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Burr Lindley Distribution: Properties, Estimation and Applications

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Abstract: In this paper, Burr Lindley (BL) distribution is constructed as a composition structure between Lindley and Burr distributions to induce a more flexible model for the hydrology data than its parents where it has decreasing and inverted bathtub shapes for the hazard rate function. Several statistical and reliability properties of the BL distribution are obtained, such as the behavior of its density and hazard rate functions, survival and residual life functions, Shannon entropy, limiting distributions and characterization via right truncated moments. For parameter estimation, the maximum likelihood estimators as well as the Fisher information matrix are investigated. Moreover, the simulation study is performed to examine the performance of the parameter estimates in terms of bias and mean square error. A real data application in hydrology field is modeled to the BL distribution and compared with other well-known distributions, to illustrate its performance. Based on goodness of fit tests, BL distribution has a superior fitting performance to hydrology data than the compared distributions.

Keywords: Burr distribution; Lindley distribution, Characterization, Maximum Likelihood, Hydrology data.

1 Introduction

Modeling of lifetime data is essential in applied sciences including, biomedical science, engineering, finance and insurance, amongst others. Several continuous distributions for modeling lifetime data has been introduced in statistical literature including exponential, Lindley, Gamma and Weibull. Many methods have been developed to generate statistical distributions in the literature. Some well-known methods for generating univariate continuous distributions including methods based on differential equations and methods of translation were developed.

The interest in developing new methods for generating new or more flexible distributions continue to be active in recent decades. [1] indicated that the majority of methods developed after the 1980s are the methods of 'combination' for the reason that these new methods are based on the idea of combining two existing distributions or by adding extra parameters to an existing distribution to generate a new family of distributions. [2] introduced a new family called the transformed transformer (T-X) family. The T-X family is explained in more detail manner in [3]. Later in the same year, [4] introduced the extended T-X families of the first kind and second kind, such that the extended T-X (II) family is the extension form of the T-X family.

The other well-known families are: Logistic-X Family (see [5]) and New Weibull-X Family (see [6]). [7] proposed a new family of distributions, namely, a modified T-X family of distributions.

Lindley distribution was initially introduced by Lindley [8, 9] to analyze failure time data, especially in applications of modeling stress-strength reliability. [10] showed that it is especially useful for modeling in mortality studies. It was used in many fields including biology, engineering and medicine. Lindley distribution belongs to an exponential family and its construction is mainly a mixture of exponential and Gamma distributions. The probability density function of Lindley distribution with parameter m is:

$$f_{lindely}(x;m) = \frac{m^2}{(m+1)}(1+x)e^{-mx}; \qquad x>0, m>0.$$

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Burr distribution (sometimes called the Burr Type XII distribution or Singh-Maddala distribution) is a unimodal family of distributions with a wide variety of shapes. Burr distribution was first discussed by [11] as a two-parameter family. An additional scale parameter was introduced by [12]. This distribution is used to model a wide variety of phenomena including crop prices, household income, option market price distributions, risk (insurance) and travel time. The probability density function for the Burr XII distribution with parameter c and k is:

$$f_{Burr}(x;c,k) = ck\frac{x^{c-1}}{(1+x^c)^{k+1}}; \qquad x>0,\ c,k>0.$$

This paper introduces the Burr Lindley (BL) distribution, its probability density function, cumulative distribution function and survival function in Section 2. The structural characteristics of BL distribution such as the behavior of the probability density function, the hazard rate function, the entropy measures, the moments and the associated moments are obtained in Section 3. Section 4 discussed the point estimation by the maximum likelihood and the method of moments furthermore the interval estimation. Simulation schemes are obtained in Section 5. Characterization of the BL distribution by Right Truncated Moments is discussed in Section 6. Finally, a real-life data application in hydrology is modeled to the BL distribution and compared with other well-known distributions in Section 7.

2 Model Formulation

The BL distribution is defined by using the concept of composition of distribution which extended by [4] in T-X families as:

$$G(x) = \int_0^{-(\log 1-F(x)\ )} l(t)\ dt,$$

where, $F(x)$ is the cumulative distribution function of Burr distribution, and $l(t)$ is the probability density function of Lindley distribution.

The probability density function (PDF) of the BL distribution is defined by

$$g(x) = \frac{x^{-1+\alpha}(1+x^{\alpha})^{-1-\beta\theta}\alpha\beta\theta^2(1+\beta\mathrm{Log}(1+x^{\alpha}))}{1+\theta}, \qquad x>0,\ \ \alpha,\beta,\theta>0. \tag{1}$$

The corresponding reliability (survival) function of the BL distribution is given by

$$S(x) = \frac{(1+x^{\alpha})^{-\beta\theta}(1+\theta+\beta\theta Log(1+x^{\alpha}))}{1+\theta}, \qquad x>0,\ \ \alpha,\beta,\theta>0. \tag{2}$$

Some of the possible shapes of density function in Equation (1) for the selected parameter values are illustrated in Figure 1; the density function can take various forms depending on the parameter values.

3 Structural Characteristics

This section concerned with the structural characteristics of BL distribution. In particular, the functional behavior of the density function, hazard function, reversed hazard function, residual life functions and others.

3.1 Behavior of the probability density function (PDF) of BL distribution

Theorem 1.*The behavior of PDF of the BL distribution $g(x)$ in Equation (1) is*

a.Unimodal if $\alpha > 1,\ \beta,\ \theta > 0.$
b.Decreasing if $0 < \alpha \le 1,\ \beta,\ \theta > 0.$

*Proof.*The first derivative of $g(x)$ can be written as:

$$g'(x) = -\frac{x^{-2+\alpha}(1+x^{\alpha})^{-2-\beta\theta}\alpha\beta\theta^2(1-\alpha+x^{\alpha}(1+\alpha\beta(-1+\theta))+\beta(1-\alpha+x^{\alpha}(1+\alpha\beta\theta))\mathrm{Log}(1+x^{\alpha}))}{1+\theta}.$$

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For $\alpha > 1$, the PDF is unimodal as the function $g'(x)$ changes its sign from positive to negative; as x increases when $\theta, \beta > 0$. In other words, the behavior of $g(x)$ is changed from increasing to decreasing. In addition, the critical point of the PDF is approximately obtained by solving the equation $g'(x) = 0$ numerically at specific initial point of the parameter, the solution is found approximately that, in case of $\alpha > 1$ there is a one critical point at which $g'(x) > 0$, then the PDF is unimodal at this point. On the other hand, when $0 < \alpha \leq 1$ *at* $\beta, \theta > 0$, we have $g'(x) < 0$ which implies that $g(x)$ is monotonically decreasing for all x.

