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Discrete Mean Inactivity Time Function, Associated Orderings, and Classes of Life Distributions

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Abstract: The concept of mean inactivity time plays an important role in reliability and life testing. In this investigation, based on the comparison of mean inactivity times of a certain function of two lifetime random variables, we introduce and study a new stochastic order. This new order lies between the reversed hazard rate and the mean inactivity time orders. Several characterizations and preservation properties of the new order under reliability operations of monotone transformation, mixture, and shock models are discussed. In addition, a new class of life distributions called strong increasing mean inactivity time is proposed, and some of its reliability properties are investigated. Finally, to illustrate the concepts, some applications in the context of reliability theory are included.

Keywords: Discrete mean inactivity time order, discrete reversed hazard rate order, preservation, mixture, shock models, strong increasing mean inactivity time class.

1 Introduction and Motivation

The mean inactivity time (MIT) function, also known as the mean past lifetime and the mean waiting time functions, is a well-known reliability measure which has applications in many disciplines such as reliability theory, survival analysis, and actuarial studies. Let be a lifetime random variable with distribution function $F(x)$. The MIT function of is defined by

$$m_X(t) = \begin{cases} \sum_{x=0}^{t} \frac{F(x)}{F(t)} & t = 1,2,\dots \\ 0, & t = 0,-1,-2,\dots \end{cases}$$

An interpretation of the MIT is as follows. Consider a situation in which we have realized a device already has failed at time, t, say. The MIT is a useful measure to predict the actual time at which the failure of the device occurs. Apart from applications in reliability engineering, reliability concepts are also heavily used in biomedicine. In cases of diseases that can recur, efficiency of a treatment is determined by analyzing the remission period, i.e., disease free survival time. Often the true remission period is unknown due to an inability to continuously monitor patients because of the high cost and effort involved. In such circumstances, the true remission period can be estimated using an MIT function. Furthermore, in actuarial science, these concepts are used to calculate optimal premiums for life insurance policies.

The MIT function was studied by Kayid and Ahmad [1], and Ahmad et al. [2] to establish several properties of stochastic comparisons based on the MIT function under the reliability operations of convolution and mixture. Badia and Berrade [3] gave an insight into properties of the MIT in mixtures of distributions. What is more important is the inclusion of some interesting characterizations based on the reliability properties of the MIT function. In Finkelstein [4], the MIT function has been used for describing different maintenance policies in reliability. Asadi [5] obtained the MIT of the components of a parallel system. Several properties of the MIT for discrete random variables are studied in Goliforushani and Asadi [6]. The MIT has also been applied to risk theory and econometrics (cf. Eeckhoudt and Gollier [7], Kijima and Ohnishi [8]). Kundu et al. [9] considered the MIT function for characterizations of quite a few distributions, and Kundu and Nanda [10] discussed its higher-order moments. Ortega [11] obtained some characterizations for comparison of lifetime random variables concerning the MIT function. Recently, Izadkhah and Kayid [12] used the harmonic mean average of the MIT function to propose a new stochastic order. It is thus seen that the results which discuss the behavior of the MIT function can be very useful. Therefore, it is not surprising that recently several researchers have devoted their efforts to obtain such results.

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The comparison of the MIT functions of two lifetime random variables provides a stochastic order between those random variables which has been defined and studied in the literature. Let Y and X have distribution functions F and G respectively. The lifetime random variable X is said to be smaller than Y in the MIT order (denoted as $X \leq_{D-MIT} Y$).

$$\frac{\sum_{u=0}^{t} F(u)}{\sum_{u=0}^{t} G(u)} \text{ is non} - \text{increasing in } t \in \mathbb{R}^{+}.$$

