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Some Parallel Algorithms for a New System of Quasi Variational Inequalities

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Abstract: In this paper, we introduce a new system of quasi variational inequalities. The projection technique is used to establish the equivalence between this new system of quasi variational inequalities and the fixed point problem. The fixed point formulation enables us to suggest some parallel projection iterative methods for solving the system of quasi variational inequalities. Convergence analysis of the proposed methods is investigated. Several special cases are discussed. Results proved in this paper continue to hold for these problems.

Keywords: Variational inequalities, fixed point problems, parallel algorithms, convergence.
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1 Introduction

Variational inequalities theory, which was introduced by Stampacchia [24], is rich in contents. It offers many beautiful results that are simple and yet striking in their formulation, diverse as well as powerful in their applications. Variational inequalities have been generalized and extended in several directions using innovative and novel techniques. A useful generalization of the variational inequalities is called quasi variational inequalities, the origin of which can be traced back to Bensoussan and Lions [2, 3]. It is worth mentioning that the involved convex set in the formulation of quasi variational inequalities depends upon the solution implicity or explicitly. For the recent applications, numerical methods, and other aspects of quasi variational inequalities, see ([1]- [25]) and the reference their in.

It turned out that quasi variational inequality is very difficult class of problems. To develop implementable and efficient methods for solving new quasi variational inequalities is still a challenging task. The normal technique is to show that the quasi variational inequalities are equivalent to the fixed point problem. This alternative equivalent formulation is used to propose some projection type methods for solving the quasi variational inequalities, see [4, 12, 13, 18, 22] for a special class of convex-valued set.

Motivated and inspired by the recent research going in this field, we introduce and consider a new system of quasi variational inequalities. The projection method is used to establish the equivalence between the system of quasi variational inequalities and the fixed point problems. This alternative equivalence is used to suggest and analyze some parallel projection algorithms for solving this system of quasi variational inequalities. The convergence analysis of the proposed parallel algorithm is consider under suitable conditions. Some special cases are discussed. Results proved in this paper continue to hold for these cases. Our results may be viewed as a refinement of the known results for quasi variational inequalities and related optimization problems. It is expected that the ideas and techniques of this paper stimulate further research in this field.

2 Preliminaries and Basic results

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed and convex set in H and $T_1, T_2 : H \to H$ be two different operators.

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Given two point-to-set mappings $K_1 : x \to K_1(x)$ and $K_2 : y \to K_2(y)$, which associate two closed convex sets $K_1(x)$ and $K_2(y)$ with any elements x,y of H, we consider the problem of finding $(x,y) \in K_1(x) \times K_2(y)$ such that

$$\left.\begin{array}{l}\langle \rho_1 T_1 y + x - y, x_1 - x\rangle \geq 0, \ \forall x_1 \in K_1(x) \\ \langle \rho_2 T_2 x + y - x, x_2 - y\rangle \geq 0, \ \forall x_2 \in K_2(y)\end{array}\right\}, \quad (1)$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are constants. The system (1) is called a system of quasi variational inequalities.

I. If $T_1 = T_2 = T$, an operator, then problem (1) is to find $(x,y) \in K_1(x) \times K_2(y)$ such that

$$\left.\begin{array}{l}\langle \rho_1 T y + x - y, x_1 - x\rangle \geq 0, \ \forall x_1 \in K_1(x) \\ \langle \rho_2 T x + y - x, x_2 - y\rangle \geq 0, \ \forall x_2 \in K_2(y)\end{array}\right\}, \quad (2)$$

is also called the system of quasi variational inequalities and appears to be a new one.

II. If $K_1(x) = K_2(y) \equiv K(x)$ and $T_1 = T_2$, then problem (1) reduces to finding $x \in K(x)$ such that

$$\langle Tx, v - x\rangle \geq 0, \ \forall v \in K(x), \quad (3)$$

is known as quasi variational inequality, introduced and studied by Bensoussan and Lions [2, 3]. For the formulation, applications, numerical methods and other aspects of the quasi variational inequalities, see ([1]- [23]) and the references therein.

III. If $K_1(x) = K_1$, a closed convex set in H and $K_2(y) = K_2$, a closed convex set in H, then problem (1) collapses to: Find $(x,y) \in K_1 \times K_2$ such that

$$\left.\begin{array}{l}\langle \rho_1 T_1 y + x - y, x_1 - x\rangle \geq 0, \ \forall x_1 \in K_1 \\ \langle \rho_2 T_2 x + y - x, x_2 - y\rangle \geq 0, \ \forall x_2 \in K_2\end{array}\right\}, \quad (4)$$

which is called a system of variational inequalities and appears to be new.