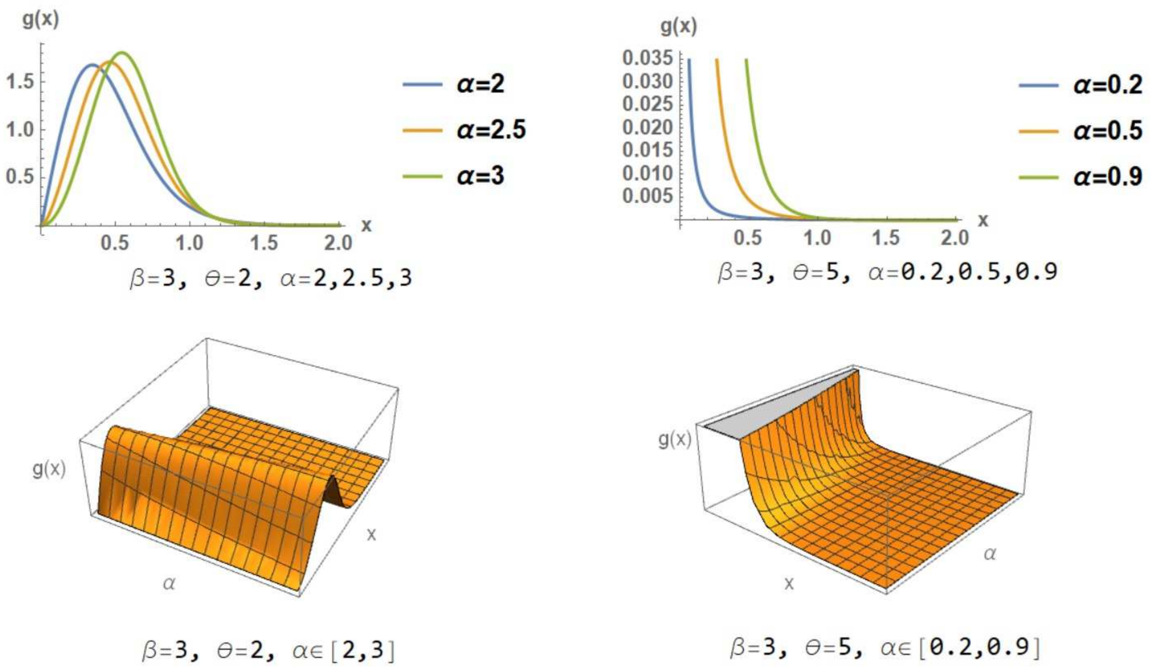

Fig. 1: PDF of the BL distribution

3.2 Hazard rate function

The hazard rate function (HRF) measures the tendency to fail or to die depending on the age reached and it thus plays a key role in classifying lifetime distributions. Generally, hazard rates are monotonic (increasing or decreasing) or non-monotonic (bathtub or inverted bathtub) functions (see [13]).

The hazard rate function of the BL distribution is given by:

$$h(x) = \frac{x^{-1+\alpha}\alpha\beta\theta^2(1+\beta\text{Log}(1+x^\alpha))}{(1+x^\alpha)(1+\theta+\beta\theta\text{Log}(1+x^\alpha))}, \qquad x > 0, \ \alpha, \beta, \theta > 0. \tag{3}$$

The cumulative hazard function of the BL distribution is defined by:

$$H(x) = \beta\theta\text{Log}(1+x^\alpha) + \text{Log}(1+\theta) - \text{Log}(1+\theta+\beta\theta\text{Log}(1+x^\alpha)), \qquad x > 0, \ \alpha, \beta, \theta > 0.$$

Theorem 2. *The behavior of the hazard rate function* h(x) *in Equation* (3) *can be summarized as follows:*

a. h(x) *is decreasing when* $\{0 < \alpha \leq 1 \ \text{ at } \ \beta, \theta > 0\}$.
b. h(x) *is inverted bathtub (IBT) when* $\{\alpha > 1 \ \textit{ at } \ \beta, \theta > 0\}$.

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*Proof.*Since The first derivative of $h(x)$ is obtained as:

$$h^{'}(x)=\frac{x^{-2+\alpha}\alpha\beta\theta^{2}\left(x^{\alpha}(-1+\alpha\beta-\theta)+(-1+\alpha)(1+\theta)-(1+x^{\alpha}-\alpha)\beta(1+2\theta)\text{Log}\,(1+x^{\alpha})-(1+x^{\alpha}-\alpha)\beta^{2}\theta\text{Log}\,(1+x^{\alpha})^{2}\right)}{(1+x^{\alpha})^{2}(1+\theta+\beta\theta\text{Log}\,(1+x^{\alpha}))^{2}}.$$

For $\alpha>1$, $h(x)$ is inverted bathtub (IBT) also named by (IDHR Increasing-Decreasing Hazard Rate), where, $h'(x)$ changes its sign from positive to negative; as x increases at $\alpha>1$ and $\theta,\beta>0$. In other words, the behavior of $h(x)$ is changed from increasing to decreasing. On the other hand, when $0<\alpha\leq 1$ at $\beta,\theta>0$, we have $h^{'}(x)<0$, $\forall x$, which implies that $h(x)$ is monotonically decreasing $\forall x$. The HRF of the BL distribution are displayed in Figure 2 for different values of α,β and θ.

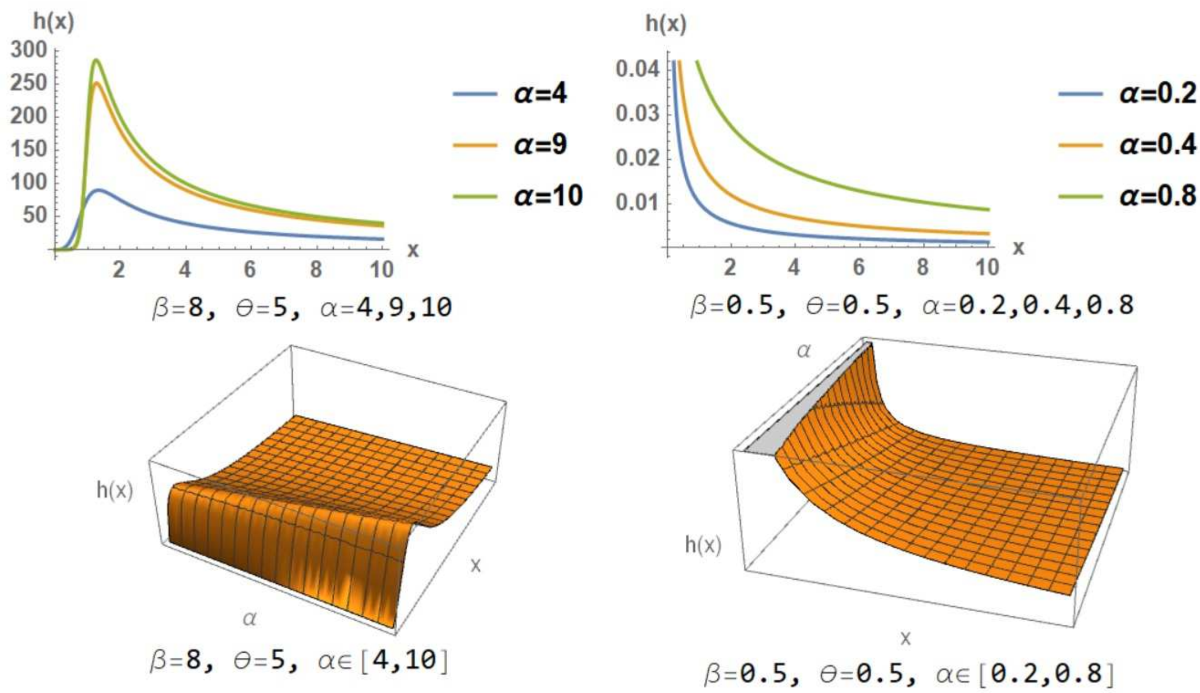

Fig. 2: HRF of the BL distribution

3.3 Reversed hazard rate function

The reversed hazard rate can be defined as the conditional random variable $[t-X|X\leq t]$ which denotes the time elapsed from the failure of a component given that its life is less than or equal to t. This random variable is also called the inactivity time or time since failure. In reliability, it is well known that the mean reversed residual life and ratio of two consecutive moments of reversed residual life characterize the distribution uniquely; for more details see [14, 15].