The reversed hazard rate (RHR) function of X is given by $r_X(t) = f(t)/F(t)$ where f is the density function of X which is closely related to the MIT function. The random variable X is said to be smaller than in the RHR order (denoted as $X \leq_{RHR} Y$) if

$$\frac{G(t)}{F(t)} \text{ is non} - \text{decreasing in } t > 0.$$

Applications, properties, and interpretations of the RHR order can be found in Shaked and Shanthikumar [13]. Recently, Li and Xu [14] proved that, for any strictly increasing, concave transformation with $\phi(0) = 0$,

$$X \leq_{D-MIT} Y \Rightarrow \phi(X) \leq_{D-CMRL} \phi(Y)$$

Because the inverse of a strictly increasing convex function is strictly increasing and concave, one can also develop that, if is strictly increasing and convex such that $\phi(0) = 0$ then

$$\phi(X) \leq_{D-CMRL} \phi(Y) \Rightarrow X \leq_{D-MIT} Y$$

One important choice for a strictly increasing convex function is $\phi(x) = x^2$, which proposes a new stochastic order called strong mean inactivity time (SMIT) order.

Definition 1:

The lifetime random variable X is said to be smaller than Y in the SMIT order (denoted as $X \leq_{D-SMIT} Y$) if

$$\frac{\sum_{x=t}^{\infty} x\bar{F}(x)}{\bar{F}(t)} \geq \frac{\sum_{x=t}^{\infty} x\bar{G}(x)}{\bar{G}(t)}, \text{ for all } t \in \mathbb{R}^{+}.$$

or equivalently, $X \leq_{D-SMIT} Y$ iff

$$\frac{\sum_{x=t}^{\infty} x\bar{F}(x)}{\sum_{x=t}^{\infty} x\bar{G}(x)} \text{is non} - \text{increasing in} t \in \mathbb{R}^{+}.$$

As demonstrated in Theorem 1 of Section 2 below, the SMIT order lies in the framework of the RHR and the MIT orders. As a result, the study of the SMIT order is meaningful because it throws an important light on the understanding of the properties of the MIT and the RHR orders, and of the relationships among these two orders and other related stochastic orders. Furthermore, the SMIT order enjoys several reliability properties which provide some applications in reliability and survival analysis. On the other hand, statisticians and reliability analysts have shown a growing interest in modeling survival data using classifications of life distributions by means of various stochastic orders.

These categories are useful for modeling situations, maintenance, inventory theory, and biometry. Consider the situation wherein X denotes the risk that the direct insurer faces, and ϕ the corresponding reinsurance contract. One important reinsurance agreement is quota-share treaty defined as $\phi(x) = \alpha X$, for all $\alpha \in (0,1)$. The stochastic comparison between the quata-share treaty and the risk in the MIT order is equivalent to say that $m_X(t)/t$ is non-decreasing in $t > 0$. It is well-known that $m_X(t)/t$ is non-decreasing in t implies that $m_X(t)$ is also non-decreasing in$t > 0$. As a result, a new class of life distributionscalled strong increasing mean inactivity time (SIMIT) is proposed as follows.

Definition 2:

The lifetime random variable X is said to be in the SIMIT class if

$$\frac{m_X(t)}{t} \text{ is non} - \text{decreasing in } t > 0.$$

The purpose of this paper is to achieve two goals. The first goal is to provide some characterizations, preservation results, and applications for the SMIT order. The second goal is to study some reliability properties of the SIMIT class, and to provide some applications of it in the context of reliability.

The paper is organized as follows. In Section 2, some characterizations and implications regarding the SMIT order are provided. Preservation properties under some reliability operations such as monotonic transformation and mixture are discussed in Section 3. In that section, we provide some examples to demonstrate the usefulness of the obtained results in recognizing the SMIT ordered random variables. In Section 4, some reliability properties of the SIMIT class including some characterization and preservation properties under Poisson shock models are discussed. To illustrate the concepts, some applications in the context of reliability theory are included in Section 5. Finally, in Section 6, we give a brief conclusion, and some remarks of the current and future of this research.

In the rest of the paper, we have two important assumptions.

• The non-negative random variables X and Y will have $\overline{F} = 1 - F$ and $\overline{G} = 1 - G$, respectively as their respective survival functions, and as their corresponding densities.