IV. If $K_1 = K_2 = K$, a closed convex set in K, then problem (4) reduces to finding $(x,y) \in K$ such that

$$\left.\begin{array}{l}\langle \rho_1 T_1 y + x - y, x_1 - x\rangle \geq 0, \ \forall x_1 \in K \\ \langle \rho_2 T_2 x + y - x, x_2 - y\rangle \geq 0, \ \forall x_2 \in K\end{array}\right\}, \quad (5)$$

which has been studied extensively in recent years.

V. If $K_1(x) = K_2(y) = K(x)$, a closed convex-valued set in H, then problem (1) is equivalent to finding $(x,y) \in K(x)$ such that

$$\left.\begin{array}{l}\langle \rho_1 T_1 y + x - y, x_1 - x\rangle \geq 0, \ \forall x_1 \in K(x) \\ \langle \rho_2 T_2 x + y - x, x_2 - y\rangle \geq 0, \ \forall x_2 \in K(x)\end{array}\right\}, \quad (6)$$

which is called the system of quasi variational inequalities and appears to be new one.

VI. If $T_1 = T_2 = T$, an operator and $K_1 = K_2 = K$, then problem (5) is equivalent to finding $x \in K$ such that

$$\langle Tx, x_1 - x\rangle \geq 0, \ \forall x_1 \in K, \quad (7)$$

which is known as the original variational inequality introduced and studied by Stampachia [24] in 1964. For the applications, generalizations, numerical methods and related optimization problems, see ([1]- [25]).
We now recall some basic results and concepts.

Lemma 2.1 [9]. Let K be a closed and convex set in H. Then, for a given $z \in H$, $u \in K$ satisfies the inequality

$$\langle u - z, v - u\rangle \geq 0, \forall v \in K,$$

if and only if,

$$u = P_K z,$$

where P_K is the projection of H onto the closed convex set.
It is known that the projection operator P_K is nonexpensive, that is

$$\|P_K u - P_K v\| \leq \|u - v\|, \ \forall u, v \in H.$$

The projection operator of H onto the closed convex-valued set $K(u)$ is denoted by $P_{K(u)}$. It is known that the projection operator $P_{K(u)}$ is not nonexpensive. However, it satisfies Lipschitz type continuity condition.
We need the following assumption for the operators $P_{K_1(x)}$ and $P_{K_2(y)}$, see [14]

Assumption 2.1. The operators $P_{K_1(x)}$ and $P_{K_2(y)}$ satisfy the conditions:

$$\|P_{K_1(x_1)} w - P_{K_1(x_2)}\| \leq \upsilon_1 \|x_1 - x_2\|, \ \forall x_1, x_2, w \in H$$

and

$$\|P_{K_2(y_1)} w - P_{K_2(y_2)}\| \leq \upsilon_2 \|y_1 - y_2\|, \ \forall y_1, y_2, w \in H$$

where $\upsilon_1 > 0$ and $\upsilon_2 > 0$ are constants.
Assumption 2.1 plays an important role in the investigation of the convergence analysis of the iterative methods.

Definition 2.1. An operator $T : H \to H$ is said to be:
(i) *strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tu - Tv, u - v\rangle \geq \alpha ||u - v||^2, \ \forall u, v \in H.$$

(ii) *Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$||Tu - Tv|| \leq \beta ||u - v||, \ \forall u, v \in H.$$

Note that, if T satisfies (i) and (ii), then $\alpha \leq \beta$.

3 Main Results

In this section, we first show that the system of quasi variational inequalities (1) is equivalent to a system of fixed point problems. This alternative equivalent formulation is used to suggest a parallel projection iterative methods for solving (1).

Lemma 3.1. The system of quasi variational inequalities (1) has a solution, $(x,y) \in K_1(x) \times K_2(y)$, if and only if, $(x,y) \in K_1(x) \times K_2(y)$ satisfies the relations.

$$x = P_{K_1(x)}[y - \rho_1 T_1 y] \quad (8)$$
$$y = P_{K_2(y)}[x - \rho_2 T_2 x], \quad (9)$$

where $\rho_1 > 0$ and $\rho_2 > 0$ are constants. Here $P_{K_1(x)}$ and $P_{K_2(y)}$ are projection of H onto the closed convex-valued sets $K_1(x)$ and $K_2(y)$, respectively.