Using Equations (1) and (2), the reversed hazard function of the BL distribution is defined as

$$r(x)=\frac{-x^{-1+\alpha}\alpha\beta\theta^{2}\,(1+\beta Log\,(1+x^{\alpha}))}{(1+x^{\alpha})\left(\left(1-(1+x^{\alpha})^{\beta\theta}\right)(1+\theta)+\beta\theta Log\,(1+x^{\alpha})\right)},\qquad x>0,\ \ \alpha,\beta,\theta>0. \tag{4}$$

3.4 Residual (reversed) life functions

Residual life and reversed residual life random variables are used extensively in risk analysis. Accordingly, we investigate some related statistical functions, such as survival function mean and variance in connection with the BL distribution. The

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residual life is described by the conditional random variable $R_{(t)} = X - t|X > t,\ t \geq 0$, and defined as the period from time t until the time of failure. Analogously, the reversed residual life can be defined as $\overline{R}_{(t)} = t - X|X \leq t$, which denotes the time elapsed from the failure of a component given that its life is less than or equal to t.

i. **Mean residual life function**

The mean residual life (MRL) function $MRL = E(X - x|X > x)$ of the BL distribution is given by

$$
\begin{aligned}
\boldsymbol{MRL} &= \frac{1}{S(x)} \int_x^{\infty} S(t)\,dt. \\
&= \left(\frac{1}{(1+x^{\alpha})^{-\beta\theta}\,(1+\theta+\beta\theta \mathrm{Log}\left[1+t^{\alpha}\right])} \right) \left(-\sum_{j=0}^{\infty} \binom{-\beta\theta}{j} \times \right. \\
&\left(\frac{-1+j\alpha(1+\theta)+\theta(-1+\alpha\beta(2+\theta))+\beta\theta(-1+j\alpha+\alpha\beta\theta)\left(-\mathrm{L}\left(-1,1,1+j-\frac{1}{\alpha}+\beta\theta\right)+\mathrm{Log}(2)\right)}{(-1+j\alpha+\alpha\beta\theta)^2} \right. \\
&- \frac{x^{1+\alpha j}\left(1+\alpha j+\theta+\alpha j\theta-\alpha\beta\theta+\alpha\beta\theta\,{}_2\mathrm{F}_1\left(1,\frac{1+\alpha j}{\alpha},\frac{1+\alpha+\alpha j}{\alpha},-x^{\alpha}\right)+(1+\alpha j)\beta\theta \mathrm{Log}(1+x^{\alpha})\right)}{(1+\alpha j)^2} \\
&+ \left.\left. \frac{2+2\alpha j+2\theta+2\alpha j\theta-2\alpha\beta\theta+\beta\theta \mathrm{Log}(4)+\alpha j\beta\theta \mathrm{Log}(4)-(1+\alpha j)\beta\theta\psi\left(\frac{1+\alpha j}{2\alpha}\right)+(1+\alpha j)\beta\theta\psi\left(\frac{1+\alpha+\alpha j}{2\alpha}\right)}{2(1+\alpha j)^2} \right) \right)
\end{aligned}
$$

where $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the Digamma function, $L(\lambda,\alpha,s) = \sum_{n=0}^{\infty} \frac{\lambda^n}{(n+\alpha)^s}$ is the Hurwitz-Lerch function, and ${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$ is the Hypergeometric function. The series which used to obtain the MRL function is

$$
\left(1+x^b\right)^{-a} = \begin{cases} \sum_{j=0}^{\infty} \binom{-a}{j} x^{-b(j+a)} & ;|x|^b > 1 \\ \sum_{j=0}^{\infty} \binom{-a}{j} x^{bj} & ;|x|^b < 1 \end{cases}
$$

where $\binom{-a}{j} = (-1)^j \binom{a+j-1}{j}$.

ii. **Residual lifetime function**

The survival function of the residual lifetime $R(t),\ t \geq 0$, for the BL distribution is given by

$$
\boldsymbol{S_{R(t)}(x)} = \frac{S(x+t)}{S(t)} = \frac{(1+(x+t)^{\alpha})^{-\beta\theta}(1+\theta+\beta\theta \mathrm{Log}\left(1+(x+t)^{\alpha}\right))}{(1+t^{\alpha})^{-\beta\theta}(1+\theta+\beta\theta \mathrm{Log}(1+t^{\alpha}))}, \qquad x > 0,
$$

and its PDF is

$$
\boldsymbol{f_{R(t)}(x)} = \frac{(1+t^{\alpha})^{\beta\theta}(t+x)^{-1+\alpha}(1+(t+x)^{\alpha})^{-1-\beta\theta}\alpha\beta\theta^2(1+\beta \mathrm{Log}\left(1+(t+x)^{\alpha}\right))}{1+\theta+\beta\theta \mathrm{Log}(1+t^{\alpha})}.
$$

Consequently, the hazard rate function of $R(t)$ has the following form

$$
\boldsymbol{h_{R(t)}(x)} = \frac{(t+x)^{-1+\alpha}\alpha\beta\theta^2(1+\beta \mathrm{Log}\left(1+(t+x)^{\alpha}\right))}{(1+(t+x)^{\alpha})(1+\theta+\beta\theta \mathrm{Log}\left(1+(t+x)^{\alpha}\right))}.
$$

iii. **Reversed residual life function**

The survival function of the reversed residual lifetime $\boldsymbol{\overline{R}(t)}$ for the BL distribution is given by

$$
\boldsymbol{S_{\overline{R}(t)}(x)} = \frac{F(t-x)}{F(t)} = \frac{(1+\theta)-(1+(t-x)^{\alpha})^{-\beta\theta}(1+\theta+\beta\theta \mathrm{Log}\left(1+(t-x)^{\alpha}\right))}{(1+\theta)-(1+t^{\alpha})^{-\beta\theta}(1+\theta+\beta\theta \mathrm{Log}(1+t^{\alpha}))},\ 0 \leq x \leq t,
$$

hence the probability density function of $\boldsymbol{\overline{R}(t)}$ takes the following form

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$$f_{\overline{R}(t)}(x)=\frac{(1+t^{\alpha})^{\beta\theta}(1+(t-x)^{\alpha})^{-1-\beta\theta}(t-x)^{-1+\alpha}\alpha\beta\theta^{2}(1+\beta\mathrm{Log}(1+(t-x)^{\alpha}))}{(-1+(1+t^{\alpha})^{\beta\theta})(1+\theta)-\beta\theta\mathrm{Log}(1+t^{\alpha})}.$$

Consequently, the hazard rate function of the reversed residual lifetime $\overline{R}(t)$ has the following form

$$h_{\overline{R}(t)}(x)=-\frac{(t-x)^{-1+\alpha}\alpha\beta\theta^{2}\left(1+\beta\mathrm{Log}\left(1+(t-x)^{\alpha}\right)\right)}{\left(1+(t-x)^{\alpha}\right)\left(-\left(-1+\left(1+(t-x)^{\alpha}\right)^{\beta\theta}\right)(1+\theta)+\beta\theta\mathrm{Log}\left(1+(t-x)^{\alpha}\right)\right)}.$$

3.5 Moments and the associated measures

The rth raw moments (about the origin) of the BL distribution by using S(x) is given by

$$\begin{aligned}\mu_r^{'}&=\int_0^{\infty}\frac{1}{1+\theta}rx^{r-1}(1+x^{\alpha})^{-\beta\theta}\left(1+\theta+\beta\theta Log\left(1+x^{\alpha}\right)\right)dx\\&=\frac{\mathrm{r}}{1+\theta}\frac{1}{\alpha\Gamma(\beta\theta)}\Gamma\left(\frac{\mathrm{r}}{\alpha}\right)\Gamma\left(-\frac{\mathrm{r}}{\alpha}+\beta\theta\right)\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{\mathrm{r}}{\alpha}+\beta\theta\right)\right)\right).\end{aligned}$$