• All integrals, expectations, and derivatives are implicitly assumed to exist wherever they are given.

Proofs are given in the Appendix.

2 Characterizations and Implications

The objective of this section is to concentrate on the relations between SMIT order with other well-known stochastic orders. A characterization result based on the excess lifetime in a renewal process is also discussed. For an exhaustive monograph on the definitions and properties of stochastic orders, we refer to Müller and Stoyan [15], and Shaked and Shanthikumar [13]. For ease of reference, before stating our main results, we present some definitions and basic properties which will be used in the sequel.

Definition 3:

The lifetime random variable is said to be decreasing reversed hazard rate (DRHR) if

$$r_X(t) \text{is non} - \text{increasing in } t > 0$$

Definition 4:

A non-negative function $\beta(x, y)$ is said to be totally positive of order 2 (TP2) in $(x, y) \in \chi \times \gamma$, if

$$\begin{vmatrix} \beta(x_1, y_1) & \beta(x_1, y_2) \\ \beta(x_2, y_1) & \beta(x_2, y_2) \end{vmatrix} \geq 0,$$

For all$x_1 \leq x_2 \in \chi$, and $y_1 \leq y_2 \in \gamma$, in which χ and γ are two real subsets of $\mathbb{R}$.

For more details and properties about the totally positive functions, we refer to Karlin [16].

Definition 5:

A probability vector $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i > 0$ for $i = 1,2, \ldots, n$ is said to be smaller than another probability vector $\underline{\beta} = (\beta_1, \ldots, \beta_n)$ in the sense of discrete likelihood ratio order (denoted as $\underline{\alpha} \leq_{D-DLR} \underline{\beta}$) if

$$\frac{\beta_i}{\alpha_i} \leq \frac{\beta_j}{\alpha_j} \; for \; all \; 1 \leq i \leq j \leq n. \tag{1}$$

The next theorem shows that the SMIT order lies between RHR and MIT orders.

Theorem 1:

Let X and Y be two non-negative random variables. Then

$$X \leq_{D-RHR} Y \Rightarrow X \leq_{D-SMIT} Y, \text{ and}$$
$$X \leq_{D-SMIT} Y \Rightarrow X \leq_{D-MIT} Y$$

In the context of theorem 1 (i), the following counter example shows that $X \leq_{D-SMIT} Y$ does not imply $X \leq_{D-RHR} Y$.

Counter example 1: Let X, and Y have distribution functions F, and G, respectively, which are given by

$$F(x) = \begin{cases} \frac{x}{2}, & 0 \leq x \leq 2 \\ 1, & x > 2 \end{cases}$$

and

$$G(x) = \begin{cases} \frac{x^2}{2}, & 0 \leq x \leq 1 \\ \frac{x^2+2}{6}, & 1 < x \leq 2 \\ 1, & x > 2. \end{cases}$$

Direct calculations give

$$\frac{\sum_{x=0}^{t} x\, G(x)}{\sum_{x=0}^{t} x\, F(x)} = \begin{cases} \frac{3t}{4}, & 0 \leq t \leq 1 \\ \frac{t}{4} + \frac{1}{t} - \frac{1}{2t^3}, & 1 < x \leq 2 \\ \frac{6t^2-9}{6t^2-8}. & t > 2. \end{cases}$$

Which is non-decreasing in t, and hence, $X \leq_{D-SMIT} Y$. In parallel, Nada et al. [17] showed in their counterexample 1 that $X \leq_{D-RHR} Y$ does not hold in this case. In general, the MIT order does not imply the SMIT order. However, the next theorem presents a sufficient condition under which $X \leq_{D-MIT} Y$ impliesX $\leq_{D-SMIT} Y$.

Theorem 2:

Let for all $t \geq 0$

$$\frac{\sum_{x=0}^{t}\sum_{u=0}^{x} F(u)}{F(t)} \leq \frac{\sum_{x=0}^{t}\sum_{u=0}^{x} G(u)}{G(t)}$$

Then $X \leq_{D-MIT} Y \Rightarrow X \leq_{D-SMIT} Y$.