*Proof.*Let $(x,y) \in K_1(x) \times K_2(y)$ be a solution of (1). Then,

$$\langle \rho_1 T_1 y + x - y, x_1 - x \rangle \geq 0, \ \forall x_1 \in K_1(x)$$

and

$$\langle \rho_2 T_2 y + y - x, x_2 - y \rangle \geq 0, \ \forall x_2 \in K_2(x).$$

Using Lemma 2.1, we have (8) and (9). □

Lemma 3.1 implies that the system (1)is equivalent to the fixed point problems (9) and (8). We can rewrite (9) and (8) in the following equivalent forms.

$$x = (1-\alpha_n)x + \alpha_n P_{K_1(x)}[y - \rho_1 T_1 y] \quad (10)$$
$$y = (1-\beta_n)y + \beta_n P_{K_2(y)}[x - \rho_2 T_2 x], \quad (11)$$

where $\alpha_n, \beta_n \in [0,1]$ for all $n \geq 0$.
This equivalent formulation is used to suggest the following parallel projection iterative method for solving system of quasi variational inequalities (1).

Algorithm 3.1. For a given $(x_0, y_0) \in K_1(x_0) \times K_2(y_0)$, find (x_{n+1}, y_{n+1}) by the iterative schemes

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_{K_1(x_n)}[y_n - \rho_1 T_1 y_n] \quad (12)$$
$$y_{n+1} = (1-\beta_n)y_n + \beta_n P_{K_2(y_n)}[x_n - \rho_2 T_2 x_n]. \quad (13)$$

Algorithm 3.1 is called the parallel projection method, which is suitable for implementation on two different processor computers. It is well known that parallel projection methods are better than the sequential iterative methods. To the best of our knowledge, Algorithm 3.1 has not been studied previously for solving the system of quasi variational inequalities.

I. If $T_1 = T_2 = T$, then Algorithm 3.1 reduces to

Algorithm 3.2. For a given $(x_0, y_0) \in K_1(x_0) \times K_2(y_0)$, find the approximate solution by the iterative schemes

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_{K_1(x_n)}[y_n - \rho T y_n]$$
$$y_{n+1} = (1-\beta_n)y_n + \beta_n P_{K_2(y_n)}[x_n - \rho T x_n],$$

where $\alpha_n, \beta_n \in [0,1]$ for all $n \geq 0$.

II. If $K_1(x) = K_1$ and $K_2(y) = K_2$ are closed convex sets in H, then Algorithm reduces to the following parallel algorithm for solving the system of variational inequalities (3).

Algorithm 3.3. For a given $(x_0, y_0) \in K_1 \times K_2$, find the approximate solution (x_n, y_n) by the iterative schemes

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n P_{K_1}[y_n - \rho_1 T_1 y_n]$$
$$y_{n+1} = (1-\beta_n)y_n + \beta_n P_{K_2}[x_n - \rho_2 T_2 x_n],$$

where $\alpha_n, \beta_n \in [0,1]$ for all $n \geq 0$.

For suitable and appropriate choice of the operators, convex sets and space, one can obtain several new and known iterative methods for solving system of (quasi) variational inequalities and related problems.

We now investigate the convergence analysis of Algorithm 3.1 and this is the main motivation of our next result.

Theorem 3.1. Let T_1, T_2 be strongly monotone with constants $\alpha_1 > 0, \alpha_2 > 0$ and Lipschitz continuous with constants $\beta_1 > 0, \beta_2 > 0$, respectively. If Assumption 2.1 and following conditions hold:

(i) $\theta_1 = \sqrt{1 - 2\rho_1\alpha_1 + \beta_1^2\rho_1^2}$ such that $0 < \theta < 1$.

(ii) $\theta_2 = \sqrt{1 - 2\rho_2\alpha_2 + \beta_2^2\rho_2^2}$ such that $0 < \theta < 1$.

(iii) $0 \leq \alpha_n, \beta_n \leq 1, (\alpha_n(1-\upsilon_1) - \theta_2\beta_n) \geq 0$ and

$$(\beta_n(1-\upsilon_2) - \theta_1\alpha_n) \geq 0,$$

such that

$$\sum_{n=0}^{\infty}(\alpha_n(1-\upsilon_1) - \theta_2\beta_n) = \infty,$$

$$\sum_{n=0}^{\infty}(\beta_n(1-\upsilon_2) - \theta_1\alpha_n) = \infty,$$

then sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 3.1 converge to x and y respectively.