The mean and variance of the BL distribution respectively, are as follows

$$\mu=\frac{1}{1+\theta}\frac{1}{\alpha\Gamma(\beta\theta)}\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(-\frac{1}{\alpha}+\beta\theta\right)\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{1}{\alpha}+\beta\theta\right)\right)\right),$$

$$\begin{aligned}\mathrm{v}(\mathrm{x})=&\frac{2}{1+\theta}\frac{1}{\alpha\Gamma(\beta\theta)}\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(-\frac{2}{\alpha}+\beta\theta\right)\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{2}{\alpha}+\beta\theta\right)\right)\right)\\&-\left(\frac{1}{1+\theta}\frac{1}{\Gamma(\beta\theta)}\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(-\frac{1}{\alpha}+\beta\theta\right)\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{1}{\alpha}+\beta\theta\right)\right)\right)\right)^{2}.\end{aligned}$$

The skewness and kurtosis can be obtained using the third and fourth moments about the mean

$$\begin{aligned}\mu_3=&\frac{3\Gamma\left(\frac{3}{\alpha}\right)\Gamma\left(-\frac{3}{\alpha}+\beta\theta\right)\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{3}{\alpha}+\beta\theta\right)\right)\right)}{\alpha(1+\theta)\Gamma(\beta\theta)}\\&-\frac{1}{\alpha^{2}(1+\theta)^{2}\Gamma(\beta\theta)^{2}}6\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(-\frac{2}{\alpha}+\beta\theta\right)\Gamma\left(-\frac{1}{\alpha}+\beta\theta\right)\times\\&\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{2}{\alpha}+\beta\theta\right)\right)\right)\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{1}{\alpha}+\beta\theta\right)\right)\right)\\&+\frac{2\Gamma\left(\frac{1}{\alpha}\right)^{3}\Gamma\left(-\frac{1}{\alpha}+\beta\theta\right)^{3}\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{1}{\alpha}+\beta\theta\right)\right)\right)^{3}}{\alpha^{3}(1+\theta)^{3}\Gamma(\beta\theta)^{3}},\end{aligned}$$

and

$$\begin{aligned}\mu_4=&\frac{4\Gamma\left(\frac{4}{\alpha}\right)\Gamma\left(-\frac{4}{\alpha}+\beta\theta\right)\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{4}{\alpha}+\beta\theta\right)\right)\right)}{\alpha(1+\theta)\Gamma(\beta\theta)}\\&-\frac{1}{\alpha^{2}(1+\theta)^{2}\Gamma(\beta\theta)^{2}}12\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{3}{\alpha}\right)\Gamma\left(-\frac{3}{\alpha}+\beta\theta\right)\Gamma\left(-\frac{1}{\alpha}+\beta\theta\right)\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{3}{\alpha}+\beta\theta\right)\right)\right)\times\\&\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{1}{\alpha}+\beta\theta\right)\right)\right)+\frac{1}{\alpha^{3}(1+\theta)^{3}\Gamma(\beta\theta)^{3}}12\Gamma\left(\frac{1}{\alpha}\right)^{2}\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(-\frac{2}{\alpha}+\beta\theta\right)\Gamma\left(-\frac{1}{\alpha}+\beta\theta\right)^{2}\times\\&\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{2}{\alpha}+\beta\theta\right)\right)\right)\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{1}{\alpha}+\beta\theta\right)\right)\right)^{2}\\&-\frac{3\Gamma\left(\frac{1}{\alpha}\right)^{4}\Gamma\left(-\frac{1}{\alpha}+\beta\theta\right)^{4}\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{1}{\alpha}+\beta\theta\right)\right)\right)^{4}}{\alpha^{4}(1+\theta)^{4}\Gamma(\beta\theta)^{4}},\end{aligned}$$

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where ψ is the Digamma function.

3.6 Shannon entropy measure

The entropy is the measurement of that change or the loss of information. It is used to calculate the average amount of information resulted from a random experiment. For a continuous random variable X with PDF $f(x)$, the Shannon entropy is defined by

$$S_H = -\int_0^\infty f(x)\log f(x)\,dx\,.$$

For the BL distribution Shannon's entropy is defined by

$$\begin{aligned} S_H = &-\frac{\alpha\beta\theta^2}{1+\theta}\int_0^\infty x^{\alpha-1}(1+x^\alpha)^{-1-\beta\theta}(1+\beta\mathrm{Log}(1+x^\alpha))\times \\ &\left(\mathrm{Log}\left(\frac{\alpha\beta\theta^2}{1+\theta}\right)+(\alpha-1)\mathrm{Log}(x)-(1+\beta\theta)\mathrm{Log}(1+x^\alpha)+\mathrm{Log}(1+\beta\mathrm{Log}(1+x^\alpha))\right)dx \\ = &-\mathrm{Log}\left(\frac{\alpha\beta\theta^2}{1+\theta}\right)+\frac{(\alpha-1)}{\alpha}\left(-(1+\theta)(0.577216+\psi(\beta\theta))+\beta\theta\psi'(\beta\theta)\right) \\ &-\frac{\alpha\beta\theta^2}{1+\theta}\int_0^\infty\left(x^{\alpha-1}(1+x^\alpha)^{-1-\beta\theta}(1+\beta\mathrm{Log}(1+x^\alpha))\right)\mathrm{Log}(1+\beta\mathrm{Log}(1+x^\alpha))dx. \end{aligned}$$

Here S_H can be interpreted as the average level of uncertainty (information) in the possible outcomes of a random variable that follows BL distribution. Some numerical values of the Entropy for the BL distribution for several arbitrary parameter values are given in Table 1.

The results for some numerical values for Shannon's entropy seem to indicate that the entropy increases with increasing á, β θ.

Table 1: Entropy for several arbitrary parameter values

Parameters	$\beta=0.5,\theta=0.5$	$\alpha=0.5,\beta=2$		$\alpha=0.3,\theta=1$	
$\alpha\uparrow$	$\uparrow$ Entropy	$\theta\uparrow$	$\uparrow$ Entropy	$\beta\uparrow$	$\uparrow$ Entropy
0.1	−84.460	0.1	−8.476	0.5	−10.430
0.5	−7.878	0.5	−1.135	1.5	0.285
1.5	4.055	1.5	2.091	2.5	3.297
2.0	5.397	3.0	6.918	4.0	5.619
4.0	7.147	4.0	10.490	5.0	6.629
6.0	7.556	5.5	16.270	6.5	7.765
9.0	7.694	7.0	22.450	8.0	8.634

3.7 Order statistics and limiting distributions

The distribution of extreme values plays an important role in statistical applications. In this section the probability and cumulative function of order statistics are introduced and the limiting distribution of minimum and the maximum arising from the BL distribution can then be derived.

i. **Probability and cumulative function of order statistics**
Suppose $X_1, X_2, \ldots, X_n$ is a random sample from the BL distribution. Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the corresponding order statistics. The probability density function and the cumulative distribution function of the kth order statistic of the BL distribution say $Y = X_{k:n}$ are given by

