	0.00486	0.26848	0.00867	0.00874
0.90	1	MLE	0.01844	0.03390	0.71849	0.02347	0.02264
		MPS	-0.00194	0.03265	0.70867	0.02322	0.02248
		Bayes	0.01056	0.00425	0.25224	0.00820	0.00809
	2	MLE	0.01896	0.03329	0.71167	0.02221	0.02283
		MPS	-0.01851	0.03181	0.69568	0.02280	0.02179
		Bayes	0.01084	0.00429	0.25337	0.00792	0.00832
	3	MLE	0.01821	0.03367	0.71614	0.02213	0.02480
		MPS	-0.00716	0.03231	0.70437	0.02323	0.02236
		Bayes	0.01050	0.00422	0.25133	0.00806	0.00757

inches reworded during the month of search within the Minneapolis-St. Paul area over a 30-year period. We computed the Kolmogorov-Smirnov (KS) distance (D) between the fitted and therefore the empirical distribution functions for the data, where KS=0.11763 and its corresponding p-value=0.8008. Figure 1 displays the plots of estimated CDF, fitted PDF, PP-plot and QQ-plot for the Akshaya distribution for complete data. Figure 1 indicates that the Akshaya distribution provides better fits to the present data. For convergence see Figure 2.

The censored data when m=20: in case of scheme I, $R = (0^{*19}, 10)$ is 0.32 0.47 0.52 0.59 0.77 0.81 0.81 1.20 1.20 1.31 1.43 1.51 1.62 1.74 1.95 2.10 F2.20 3.00 3.09 3.37. in case of scheme II $R = (10, 0^{*19})$ is 0.47 0.59 0.77 0.81 0.96 1.18 1.20 1.35 1.51 1.62 1.89 1.95 2.05 2.10 2.20 2.48 2.81 3.00 3.37 4.75. In case of scheme III $R = (5, 0^{*18}, 5)$ is 0.47 0.59 0.77 0.81 0.81 1.18 1.20 1.31 1.35 1.43 1.51 1.62 1.87 1.95 2.10 2.20 2.81 3.00 3.09 3.37. Table 3 shows that MLE is very similar to the Bayes estimate Complete sample. In addition, estimates obtained from the Bayes supported censored data of progressive type II are closer to the estimates than the MLEs Complete data collected. The point estimates are nevertheless not appropriate to determine the best estimation method, because the specific values of unknown parameters are not well understood. Interval estimation is one of the comparison tools. The results of Table 4 show that the credible intervals of Bayesian θ is marginally shorter than other intervals.

Table 2: The MLE, MPS and Bayesian estimation and interval estimation of parameter of Akshaya distribution based on progressive type-II censoring with different ratio when $\theta = 2.2$ and n=100 .

ratio	scheme	n=100					
			Bias	MSE	L.CI	B-t	B-p
0.60	1	MLE	0.01432	0.02144	0.57146	0.01904	0.01750
		MPS	0.00896	0.02112	0.56888	0.01817	0.01874
		Bayes	0.00697	0.00263	0.19907	0.00638	0.00606
	2	MLE	0.02405	0.02657	0.63231	0.02001	0.02047
		MPS	-0.00624	0.02511	0.62101	0.01975	0.02039
		Bayes	0.01092	0.00332	0.22193	0.00704	0.00687
	3	MLE	0.01897	0.02355	0.59730	0.01876	0.01813
		MPS	0.01013	0.02300	0.59340	0.01904	0.02038
		Bayes	0.00886	0.00285	0.20665	0.00658	0.00677
0.75	1	MLE	0.00812	0.01789	0.52358	0.01606	0.01633
		MPS	0.00102	0.01764	0.52092	0.01657	0.01714
		Bayes	0.00505	0.00216	0.18118	0.00589	0.00571
	2	MLE	0.01176	0.02093	0.56557	0.01738	0.01796
		MPS	-0.01373	0.02032	0.55647	0.01790	0.01759
		Bayes	0.00693	0.00253	0.19530	0.00635	0.00596
	3	MLE	0.01432	0.01923	0.54093	0.01740	0.01695
		MPS	0.00385	0.01877	0.53710	0.01709	0.01676
		Bayes	0.00764	0.00233	0.18688	0.00592	0.00590
0.90	1	MLE	0.00747	0.00874	0.36543	0.01173	0.01140
		MPS	0.00389	0.00865	0.36448	0.01187	0.01204
		Bayes	0.00384	0.00104	0.12563	0.00405	0.00383
	2	MLE	0.00747	0.00874	0.36543	0.01173	0.01140
		MPS	0.00389	0.00865	0.36448	0.01187	0.01204
		Bayes	0.00384	0.00104	0.12563	0.00405	0.00383
	3	MLE	0.00849	0.01010	0.39279	0.01327	0.01284
		MPS	0.00317	0.00997	0.39144	0.01300	0.01234
		Bayes	0.00428	0.00118	0.13367	0.00424	0.00404