Proof. Let $(x,y) \in K_1(x) \times K_2(y)$ be a solution of (1). Then, from (9), (10) and (12), we have

$$\begin{aligned}
&\|x_{n+1} - x\| \\
&\leq (1-\alpha_n)\|x_n - x\| + \alpha_n\|P_{K_1(x_n)}[y_n - \rho_1 T_1 y_n] \\
&\quad - P_{K_1(x)}[y - \rho_1 T_1 y]\| \\
&\leq (1-\alpha_n)\|x_n - x\| + \alpha_n\|P_{K_1(x_n)}[y_n - \rho_1 T_1 y_n] \\
&\quad - P_{K_1(x)}[y_n - \rho_1 T_1 y_1]\| \\
&\qquad + \alpha_n\|P_{K_1(x)}[y_n - \rho_1 T_1 y_n] - P_{K_1(x)}[y - \rho_1 T_1 y]\| \\
&\leq (1-\alpha_n)\|x_n - x\| + \alpha_n\upsilon_1\|x_n - x\| \\
&\quad + \alpha_n\|y_n - y - \rho_1(T_1 y_n - T_1 y)\| \\
&= (1-\alpha_n(1-\upsilon_1))\|x_n - x\| \\
&\quad + \alpha_n\|y_n - y - \rho_1(T_1 y_n - T_1 y)\|. \qquad (14)
\end{aligned}$$

Since T_1 is strongly monotone with constants $\alpha_1 > 0$ and Lipschitz continuous with constant $\beta_1 > 0$, so

$$\|y_n - y - \rho_1(T_1 y_n - T_1 y)\|^2$$

$$\leq \|y_n - y\|^2 - 2\rho_1\langle T_1y_n - T_1y, y_n - y\rangle + \rho^2\|T_1y_n - T_1y\|^2,$$
$$\leq (1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2)\|y_n - y\|^2. \quad (15)$$

From (15) and (14), we obtain

$$\begin{aligned}
&\|x_{n+1} - x\| \\
&\leq (1 - \alpha_n(1 - \upsilon_1))\|x_n - x\| + \\
&\alpha_n\sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_2}\|y_n - y\| \quad (16)\\
&= (1 - \alpha_n(1 - \upsilon_1))\|x_n - x\| + \alpha_n\theta_1\|y_n - y\|.
\end{aligned}$$

In a similar way, from (11), (13) and (15), we have

$$\begin{aligned}
&\|y_{n+1} - y\| \\
&\leq (1 - \beta_n)\|y_n - y\| + \beta_n\|P_{K_2(y_n)}[x_n - \rho_2T_2x_n] \\
&\quad - P_{K_2(y)}[x - \rho_2T_2x]\| \\
&\leq (1 - \beta_n)\|y_n - y\| + \beta_n\|P_{K_2(y_n)}[x_n - \rho_2T_2x_n] \\
&\quad - P_{K_2(y)}[x_n - \rho_2T_1x_n]\| \\
&\qquad + \beta_n\|P_{K_2(y)}[x_n - \rho_2T_2x_n] - P_{K_2(y)}[x - \rho_2T_2x]\| \\
&\leq (1 - \beta_n)\|y_n - y\| + \beta_n\upsilon_2\|y_n - y\| \\
&\quad + \beta_n\|x_n - x - \rho_2(T_2x_n - T_2x)\| \\
&= (1 - \beta_n(1 - \upsilon_2))\|y_n - y\| \\
&\quad + \beta_n\sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\beta_2}\|x_n - x\| \\
&= (1 - \beta_n(1 - \upsilon_2))\|y_n - y\| + \beta_n\theta_2\|x_n - x\|, \quad (17)
\end{aligned}$$

where we have used the fact that Assumption 2.1 holds and the operator T_2 is strongly monotone with constant $\alpha_2 > 0$ and Lipschitz continuous with constant $\beta_2 > 0$ respectively.
From (17) and (16), we have