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$$f_y(y) = \frac{n!}{(k-1)!\,(n-k)!} F^{k-1}(y)(1-F(y))^{n-k} f(y)$$
$$= \frac{n!}{(k-1)!\,(n-k)!}\left(1-\frac{(1+y^{\alpha})^{-\beta\theta}(1+\theta+\beta\theta Log(1+y^{\alpha}))}{1+\theta}\right)^{k-1}\times$$
$$\left(\frac{(1+y^{\alpha})^{-\beta\theta}(1+\theta+\beta\theta Log[1+y^{\alpha}])}{1+\theta}\right)^{n-k}\left(\frac{y^{-1+\alpha}(1+y^{\alpha})^{-1-\beta\theta}\alpha\beta\theta^{2}(1+\beta \mathrm{Log}[1+y^{\alpha}])}{1+\theta}\right)$$

Moreover,

$$F_y(y) = \sum_{m=k}^{n}\binom{n}{m} F^{m}(y)(1-F(y))^{n-m}$$
$$= \sum_{m=k}^{n}\binom{n}{m}\left(1-\frac{(1+y^{\alpha})^{-\beta\theta}(1+\theta+\beta\theta Log(1+y^{\alpha}))}{1+\theta}\right)^{m}\left(\frac{(1+y^{\alpha})^{-\beta\theta}(1+\theta+\beta\theta Log[1+y^{\alpha}])}{1+\theta}\right)^{n-m}$$

ii.**Limiting distributions of extreme values**

Let $m_n = X_{1:n} = \min(X_1, X_2, \ldots, X_n)$ and $M_n = X_{n:n} = \max(X_1, X_2, \ldots, X_n)$ be arising from the BL distribution. The limiting distributions of $X_{1:n}$ and $X_{n:n}$ can be obtained by the following theorem.

Theorem 3. *Let m_n and M_n be the minimum and the maximum of a random sample from the BL distribution respectively. Then*

1.
$$\lim_{n\to\infty} p\left(\frac{m_n - a_n}{b_n} \le x\right) = 1-\exp(-x^{\alpha})\ ; x>0\ ,$$

2.
$$\lim_{n\to\infty} p\left(\frac{M_n - c_n}{d_n} \le x\right) = \exp\left(-x^{-\alpha\beta\theta}\right)\ ; x>0\ ,$$

where $a_n = 0 = F^{-1}(0)$, $b_n = F^{-1}\left(\frac{1}{n}\right)$, $c_n = 0$, $d_n = \frac{1}{F^{-1}\left(1-\frac{1}{n}\right)}$.

Proof. 1.Using L'Hospital rule, we have

$$\lim_{\varepsilon\to 0^{+}} \frac{F(F^{-1}(0)+\varepsilon x)}{F(F^{-1}(0)+\varepsilon)} = \lim_{\varepsilon\to 0^{+}} \frac{F(\varepsilon x)}{F(\varepsilon)} = \lim_{\varepsilon\to 0^{+}} \frac{x f(\varepsilon x)}{f(\varepsilon)} = x^{\alpha}.$$

Therefore, by Theorem (8.3.6) of [16], the minimal domain of attraction of the BL distribution is the Weibull distribution, and thus (1) is proved.

2.Using L'Hospital rule, we have

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = \lim_{t\to\infty} \frac{x f(tx)}{f(t)} = x^{-\alpha\beta\theta},$$

therefore, by Theorem (1.6.2) and Corollary (1.6.3) in [17], the maximal domain of attraction of the BL is the Fréchet distribution.

4 Methods of Estimation

The methods of Maximum Likelihood and Moments are explained here in this section as possible methods for estimating the unknown parameters of the BL distribution. Moreover, the asymptotic distribution of $\widehat{\boldsymbol{\Theta}} = (\widehat{\alpha}, \widehat{\beta}, \widehat{\theta})$ are found using the elements of the inverse Fisher information matrix.

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4.1 Maximum likelihood estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from the BL distribution with PDF given by Equation (1). The log-likelihood function $\mathscr{L}(\alpha,\beta,\theta)$ of the BL distribution is given by,

$$\begin{aligned}\mathscr{L}(\alpha,\beta,\theta) &= n\mathrm{Log}(\alpha) + n\mathrm{Log}(\beta) + 2n\mathrm{Log}(\theta) - n\mathrm{Log}(1+\theta) + \sum_{i=1}^{n}\mathrm{Log}(1+\beta\mathrm{Log}(1+x_i^{\alpha})) \\ &+ (-1+\alpha)\sum_{i=1}^{n}\mathrm{Log}(x_i) - (1+\beta\theta)\sum_{i=1}^{n}\mathrm{Log}(1+x_i^{\alpha})\end{aligned} \tag{5}$$

The maximum likelihood estimators (MLEs), say $\widehat{\alpha}, \widehat{\beta}$ *and* $\widehat{\theta}$, are the simultaneous solutions of the equations:

$$\frac{\partial}{\partial\alpha}L(\alpha,\beta,\theta) = \frac{n}{\alpha} + \sum_{i=1}^{n}\mathrm{Log}(x_i) - (1+\beta\theta)\sum_{i=1}^{n}\frac{\mathrm{Log}(x_i)x_i^{\alpha}}{1+x_i^{\alpha}} + \sum_{i=1}^{n}\frac{\beta\mathrm{Log}(x_i)x_i^{\alpha}}{(1+\beta\mathrm{Log}[1+x_i^{\alpha}])(1+x_i^{\alpha})} = 0 \tag{6}$$

$$\frac{\partial}{\partial\beta}L(\alpha,\beta,\theta) = \frac{n}{\beta} - \theta\sum_{i=1}^{n}\mathrm{Log}(1+x_i^{\alpha}) + \sum_{i=1}^{n}\frac{\mathrm{Log}(1+x_i^{\alpha})}{1+\beta\mathrm{Log}(1+x_i^{\alpha})} = 0 \tag{7}$$

$$\frac{\partial}{\partial\theta}L(\alpha,\beta,\theta) = \frac{2n}{\theta} - \frac{n}{1+\theta} - \beta\sum_{i=1}^{n}\mathrm{Log}(1+x_i^{\alpha}) = 0 \tag{8}$$

We used this method - among others - in order to estimate the parameters in Section 7 by solving Equations (6, 7, 8) numerically since we were unable to find an explicit form of the solution.

4.2 Fisher information matrix

For interval estimation of the parameter vector $\boldsymbol{\Theta}=(\alpha,\beta,\theta)^T$ for the BL distribution, the expected Fisher information matrix is $\mathrm{I} = [I_{ij}], i,j = 1,2,3$ as follows:

$$\begin{aligned}I_{11} &= E\left(-\frac{\partial^2 lnf(x)}{\partial\alpha^2}\right) \\ &= \Big\{-((1+x^{\alpha})^2(1+\beta\mathrm{Log}[1+x^{\alpha}])^2 + x^{\alpha}\alpha^2\mathrm{Log}[x]^2(1+x^{\alpha}\beta^2+\beta(-1+\theta)+\beta(2+\beta(-1+2\theta))\times \\ &\quad \mathrm{Log}(1+x^{\alpha}) + \beta^2(1+\beta\theta)\mathrm{Log}(1+x^{\alpha})^2))\Big\}\Big\{(1+x^{\alpha})^2\alpha^2(1+\beta\mathrm{Log}(1+x^{\alpha}))^2\Big\}^{-1}\end{aligned}$$

$$I_{22} = E\left(-\frac{\partial^2 lnf(x)}{\partial\beta^2}\right) = -\frac{1}{\beta^2} - \frac{\mathrm{Log}(1+x^{\alpha})^2}{(1+\beta\mathrm{Log}(1+x^{\alpha}))^2}$$

$$I_{33} = E\left(-\frac{\partial^2 lnf(x)}{\partial\theta^2}\right) = -\frac{2+4\theta+\theta^2}{\theta^2(1+\theta)^2}$$

$$I_{12} = E\left(-\frac{\partial^2 lnf(x)}{\partial\alpha\partial\beta}\right) = -\frac{x^{\alpha}\mathrm{Log}(x)(-1+\theta+2\beta\theta\mathrm{Log}(1+x^{\alpha})+\beta^2\theta\mathrm{Log}(1+x^{\alpha})^2)}{(1+x^{\alpha})(1+\beta\mathrm{Log}(1+x^{\alpha}))^2}$$

$$I_{23} = E\left(-\frac{\partial^2 lnf(x)}{\partial\beta\partial\theta}\right) = -\mathrm{Log}(1+x^{\alpha})$$

$$I_{13} = E\left(-\frac{\partial^2 lnf(x)}{\partial\alpha\partial\theta}\right) = -\frac{x^{\alpha}\beta\mathrm{Log}(x)}{1+x^{\alpha}}$$