Table 3: Estimates, SEs, L.CI, and U.CI using the MLE, MPS and Bayesian methods for for complete data

	estimate	SE	CIL1	CIU1
MLE	1.5822	0.0231	1.5370	1.6274
MPS	1.5515	0.0218	1.5087	1.5942
Bayesian	1.5819	0.0221	1.5386	1.6255

Table 4: Estimates, SEs, L.CI, and U.CI using the MLE, MPS and Bayesian methods under progressive censored sample.

scheme		bc1.mn	bc1	CIL1	CIU1
1	MLE	1.1331	0.0140	1.1057	1.1606
	MPS	1.1257	0.0136	1.0991	1.1524
	Bayesian	1.1335	0.0138	1.1065	1.1602
2	MLE	1.4112	0.0257	1.3607	1.4616
	MPS	1.3798	0.0241	1.3326	1.4270
	Bayesian	1.4112	0.0255	1.3613	1.4611
3	MLE	1.2477	0.0183	1.2118	1.2837
	MPS	1.2357	0.0177	1.2010	1.2703
	Bayesian	1.2475	0.0178	1.2125	1.2827

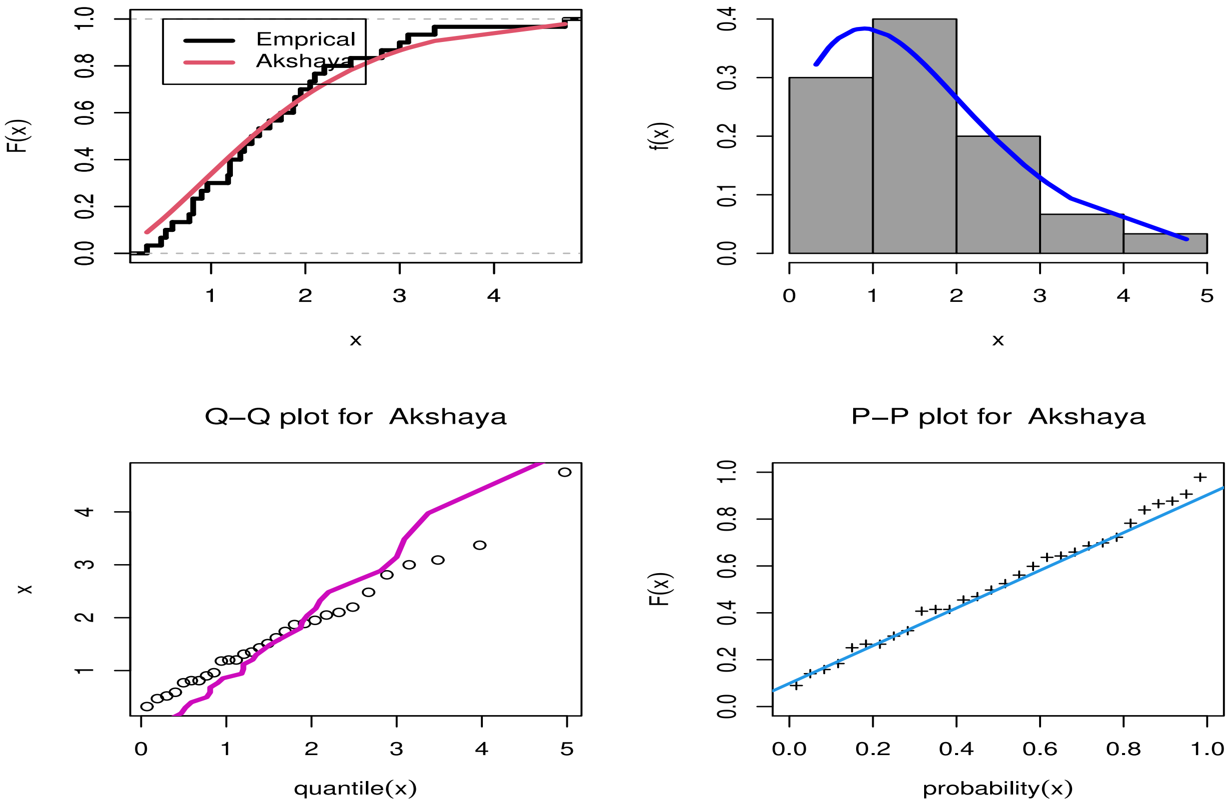

Fig. 1: The estimated CDF, fitted PDF, PP-plot and QQ-plot of the Akshaya distribution for complete data.

6 Conclusion