$$\begin{aligned}
&\|x_{n+1} - x\| + \|y_{n+1} - y\| \\
&\leq (1 - \alpha_n(1 - \upsilon_1) + \beta_n\theta_2)\|x_n - x\| \\
&\quad + (1 - \beta_n(1 - \upsilon_2) + \alpha_n\theta_1)\|y_n - y\| \\
&\leq \max\{((1 - \alpha_n(1 - \upsilon_1) - \beta_n\theta_2), \\
&\quad (1 - \beta_n(1 - \upsilon_2) - \alpha_n\theta_1))(\|x_n - x\| + \|y_n - y\|)\} \\
&\leq \max(w_1, w_2)(\|x_n - x\| + \|y_n - y\|), \quad (18)
\end{aligned}$$

where

$$w_1 = 1 - (\alpha_n(1 - \upsilon_1) - \beta_n\theta_2)$$
$$w_2 = 1 - (\beta_n(1 - \upsilon_2) - \alpha_n\theta_1).$$

Define the norm $\|(.,.)\|$ on H by

$$\|(u,v)\| = \|u\| + \|v\|, \ \forall u, v \in H \times H.$$

Using the fact that $H \times H$ is a Banach space, and from (18), we have

$$\|(x_{n+1}, y_{n+1}) - (x,y)\| \leq \max(w_1, w_2)\|(x_n, y_n) - (x,y)\|.$$

From assumption (iii), we have

$$\lim_{n\to\infty}\|(x_{n+1}, y_{n+1}) - (x,y)\| = 0.$$

This implies that

$$\lim_{n\to\infty}\|x_{n+1} - x\| = 0,$$

and

$$\lim_{n\to\infty}\|y_{n+1} - y\| = 0,$$

the required result. □

We now suggest and analyze some new iterative methods for solving system of quasi variational inequalities (1).

Using Lemma 3.1, one can easily show that $(x,y) \in K_1(x) \times K_2(y)$ is a solution of (1) if and only if, $(x,y) \in K_1(x) \times K_2(y)$ satisfies

$$x = P_{K_1(x)}z \quad (19)$$
$$y = P_{K_2(y)}w \quad (20)$$
$$z = y - \rho_1T_1y \quad (21)$$
$$w = x - \rho_2T_2x. \quad (22)$$

This alternative formulation can be used to suggest and analyze the following iterative methods for solving the system of quasi variational inequalities (1).

Algorithm 3.4. For a given (x_0, y_0), find the approximate solutions x_{n+1} and y_{n+1} by the iterative schemes

$$x_{n+1} = P_{K_1(x_n)}z_n \quad (23)$$
$$y_{n+1} = P_{K_2(y_n)}w_n \quad (24)$$
$$z_n = y_n - \rho_1T_1y_n \quad (25)$$
$$w_n = x_n - \rho_2T_2x_n, \ n = 0, 1, 2\ldots \quad (26)$$

If $T_1 = T_2 = T$, an operator, then Algorithm 3.4 reduces to:

Algorithm 3.4. For a given x_0 and y_0 find the approximate solutions x_{n+1} and y_{n+1} by the iterative schemes:

$$x_{n+1} = P_{K_1(x_n)}z_n$$
$$y_{n+1} = P_{K_2(y_n)}w_n$$
$$z_n = y_n - \rho_1y_n$$
$$w_n = x_n - \rho_2Tx_n, \ n = 0, 1, 2\ldots$$

For appropriate and suitable choice of the operators, convex sets and spaces, one can obtain several new and previously known iterative methods for solving system of quasi variational inequalities and related optimization problems.

We now consider the convergence of Algorithm 3.4 and this is the main motivation of our next result.

Theorem 3.2. Let T_1, T_2 be strongly monotone with constants $\alpha_1 > 0, \alpha_2 > 0$ and Lipschitz continuous with $\beta_1 > 0, \beta_2 > 0$, respectively. If the Assumption 2.1 and the following conditions hold:

(i) $\theta_1 = \nu_1 + \sqrt{1 - 2\rho_2\alpha_2 + \rho_2^2\beta_2^2} < 1$

(ii) $\theta_2 = \nu_2 + \sqrt{1 - 2\rho_1\alpha_1 + \rho_1^2\beta_1^2} < 1,$

then the approximate solutions x_n+1 and y_n+1 obtained from Algorithm 3.4 converge to the exact solution x and y, respectively.