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Under regularity conditions, see[18] showed that as $n \to \infty, \sqrt{n}(\widehat{\boldsymbol{\Theta}} - \boldsymbol{\Theta})$ is asymptotically normal 3-variate with (vector) mean zero and covariance matrix I^{-1}.

Asymptotic variances and covariance of the elements of $\widehat{\theta}$ are obtained by:

$$var(\widehat{\alpha}) = \frac{I_{22}I_{33} - I_{23}^2}{n\Delta}, \qquad var\left(\widehat{\beta}\right) = \frac{I_{11}I_{33} - I_{13}^2}{n\Delta}, \qquad var\left(\widehat{\theta}\right) = \frac{I_{11}I_{22} - I_{12}^2}{n\Delta}$$

$$cov\left(\widehat{\alpha},\widehat{\beta}\right) = \frac{I_{13}I_{23} - I_{12}I_{33}}{n\Delta}, \qquad cov\left(\widehat{\alpha},\widehat{\theta}\right) = \frac{I_{12}I_{23} - I_{13}I_{22}}{n\Delta}, \qquad cov\left(\widehat{\beta},\widehat{\theta}\right) = \frac{I_{13}I_{12} - I_{11}I_{23}}{n\Delta},$$

where $\Delta = \det(I)$.

4.3 Method of moments

Consider the random sample of size n drawn from the BL distribution with PDF given by Equation (1); the raw moments and MME equations are

$$\mu_1' = \frac{1}{1+\theta}\left(\frac{\Gamma\left(1+\frac{1}{\alpha}\right)\Gamma\left(-\frac{1}{\alpha}+\beta\theta\right)\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{1}{\alpha}+\beta\theta\right)\right)\right)}{\Gamma(\beta\theta)}\right),$$

$$\mu_2' = \frac{2}{1+\theta}\left(\frac{\Gamma\left(\frac{2}{\alpha}\right)\Gamma\left(-\frac{2}{\alpha}+\beta\theta\right)\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{2}{\alpha}+\beta\theta\right)\right)\right)}{\alpha\Gamma(\beta\theta)}\right),$$

$$\mu_3' = \frac{3}{1+\theta}\left(\frac{\Gamma\left(\frac{3}{\alpha}\right)\Gamma\left(-\frac{3}{\alpha}+\beta\theta\right)\left(1+\theta+\beta\theta\left(\psi(\beta\theta)-\psi\left(-\frac{3}{\alpha}+\beta\theta\right)\right)\right)}{\alpha\Gamma(\beta\theta)}\right).$$

5 Simulation Studies

For generating data from a distribution with cumulative distribution function $F(x)$, the equation $F(x) - u = 0$ is used, where u is an observation from the uniform distribution on $(0,1)$. The simulation experiment was repeated 1000 times each with sample sizes; $n = 30, 50, 70, 90, 100$ and $(\alpha, \beta, \theta) = (1, 0.5, 1), (0.5, 0.7, 0.5), (1, 0.8, 0.5)$. The following measures are computed.

Average bias and the mean square error (MSE) of $\widehat{\gamma}$ of the parameter α, β, θ:

$$\frac{1}{N}\sum_{i=1}^{N}(\widehat{\gamma}-\gamma), \qquad \frac{1}{N}\sum_{i=1}^{N}(\widehat{\gamma}-\gamma)^2$$

Table 2 presents the average bias and the MSE of the estimates. The values of the bias and the MSEs decrease while the sample size increases.

6 Characterization of BL Distribution by Right Truncated Moments

Characterization of the BL distribution using the relation between the right truncated moments and reversed failure rate function is obtained as follows

Theorem 4. *A random variable X, has the BL distribution with parameters $\alpha, \beta, \theta > 0$ if and only if*

$$E\left((1+x^{\alpha})^{1+\beta\theta} \mid X \le x\right) = r(x) \cdot \frac{-x^{\alpha}(\beta-1) + (1+x^{\alpha})\beta \mathrm{Log}(1+x^{\alpha})}{\alpha x^{\alpha-1}(1+x^{\alpha})^{-1-\beta\theta}(1+\beta \mathrm{Log}(1+x^{\alpha}))}, for\ x > 0, \alpha, \beta, \theta > 0$$

where $r(x)$ is the BL reversed hazard function.

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Table 2: Bias and MSE for the parameters $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\theta}$ of BL Distribution

α	β	θ	n	$\widehat{\alpha}$	$\widehat{\beta}$	$\widehat{\theta}$	MSE α	MSE β	MSE θ	Bias α	Bias β	Bias θ
1	0.5	1	30	1.11326	1.50686	2.29702	0.34026	5.93109	11.3877	0.11326	1.00686	1.29702
			50	1.04815	1.20822	2.19220	0.09722	4.00385	11.2662	0.04815	0.70822	1.19220
			70	1.02298	1.19017	2.01609	0.06142	3.41799	07.5323	0.02298	0.69017	1.01609
			90	1.00897	1.11795	1.81763	0.04607	2.59672	07.0129	0.00897	0.61794	0.81763
			100	1.00139	1.05610	1.77458	0.04193	2.59336	05.9473	0.00138	0.55610	0.77458
0.5	0.7	0.5	30	0.79044	1.92450	1.00417	1.98059	7.95570	04.2203	0.21933	1.30203	0.50417
			50	0.60218	1.85390	0.92103	0.55879	6.17626	03.5082	0.10218	1.15390	0.42103
			70	0.52395	1.66835	0.87280	0.03196	5.20743	02.6845	0.02395	0.96834	0.37280
			90	0.51212	1.63587	0.70860	0.02991	4.18871	01.2892	0.01212	0.93586	0.20860
			100	0.51132	1.55122	0.67385	0.02393	3.84555	00.7639	0.01132	0.85122	0.17385
1	0.8	0.5	30	1.38402	2.05993	0.94467	3.21868	7.09281	03.1536	0.38401	1.25993	0.44467
			50	1.14576	2.01801	0.80524	2.43928	6.79011	01.7861	0.14575	1.21801	0.30524
			70	1.05304	1.79985	0.76328	0.17349	6.04904	01.4330	0.05304	1.13735	0.26328
			90	1.04094	1.71287	0.67853	0.09702	4.68233	00.9538	0.04093	0.99635	0.19187
			100	1.01160	1.67523	0.62959	0.07133	4.18001	00.7214	0.01160	0.87746	0.12959

*Proof.*Suppose X has the BL distribution with parameter $\alpha,\beta,\theta>0$.

Then,

$$E\left(f(x)\,|X\le x\right)=\frac{1}{G(x)}\int_0^x f(y)\,g(y)\,dy,$$

where $g(x)$ is the PDF and $G(x)$ is the CDF for the BL distribution defined in Equation (1) and (2).