During this paper, we introduced the estimation by using three methods, the maximum likelihood, product of spacing method and the Bayesian technique, for the one-parameter Akshaya distribution. In addition, asymptotic distribution of MLEs is expected to be provided by estimated confidence intervals (ACIs) and bootstrap confidence intervals for the unknown parameter. In addition, Bayesian estimates for symmetric loss, including squared error loss function, is obtained. Gibbs within Metropolis-Hasting method of the sampler is used to obtain the Bayes estimate of the unknown parameter and hence the corresponding credible interval of the Markov chain Monte Carlo (MCMC). Finally, the suggested approaches for example is evaluated by a real data set.

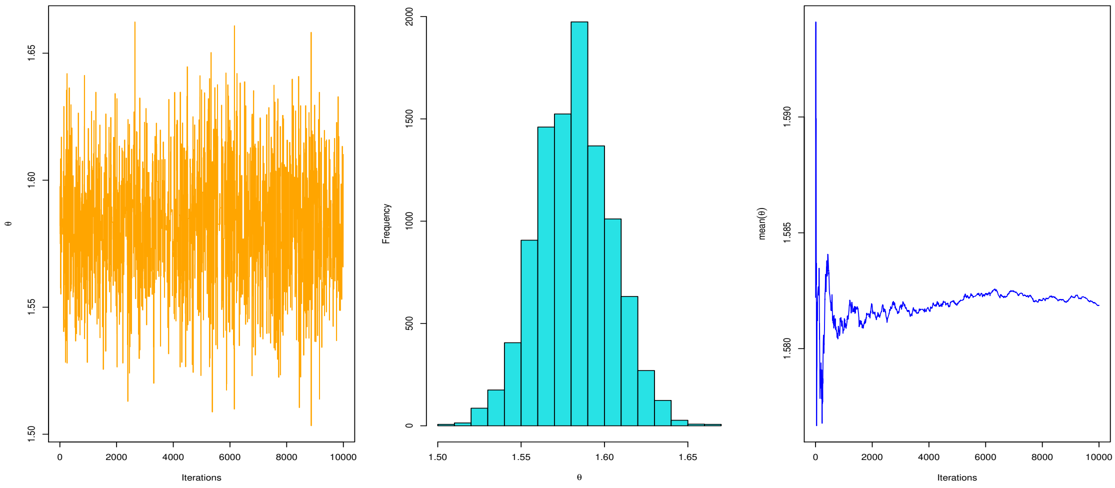

Fig. 2: Histogram plot and convergence of the Akshaya distribution for complete data.

Table 5: The MLE, MPS and Bayesian estimation and interval estimation of parameter of Akshaya distribution based on progressive type-II censoring with different ratio when $\theta = 2.2$ and n=200 .