In many reliability problems, it is interesting to study $X_Y = [X - Y|X > Y]$, the residual life of X with a random age.

The residual life at a random time (RLRT) represents the actual working time of the standby unit if X is regarded as the total random life of a warm standby unit with its age. For more details about RLRT, see Stoyan [18]. Yue and Cao [19] gave some stochastic properties of X_Y , and applied them to queuing theory. Also, Li and Zuo [20], and Misra et al. [21] obtained some new stochastic orders and aging properties regarding X_Y.Recently, Cai and Zheng [22] studied the RLRT in the context of generalized aging classes. Suppose that X and Y are s-independent. Then, the distribution function of X_Y , for any $x \geq 0$ is given by

$$P(X_Y \leq x) = \frac{\sum_{y=0}^{\infty}[F(y+x) - F(y)]\, g(y)}{\sum_{y=0}^{\infty} G(y)\, F(y)} \quad (2)$$

Theorem 3:

$X_Y \leq_{D-SMIT} X$ For any Y thas is s-independent of X iff $X_t \leq_{D-SMIT} X$ for all $t \geq 0$.

Let $\{X_n, n = 1,2, \dots\}$ be a sequence of mutually s-independent and identically distributed (i.i.d.) non-negative random variables with common distribution function F. For $n \geq 1$, denote $S_n = \sum_{i=1}^{n} X_i$ as the time of the n-Th arrival, with $S_0 = 0$, let $N(t) = \mathrm{Sup}\{n : S_n \leq t\}$ represent the number of arrivals during the interval$[0, t]$. Then, $N = \{N(t), t \geq 0\}$ is a renewal process with underlying distribution F (see Ross [23]). Let $\gamma(t)$ be the excess lifetime at time $t \geq 0$, that is, $\gamma(t) = S_{N(t)+1} - t$. In this context, we denote the renewal function by $M(t) = E[N(t)]$ which satisfies the well-known fundamental renewal equation

$$M(t) = F(t)\sum_{y=0}^{t} F(t-y)m(y)\,, t \geq 0.$$

According to Barlow and Proschan [24], it holds that, for all $t \geq 0$ and $x \geq 0$.

$$P(\gamma(t) \leq x) = F(t+x) + \sum_{u=0}^{t} F(t+x-u)m(u) - M(t). \quad (3)$$

In the literature, several results have been given to characterize the stochastic orders by the excess lifetime in a renewal process. For more details on definitions and properties, readers are referred to Barlow and Proschan [24]. Chen [25] investigated the relationship between the behavior of the renewal function $M(t) = E(N(t))$ and the aging property of the underlying distribution. Ahmad et al. [26] established some stochastic comparisons of the excess lifetime at different times of a renewal process when the inter-arrival times belong to a nonparametric aging class. Later, Belzunce et al. [27] established comparisons of expected failure times of an age (block) replacement policy to a renewal process with no planned replacements when the lifetime of the unit belongs to the new better than used in expectation class. Next, we will investigate the behavior of the excess lifetime of a renewal process with SMIT inter-arrivals.

Theorem 4:

If $X_t \leq_{D-SMIT} X$ for all $t = 0,1, \dots$, then

$$\gamma(t) \leq_{D-SMIT} \gamma(0) \quad \text{for all } t = 0,1, \dots$$

3 Preservation Properties

In this section, we develop some preservation properties of the SMIT ordering under some reliability operations such as monotone transformation and mixture. Some examples for these results are mentioned that can be useful in recognizing situations when the random variables are SMIT ordered. The next result shows that the SMIT order is preserved under monotone concave transformation.

Theorem 5:
Assume that ϕ is strictly increasing, and concave $\phi(0) = 0$. Then

$$X \leq_{D-SMIT} Y \Rightarrow \phi(X) \leq_{D-SMIT} \phi(Y)$$

In general, the SMIT order is not preserved under mixture. However, the following theorem shows that this order may be preserved under mixture, when appropriate assumptions are satisfied.