Let $f(x)=(1+x^{\alpha})^{1+\beta\theta}$, and $r(x)$ is the BL reversed hazard function, then

$$E\left((1+x^{\alpha})^{1+\beta\theta}|X\le x\right)=r(x)\,\frac{1}{g(x)}\int_0^x (1+y^{\alpha})^{1+\beta\theta}g(y)\,dy,$$

$$\int_0^x (1+y^{\alpha})^{1+\beta\theta}g(y)\,dy=\frac{-x^{\alpha}(-1+\beta)+(1+x^{\alpha})\beta\mathrm{Log}(1+x^{\alpha})}{\alpha}$$

$$E\left((1+x^{\alpha})^{1+\beta\theta}|X\le x\right)=r(x)\,\frac{-x^{\alpha}(-1+\beta)+(1+x^{\alpha})\beta\mathrm{Log}(1+x^{\alpha})}{\alpha x^{\alpha-1}(1+x^{\alpha})^{-1-\beta\theta}(1+\beta\mathrm{Log}(1+x^{\alpha}))}$$

Then,

$$w(x)=\frac{-x^{\alpha}(-1+\beta)+(1+x^{\alpha})\beta\mathrm{Log}(1+x^{\alpha})}{\alpha x^{\alpha-1}(1+x^{\alpha})^{-1-\beta\theta}(1+\beta\mathrm{Log}(1+x^{\alpha}))}$$

$$\frac{f(x)}{w(x)}=\frac{\alpha(1+\beta\mathrm{Log}(1+x^{\alpha}))}{x^{1-\alpha}(-x^{\alpha}(-1+\beta)+(1+x^{\alpha})\beta\mathrm{Log}(1+x^{\alpha}))}$$

$$\begin{aligned}\log w(x)=&\log\left(-x^{\alpha}(-1+\beta)+(1+x^{\alpha})\beta\mathrm{Log}(1+x^{\alpha})\right)\ -\log\alpha\\&-(\alpha-1)\log x\ -(-1-\beta\theta)\log(1+x^{\alpha})\ -\log\left(1+\beta\mathrm{Log}(1+x^{\alpha})\right)\end{aligned}$$

Now, $\forall x>0$,

$$\int\frac{\acute{w}(x)-f(x)}{w(x)}\,dx=(1-\alpha)\mathrm{Log}(x)+(1+\beta\theta)\mathrm{Log}(1+x^{\alpha})-\mathrm{Log}[1+\beta\mathrm{Log}(1+x^{\alpha})]$$

$$g(x)=K\exp\left[-\int\frac{\acute{w}(x)-f(x)}{w(x)}\,dx\right]\ =K\,x^{-1+\alpha}(1+x^{\alpha})^{-1-\beta\theta}(1+\beta\mathrm{Log}(1+x^{\alpha})])$$

where $K>0$ is the normalizing constant, $K=\frac{\alpha\beta\theta^2}{1+\theta}$.

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7 Application

A 34 storm events were observed from a water shed. The data are given by [19] which describes the event runoff of the hydrological characteristics of the study storms. The Bl distribution was fitted to these data and compared via goodness of fit criteria among some well-known distributions which fitted to these data. The compared distributions are; Gumbel distribution (G), Gamma Weibull distribution (GW), Modified Weibull distribution (MW), Beta Gamma distribution (BG), Beta Exponentiated Weibull (BEW) distribution see [20].

The data are:
0.9, 0.6, 16.8, 59.3, 2.0, 78.2, 30.7, 146.8, 1.8, 3.4, 1.1, 0.8, 2.5, 6.1, 17.0, 5.1, 216.2, 8.1, 1.6, 2.0, 2.0, 0.8, 0.8, 2.9, 7.3, 13.3, 181.7, 20.5, 24.1, 33.5, 89.1, 7.2, 6.0, 75.9.

First, the model selection is carried out using the AIC (Akaike Information Criterion), the BIC (Bayesian Information Criterion), the CAIC (Consistent Akaike Information Criteria) and the HQIC (Hannan Quinn Information Criterion).

$$\begin{aligned} AIC &= -2L\left(\widehat{\boldsymbol{\Theta}}\right) + 2q, \\ BIC &= -2L\left(\widehat{\boldsymbol{\Theta}}\right) + q\log(n), \\ HQIC &= -2L\left(\widehat{\boldsymbol{\Theta}}\right) + 2q\log(\log(n)), \\ CAIC &= -2L\left(\widehat{\boldsymbol{\Theta}}\right) + \frac{2qn}{(n-q-1)} \end{aligned} \tag{9}$$

where $L\left(\widehat{\boldsymbol{\Theta}}\right)$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, q is the number of parameters, and n is the sample size. Here we let $\boldsymbol{\Theta}$ denotes the parameters, i.e., $\boldsymbol{\Theta} = (\alpha,\ \beta,\ \theta)$.

The global maximum likelihood estimates of BL distribution were obtained as $\widehat{\boldsymbol{\Theta}} = (\widehat{\alpha} = 2.68276, \widehat{\beta} = 0.288275, \widehat{\theta} = 0.912788)$, using the method of Simulated Annealing which is a probabilistic method for approximating the global optimum of a given function which is in this case the Likelihood function. There are similar methods such as Differential Evolution which we also used and got the same solution. We also used Newton's method to solve Equations (6), (7) and (8) numerically, and got the same estimated parameter values. It is known that Newton's method uses initial values to search for a local solution near them that is why we only used it as a confirmation for the global solution obtained using Simulated Annealing method. These methods were applied using the computational program "Mathematica 12" which allows choosing from several numerical methods to maximize an objective function or to solve a nonlinear system of equations. The model with minimum AIC (or BIC, CAIC and HQIC) value is chosen as the best model to fit the data. The results are summarized in Table 3.

From Table 3 , we conclude that the BL distribution is best fitted comparable with the mentioned distributions.

Secondly, for an ordered random sample $X_1, X_2, \ldots, X_n$ from BL distribution $(\alpha,\ \beta,\ \theta)$, the Kolmogorov–Smirnov D_n, Cramérvon Mises W_n^2 and Watson U_n^2 tests statistics are given as follows

$$D_n = \max_i \left(\frac{i}{n} - G\left(x_i, \widehat{\alpha}, \widehat{\beta}, \widehat{\theta}\right), G\left(x_i, \widehat{\alpha}, \widehat{\beta}, \widehat{\theta}\right) - \frac{i-1}{n} \right)$$

$$W_n^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left(G\left(x_i, \widehat{\alpha}, \widehat{\beta}, \widehat{\theta}\right) - \frac{2i-1}{2n} \right)^2$$

$$U_n^2 = W_n^2 - \sum_{i=1}^{n} \left(\frac{G(x_i, \widehat{\alpha}, \widehat{\beta}, \widehat{\theta})}{n} - \frac{1}{2} \right)^2$$

Comparison study using the goodness of fit criteria are shown in Table 4 where we use the Kolmogorov–Smirnov test to test the null hypothesis that the data is drawn from the corresponding distribution.