ratio	scheme	n=200					
			Bias	MSE	L.CI	B-t	B-p
0.60	1	MLE	0.00563	0.01108	0.41233	0.01333	0.01298
		MPS	0.00295	0.01101	0.41141	0.01340	0.01344
		Bayes	0.00319	0.00130	0.14099	0.00457	0.00455
	2	MLE	0.00257	0.01082	0.40779	0.01336	0.01250
		MPS	-0.01495	0.01079	0.40317	0.01241	0.01298
		Bayes	0.00239	0.00128	0.13997	0.00469	0.00454
	3	MLE	0.00678	0.01017	0.39455	0.01249	0.01214
		MPS	0.00230	0.01006	0.39322	0.01311	0.01211
		Bayes	0.00364	0.00121	0.13541	0.00418	0.00424
0.75	1	MLE	0.00747	0.00874	0.36543	0.01173	0.01140
		MPS	0.00389	0.00865	0.36448	0.01187	0.01204
		Bayes	0.00384	0.00104	0.12563	0.00405	0.00383
	2	MLE	0.00747	0.00874	0.36543	0.01173	0.01140
		MPS	0.00389	0.00865	0.36448	0.01187	0.01204
		Bayes	0.00384	0.00104	0.12563	0.00405	0.00383
	3	MLE	0.00849	0.01010	0.39279	0.01327	0.01284
		MPS	0.00317	0.00997	0.39144	0.01300	0.01234
		Bayes	0.00428	0.00118	0.13367	0.00424	0.00404
0.90	1	MLE	0.00063	0.00796	0.34987	0.01150	0.01081
		MPS	-0.00468	0.00792	0.34854	0.01107	0.01088
		Bayes	0.00134	0.00093	0.11934	0.00360	0.00369
	2	MLE	0.00535	0.00830	0.35676	0.01047	0.01122
		MPS	-0.00728	0.00821	0.35414	0.01103	0.01169
		Bayes	0.00283	0.00098	0.12231	0.00372	0.00382
	3	MLE	0.01023	0.00840	0.35719	0.01104	0.01112
		MPS	0.00331	0.00823	0.35555	0.01154	0.01196
		Bayes	0.00449	0.00100	0.12259	0.00403	0.00386

Table 6: The MLE, MPS and Bayesian estimation and interval estimation of parameter of Akshaya distribution based on progressive type-II censoring with different ratio when $\theta = 0.7$ and n=50 .

ratio	scheme	n=50					
			Bias	MSE	L.CI	B-t	B-p
0.60	1	MLE	0.00379	0.00403	0.24848	0.00801	0.00773
		MPS	0.00250	0.00399	0.24767	0.00790	0.00788
		Bayes	0.00228	0.00048	0.08582	0.00265	0.00279
	2	MLE	0.00524	0.00470	0.26798	0.00867	0.00856
		MPS	-0.00716	0.00455	0.26315	0.00833	0.00819
		Bayes	0.00289	0.00058	0.09384	0.00275	0.00304
	3	MLE	0.00459	0.00435	0.25811	0.00861	0.00865
		MPS	0.00154	0.00429	0.25674	0.00814	0.00780
		Bayes	0.00263	0.00053	0.08997	0.00294	0.00307
0.75	1	MLE	0.00243	0.00313	0.21916	0.00666	0.00662
		MPS	-0.00004	0.00309	0.21814	0.00721	0.00698
		Bayes	0.00181	0.00037	0.07503	0.00235	0.00243
	2	MLE	0.00761	0.00417	0.25138	0.00813	0.00766
		MPS	-0.00321	0.00398	0.24720	0.00820	0.00800
		Bayes	0.00399	0.00051	0.08705	0.00280	0.00275
	3	MLE	0.00377	0.00371	0.23834	0.00758	0.00774
		MPS	-0.00054	0.00364	0.23653	0.00709	0.00760
		Bayes	0.00244	0.00045	0.08266	0.00265	0.00254
0.90	1	MLE	0.00372	0.00295	0.21266	0.00655	0.00655
		MPS	-0.00077	0.00289	0.21099	0.00672	0.00637
		Bayes	0.00213	0.00034	0.07202	0.00217	0.00231
	2	MLE	0.00344	0.00312	0.21880	0.00705	0.00708
		MPS	-0.00595	0.00306	0.21569	0.00666	0.00723
		Bayes	0.00210	0.00037	0.07490	0.00255	0.00227
	3	MLE	0.00489	0.00348	0.23066	0.00773	0.00714
		MPS	-0.00108	0.00339	0.22840	0.00728	0.00747
		Bayes	0.00279	0.00041	0.07878	0.00253	0.00245

Table 7: The MLE, MPS and Bayesian estimation and interval estimation of parameter of Akshaya distribution based on progressive type-II censoring with different ratio when $\theta = 0.7$ and n=100 .