Theorem 6:
Let X, Y, and Θ be random variables such that $[X|\Theta = \theta] \leq_{D-SMIT} [Y|\Theta = \theta']$ for all θ, and θ' in the support of Θ. Then

$$X \leq_{D-SMIT} Y.$$

Let $X_i, i = 1, \ldots, n$ be a collection of s-independent random variables. Suppose that F_i is the distribution function of X_i. Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, and $\underline{\beta} = (\beta_1, \ldots, \beta_n)$ be two probability vectors; and let X, and Y be two random variables having the respective distribution functions F, and G defined by

$$F(x) = \sum_{i=1}^{n} \alpha_i F_i(x), and\ G(x) = \sum_{i=1}^{n} \beta_i F_i(x). \tag{4}$$

The next result gives conditions under which X and Y are comparable with respect to the SMIT order.

Theorem 7:
Let $X_1, \ldots, X_n$ be a collection of s-independent random variables with corresponding distribution functions $F_1, \ldots, F_n$, such that:

$$X_1 \leq_{D-SMIT} X_2 \leq_{D-SMIT} \cdots \leq_{D-SMIT} X_n,$$

and let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$, and $\underline{\beta} = (\beta_1, \ldots, \beta_n)$ be such that $\underline{\alpha} \leq_{D-DLR} \underline{\beta}$. Let X, and Y have distribution functions F, and G defined in (4). Then $X \leq_{D-SMIT} Y$.

To demonstrate the usefulness of theorem 7 in recognizing SMIT ordered random variables, we consider the following examples.

Example 1:

Suppose that $X_i, i = 1, \ldots, n$ are s-independent exponential random variables with means $\lambda_i, i = 1, \ldots, n$. Let X and Y be $\underline{\alpha}$ and $\underline{\beta}$ mixtures of X_i. An application of theorem 7 immediately yields $X \leq_{D-SMIT} Y$ for every two probability vector $\underline{\alpha}$ and $\underline{\beta}$ such that $\underline{\alpha} \leq_{D-DLR} \underline{\beta}$ provided that $\lambda_1 \leq \cdots \leq \lambda_n$.

Example 2:

Let $X_{\underline{\lambda}}$, and $X_{\underline{\mu}}$ denote the convolution of n exponential distributions with parameters $\lambda_1, \ldots, \lambda_n$, and $\mu_1, \ldots, \mu_n$, respectively. Without loss of generality, assume that $\lambda_1 \leq \cdots \leq \lambda_n$, and $\mu_1 \leq \cdots \leq \mu_n$. For $0 \leq q \leq p \leq 1$ and $p + q = 1$, according to theorem 7, we have

$$pX_{\underline{\lambda}} + qX_{\underline{\mu}} \leq_{D-SMIT} qX_{\underline{\lambda}} + pX_{\underline{\mu}},$$

Whenever $\lambda_i \leq \mu_i$ for $i = 1,2, \ldots, n$.

4 The SIMIT Class of Life Distributions

This section examines some properties and applications of the SIMIT class of life distributions. The next result provides a sufficient condition for a probability distribution to have the SIMIT property.

Theorem 8:
Let X be a lifetime random variable with the RHR function r_X. If $\beta(x) = xr_X(x)$ is non-increasing in $x > 0$.
Then X is D-SIMIT.

To demonstrate the usefulness of theorem 8 in recognizing parametric D-SIMIT distributions, we consider the following examples.

Example 3:
let X be an exponential random variable with pdf $f(x) = \lambda \exp(-\lambda x)$, for $x > 0$, and $\lambda > 0$. The D-RHR function of X is $r_X(x) = \lambda [\exp(\lambda x) - 1]^{-1}$. One can easily check that $xr_X(x)$ is non-increasing in x, and hence, according to theorem 8, X is D-SIMIT.