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Table 3: MLEs and the measures AIC, BIC, HQIC and CAIC

Distribution	Estimates	-L	AIC	BIC	AICC	HQIC	CAIC
BL ($\widehat{\alpha},\widehat{\beta},\widehat{\theta}$)	02.68276 0.288275 0.912787	137.323	280.450	285.224	281.445	282.207	281.445
G ($\widehat{\mu},\widehat{\sigma}$)	12.19425 25.18660	169.561	343.122	346.175	343.509	344.163	343.509
GW ($\widehat{\xi}$, $\widehat{\text{k}}$, $\widehat{\lambda}$)	1.132274 0.238854 0.007631	138.472	282.944	287.523	283.744	284.506	283.744
MW ($\widehat{\lambda}$, $\widehat{\beta}$, $\widehat{\text{k}}$)	1.2(10^{-9}) 0.173121 0.591503	140.339	286.678	291.257	287.478	288.240	287.478
BG ($\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\rho}$, $\widehat{\lambda}$)	1.314604 3.828817 0.145020 99.99009	141.560	291.120	297.225	292.499	293.202	292.499
BEW ($\widehat{\alpha}$, $\widehat{\lambda}$, $\widehat{\text{a}}$, $\widehat{\text{b}}$, $\widehat{\text{c}}$)	22.69658 0.099964 0.031300 0.094967 1.099605	137.777	285.554	293.186	287.697	288.157	287.697

Table 4: Goodness of fit tests

Distribution	D_n	W_n^2	U_n^2	p-value of K-S
BL ($\widehat{\alpha},\widehat{\beta},\widehat{\theta}$)	0.103821	0.071774	0.066923	0.857089
G ($\widehat{\mu}$, $\widehat{\sigma}$)	0.279948	0.715943	0.598105	0.009695
GW ($\widehat{\xi}$, $\hat{k}$, $\widehat{\lambda}$)	0.110768	0.097688	0.088761	0.798360
MW ($\widehat{\lambda}$, $\widehat{\beta}$, $\hat{k}$)	0.138887	0.135028	0.121132	0.528211
BG ($\widehat{\alpha}$, $\widehat{\beta}$, $\widehat{\rho}$, $\widehat{\lambda}$)	0.171925	0.214189	0.169892	0.267338
BEW ($\widehat{\alpha}$, $\widehat{\lambda}$, $\hat{a}$, $\hat{b}$, $\hat{c}$)	0.106974	0.053821	0.052665	0.831350

Table 4 indicates that the test statistics D_n, W_n^2 and U_n^2 have the smallest values for the data set under the BL distribution model regarding the other models except Cramérvon Mises W_n^2 and Watson U_n^2 for Beta exponentiated Weibull. The BL distribution is fitted to data where Kolmogorov-Smirnov is 0.1038 with its p-value of K-S statistics is 0.857 so we accept the null hypothesis. We conclude from Table 4 that the proposed model offers an alternative to the compared distributions.

The quantile-quantile or Q-Q plot is used to check the validity of the distributional assumption for the data.

Figure 3 shows that the data seems to follow the BL distribution reasonably well, except some points on extreme.

8 Conclusion

In this paper, a composition structure is performed to introduce the Burr Lindley (BL) distribution. The BL distribution can exhibit a much more flexible model for lifetime data, presenting decreasing and inverted bathtub hazard rate function. Most statistical and reliability properties are derived and studied.

Simulation schemes are formulated and provide less bias and mean square error as sample size increases for MLEs of BL parameters. Point estimation via MME and MLE methods are done moreover, the Fisher information matrix for interval estimation is studied.

A real data application in hydrology is modeled to the BL distribution and compared with other well-known distributions, to illustrate its performance. Based on goodness of fit criteria, the BL distribution has a superior fitting performance among the compared distributions.

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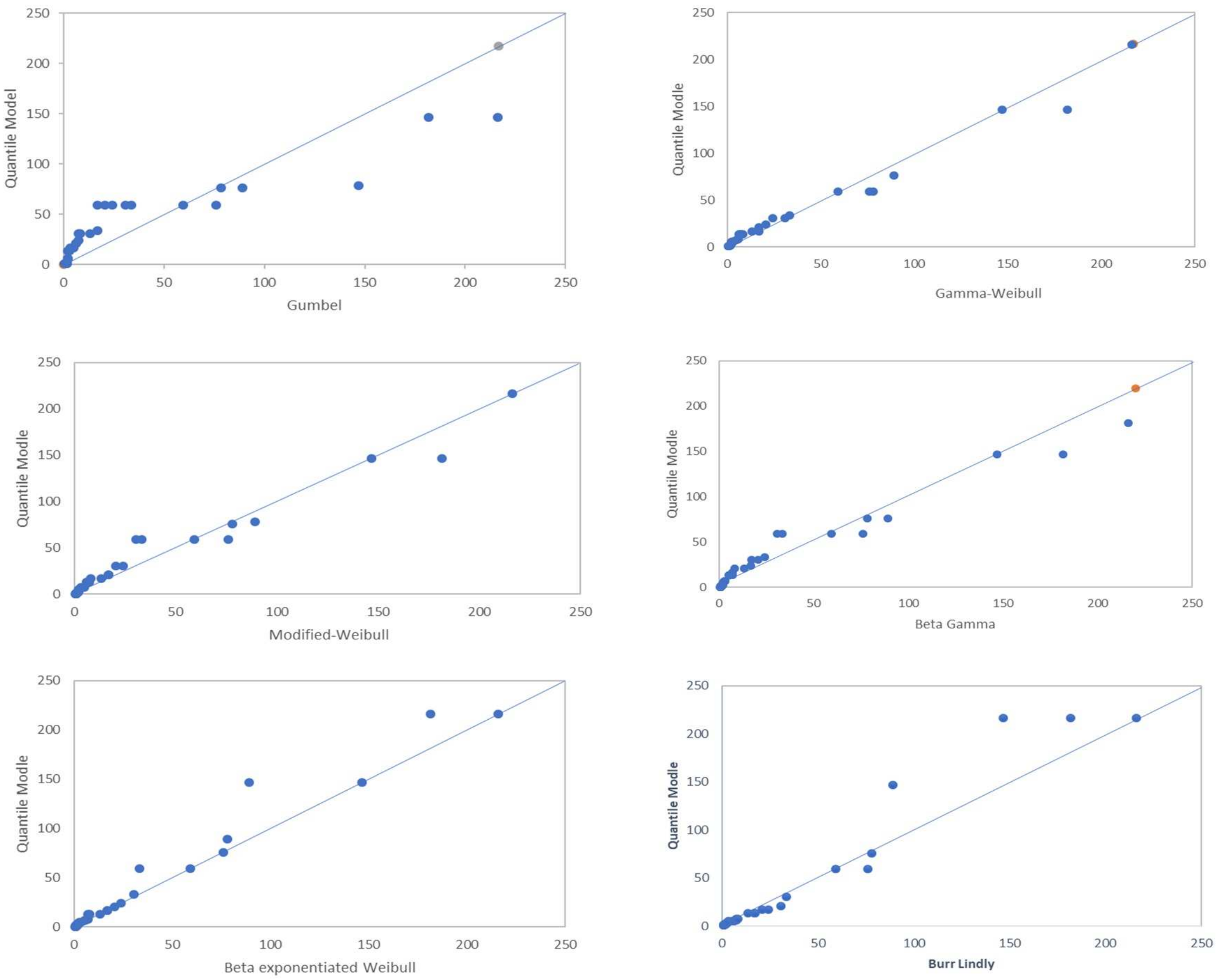

Fig. 3: The Q-Q plot for hydrological data

Declarations

Ethics approval and consent to participate

Not applicable.

Consent for publication

Not applicable.

Availability of data and material

Data set used in this paper is available in the cited reference [19] described in the Application section.

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Competing interests

The authors declare that they have no competing interests.

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Authors' contributions

Prof. W. A. Hassanein introduced the main idea of the paper along with theory and supervision. S. Yehia made the calculations, applications and general style of the paper.

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