ratio	scheme	n=100					
			Bias	MSE	L.CI	B-t	B-p
0.60	1	MLE	0.00162	0.00203	0.17651	0.00584	0.00582
		MPS	0.00101	0.00202	0.17624	0.00572	0.00547
		Bayes	0.00105	0.00024	0.06015	0.00210	0.00198
	2	MLE	0.00109	0.00248	0.19510	0.00614	0.00623
		MPS	-0.00644	0.00246	0.19280	0.00589	0.00621
		Bayes	0.00099	0.00028	0.06594	0.00205	0.00206
	3	MLE	0.00241	0.00208	0.17882	0.00579	0.00581
		MPS	0.00087	0.00207	0.17830	0.00564	0.00543
		Bayes	0.00127	0.00024	0.06118	0.00200	0.00195
0.75	1	MLE	0.00214	0.00167	0.15983	0.00491	0.00503
		MPS	0.00090	0.00165	0.15942	0.00515	0.00506
		Bayes	0.00117	0.00019	0.05431	0.00170	0.00176
	2	MLE	0.00194	0.00184	0.16813	0.00535	0.00524
		MPS	-0.00449	0.00182	0.16649	0.00569	0.00520
		Bayes	0.00133	0.00022	0.05783	0.00183	0.00188
	3	MLE	0.00178	0.00189	0.17034	0.00539	0.00561
		MPS	-0.00037	0.00187	0.16969	0.00549	0.00558
		Bayes	0.00125	0.00022	0.05839	0.00186	0.00183
0.90	1	MLE	0.00173	0.00143	0.14835	0.00468	0.00474
		MPS	-0.00060	0.00142	0.14769	0.00452	0.00479
		Bayes	0.00101	0.00017	0.05067	0.00169	0.00160
	2	MLE	0.00185	0.00162	0.15766	0.00513	0.00515
		MPS	-0.00378	0.00160	0.15637	0.00492	0.00500
		Bayes	0.00115	0.00019	0.05400	0.00167	0.00181
	3	MLE	0.00018	0.00155	0.15456	0.00505	0.00497
		MPS	-0.00294	0.00155	0.15378	0.00501	0.00501
		Bayes	0.00048	0.00018	0.05259	0.00167	0.00162

Table 8: The MLE, MPS and Bayesian estimation and interval estimation of parameter of Akshaya distribution based on progressive type-II censoring with different ratio when $\theta = 0.7$ and n=200 .

ratio	scheme	n=100					
			Bias	MSE	L.CI	B-t	B-p
0.60	1	MLE	-0.00012	0.00097	0.12190	0.00389	0.00401
		MPS	-0.00040	0.00097	0.12183	0.00378	0.00389
		Bayes	0.00022	0.00011	0.04155	0.00132	0.00132
	2	MLE	0.00272	0.00123	0.13726	0.00429	0.00432
		MPS	-0.00177	0.00121	0.13637	0.00421	0.00432
		Bayes	0.00126	0.00015	0.04712	0.00158	0.00149
	3	MLE	0.00226	0.00112	0.13072	0.00415	0.00417
		MPS	0.00150	0.00111	0.13050	0.00412	0.00392
		Bayes	0.00109	0.00013	0.04492	0.00142	0.00147
0.75	1	MLE	0.00119	0.00088	0.11624	0.00371	0.00375
		MPS	0.00057	0.00088	0.11607	0.00377	0.00371
		Bayes	0.00076	0.00010	0.03954	0.00123	0.00127
	2	MLE	0.00142	0.00097	0.12212	0.00394	0.00394
		MPS	-0.00234	0.00096	0.12145	0.00393	0.00413
		Bayes	0.00080	0.00011	0.04161	0.00133	0.00129
	3	MLE	0.00141	0.00088	0.11636	0.00384	0.00360
		MPS	0.00032	0.00088	0.11616	0.00352	0.00358
		Bayes	0.00080	0.00010	0.03969	0.00123	0.00123
0.90	1	MLE	0.00094	0.00077	0.10863	0.00349	0.00332
		MPS	-0.00023	0.00076	0.10839	0.00340	0.00337
		Bayes	0.00050	0.00009	0.03705	0.00119	0.00116
	2	MLE	0.00135	0.00079	0.10981	0.00345	0.00365
		MPS	-0.00193	0.00078	0.10933	0.00355	0.00346
		Bayes	0.00073	0.00009	0.03712	0.00124	0.00115
	3	MLE	0.00129	0.00081	0.11180	0.00360	0.00342
		MPS	-0.00032	0.00081	0.11148	0.00350	0.00348
		Bayes	0.00066	0.00009	0.03791	0.00121	0.00121

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Conflicts of Interests
The authors declare that they have no conflicts of interests

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