Example 4:

let X have a power distribution with distribution function $F(x) = (x/b)^a$, for $0 \leq x \leq b$. The D-RHR function of X is $r_X(x) = a/x$, and $xr_X(x) = a$ is non-increasing in x. According to theorem 8, X is D-SIMIT.

Example 5:

Let X have a reciprocal Weibull distribution with distribution function $F(x) = \exp[-(1/\alpha x)^{\lambda}], x > 0$, and $\sigma, \lambda > 0$. The D-RHR function is $r_X(x) = \lambda \sigma^{-\lambda} x^{-(1+\lambda)}$, and $xr_X(x) = \lambda \sigma^{-\lambda} x^{-\lambda}$ is non-increasing in x. Hence, X is D-SIMIT.

The following theorem presents a characterization property of the D-SIMIT class.

Theorem 9:

A lifetime random variable X is D-SIMIT iff $ZX \leq_{D-MIT} X$, for each random variable Z with $S_Z = [0,1]$, which is s-independent of X.

Theorem 10:

Let ϕ be non-negative function on $[0, \infty)$, strictly increasing, log-convex, and differentiable. Then

$$X \text{ is } D-DRHR \Rightarrow \phi(X) \text{ is } D-SIMIT.$$

Finally, we discuss a sufficient condition for the life distribution of a device subjected to shocks occurring randomly according to Poisson process to belong to the SIMIT class. Suppose that the system is assumed to have an ability to with stand a random number of these shocks, and it is commonly assumed that the number of shocks and the inter-arrival times of shocks are s-independent. Let N denote the number of shocks survived by the system, and let X_j denote the random inter-arrival time between the (j-1) and j-th shocks. Then the lifetime of the system is given by (t) . Therefore, shock models are particular cases of random sums. In particular, if the inter-arrivals are assumed to be s-independent and exponentially distributed (with common parameter), then the distribution function of T can be written as

$$H(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} P_k. \quad t \geq 0. \tag{5}$$

Where $P_k = P[N \leq k]$ for all $k \in \mathbb{N}(\bar{P}_0 = 1)$ (cf. Pellerey [28]).

Definition 6:

The discrete probability distribution P_k is said to be discrete strong increasing in mean inactivity time (D-SIMIT) if

$$\sum_{j=0}^{k-1} P_j / kP_{k-1}, is\ non-decreasing\ in\ k \in \mathbb{N}.$$

Theorem 11:

If $P_k, k \in \mathbb{N}$ is D-SIMIT, then T with distribution function H as given in (5) is SIMIT

5 Reliability Applications

In this section, we discuss some relevant applications in reliability theory involving the SMIT order, and the SIMIT class of life distributions. Suppose that $X_1, X_2, \ldots, X_n$ are i.i.d. life time random variables from SIMIT class, and that $Y_1, Y_2, \ldots, Y_n$ are also i.i.d. life time random variables from SIMIT class. Denote by

$$T_1 = \max\{X_1, X_2, \ldots, X_n\}$$

and

$$T_2 = max\{Y_1, Y_2, \ldots, Y_n\}$$

The lifetimes of the two associated parallel systems. In the following result, we show that if the life times of two parallel systems with i.i.d. components are SMIT ordered, then their components are also SMIT ordered.

Theorem 12:

If $T_1 \leq_{D-SMIT} T_2$, then $X_i \leq_{D-SMIT} Y_i$, for all $i = 1,2, \ldots, n$.

Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables, and let N be a positive integer-valued random variable, which is s-independent of theX_i. Denote by

$$X_{N:N} = \max\{X_1, X_2, \ldots, X_N\}$$

the maximum extreme order statistic.