Proof. Let $(x,y) \in K_1(x) \times K_2(y)$ be a solution of (1). Then, from (15), (21) and (25), we have

$$\begin{aligned} & \|z_n - z\| \\ = & \|y_n - y - \rho_1(T_1y_n - T_1y)\| \\ \leq & \sqrt{1-2\rho_1\alpha_1+\rho_1^2\beta_1^2}\|y_n - y\|. \end{aligned} \tag{27}$$

In a similar way, from (15), (22) and (26),

$$\begin{aligned} & \|w_n - w\| \\ = & \|x_n - x - \rho_2(T_2x_n - T_2x)\| \\ \leq & \sqrt{1-2\rho_2\alpha_2+\rho_2^2\beta_2^2}\|x_n - x\|. \end{aligned} \tag{28}$$

Using Assumption 2.1, from (19), (23) and (27), we have

$$\begin{aligned} & \|x_{n+1} - x\| \\ = & \|P_{K_{1(x_n)}(z_n)} - P_{K_{1(x)}(z)}P_{K_1(z_n)}\| \\ \leq & \|P_{K_{1(x_n)}(z_n)} - P_{K_{1(x)}(z)}P_{K_1(z_n)}\| \\ & +\|P_{K_{1(x_n)}(z_n)} - P_{K_{1(x)}(z)}P_{K_1(z_n)}\| \\ \leq & \|z_n - Z\| + \nu_1\|x_n - x\| \\ \leq & \nu_1\|x_n - x\| + (\sqrt{1-2\rho_1\alpha_1+\rho_1^2\beta_1^2})\|y_n - y\|. \end{aligned} \tag{29}$$

Similarly, from (20), (24), (28) and using Assumption 2.1, we have

$$\begin{aligned} & \|y_{n+1} - y\| \\ = & \|P_{K_{2(y_n)}(w_n)} - P_{K_{2(y)}(w)}P_{K_2(w_n)}\| \\ \leq & \|P_{K_{2(y_n)}(w_n)} - P_{K_{2(y)}(w)}P_{K_2(w_n)}\| \\ & +\|P_{K_{2(y_n)}(w_n)} - P_{K_{2(y)}(w)}P_{K_2(w_n)}\| \\ \leq & \|w_n - w\| + \nu_2\|y_n - y\| \\ \leq & \nu_2\|y_n - y\| + (\sqrt{1-2\rho_2\alpha_2+\rho_2^2\beta_2^2})\|x_n - x\|. \end{aligned} \tag{30}$$

From (29) and (30), we have

$$\begin{aligned} & \|x_{n+1} - x\| + \|y_{n+1} - y\| \\ \leq & (\nu_1 + \sqrt{1-2\rho_2\alpha_2+\rho_2^2\beta_2^2})\|x_n - x\| \\ & +(\nu_2 + \sqrt{1-2\rho_1\alpha_1+\rho_1^2\beta_1^2})\|y_n - y\| \\ \leq & \theta_1\|x_{n+1} - x\| + \theta_2\|y_{n+1} - y\| \\ \leq & \max(\theta_1,\theta_2)(\|x_n - x\| + \|y_n - y\|), \end{aligned}$$

where

$$\theta_1 = \nu_1 + \sqrt{1-2\rho_2\alpha_2+\rho_2^2\beta_2^2}$$

$$\theta_2 = \nu_2 + \sqrt{1-2\rho_1\alpha_1+\rho_1^2\beta_1^2}.$$

Using the technique of Theorem 3.1, it follows that

$$\|(x_{n+1},y_{n+1} - (x,y))\| \leq \max\theta\|(x_n,y_n) - (x,y)\|,$$

where

$$\theta = \max(\theta_1,\theta_2).$$

From condition (i) and (ii), it follows that $\theta_1 < 1$ and $\theta_2 < 1$. This implies that $\theta < 1$. Thus, one can conclude that

$$\lim_{n\to\infty}\|(x_{n+1},y_{n+1} - (x,y))\| = 0.$$

This implies that

$$\lim_{n\to\infty}\|(x_{n+1} - x\| = 0,$$

and

$$\lim_{n\to\infty}\|(y_{n+1} - y\| = 0;$$

the required result. □

4 Conclusion

In this paper, we have introduced a new system of quasi variational inequalities. We have established the equivalence between the system of quasi variational inequalities and the fixed point problems. This alternative formulation is used to suggest and analyze some parallel algorithms for solving the system of quasi variational inequalities. Convergence analysis of the proposed iterative methods investigated. Several special cases are discussed. The comparison of the proposed methods with other methods is an open problem. The interested readers are encouraged to find novel and new applications of the quasi variational inequalities in pure and applied sciences.

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