This random variable arises naturally in reliability theory as the lifetime of a parallel system with the random number of identical components with life times $X_1, X_2, \dots, X_N$. In life testing, if a random censoring is adopted, then the completely observed data constitute a sample of random size N, say $X_1, X_2, \dots, X_N$, where $N > 0$ is a random variable of integer values. In actuarial science, the claims received by an insurer in a certain time interval should also be a sample of random size, and $X_{N:N}$ denotes the largest claim amount of the period. Let $X_{N_i:N_i}$ denote the maximum order statistic among $X_1, X_2, \dots\dots, X_{N_i}$ where N_i is a positive integer-valued random variable which is s-independent from the sequence of $X_1, X_2, \dots$ for each. Below we discuss the SMIT order between two such extreme order statistics. To state and prove Theorem 13, we need to recall the definition of the hazard rate (HR) order (Ahmad and Kayid [29], and Kayid et al. [30]). The random variable X is said to be smaller than Y in the hazard rate order (denoted as $X \leq_{D-HR} Y$) if

$$\frac{\overline{G}(x)}{\overline{F}(x)} \geq \frac{g(x)}{f(x)} \text{for all } x > 0$$

Theorem 13:

Let $N_1 \leq_{D-HR} N_2$, then $X_{N_1:N_1} \leq_{D-SMIT} X_{N_2:N_2}$.

Consider two devices with random life times T_1, and T_2 subject to Poisson shocks occurring randomly with respect to random numbers N_1, and N_2, respectively. Denote $P_k^{[i]} = P(N_i \leq k)$, for all $k \in \mathbb{N}$, and for each $i = 1,2$. It is well-known that, if T_i has the distribution function H_i, then (cf. Pellerey [28]).

$$H_i(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!} P_k^{[i]}. \quad t \geq 0. \qquad (6)$$

The following result presents another application in reliability theory.

$$\frac{\sum_{k=1}^{j-1} k P_{k-1}^{[2]}}{\sum_{k=1}^{j-1} k P_{k-1}^{[1]}}, \text{is non} - \text{decreasing in } j \in \mathbb{N}.$$

Then

$$T_1 \leq_{D-SMIT} T_2$$

In the next example, we discuss the reversed preservation property of the SIMIT class under the formation of the parallel systems.

Example 6:

Let $X_1, X_2, \dots\dots, X_n$ be i.i.d. random lifetimes such that $T = \max\{X_1, X_2, \dots\dots, X_n\}$ belongs to the SIMIT class. Then, we have $aT \leq_{D-MIT} T$, for all $a \in (0,1]$. This result means that, for all $a \in (0,1]$.

$$\max\{aX_1, aX_2, \dots\dots, aX_n\} \leq_{D-MIT} \max\{X_1, X_2, \dots\dots, X_n\}. \qquad (7)$$

In view of theorem 3.2 of Li and Xu [14], and from (7), we deduce that $aX_i \leq_{D-SMIT} X_i, i = 1,2, \dots\dots, n$ for all $a \in (0,1]$. That is $X_i, i = 1,2, \dots\dots, n$ is SIMIT.

As an application of theorem 9 in accelerated life models, we consider the following example.

Example 7:

Consider n units (not necessarily s-independent) with lifetimes $T_i, i = 1,2, \dots\dots, n$. Suppose that the units are working in a common operating environment, which is represented by a random vector $\Theta = (\Theta_1, \Theta_2, \dots, \Theta_n)$, s-independent of $T_1, T_2, \dots, T_n$, and has an effect on the units of the form

$$X_i = \frac{T_i}{\Theta_i}, i = 1,2, \dots, n. \qquad (8)$$

If Θ has support on $(1, \infty)^n$, then the components are working in a harsh environment; and, if they have support on $(0,1)^n$, then the components are working in a gentler environment (see Ma [31]). In a harsh environment, let T_j be SIMIT for some $j = 1,2, \dots\dots, n$. Then, theorem 9 tells that, for each Z with support on $[0,1]$, which is s-independent of T_j, we have $ZT_j \leq_{D-MIT} T_j$. Thus, $Z = 1/\Theta_j$ implies that $T_j/\Theta_j \leq_{D-MIT} T_j$. With a similar discussion, in a gentler environment, if X_j is SIMIT for some $j = 1,2, \dots\dots, n$, then we conclude that $T_j \leq_{D-MIT} X_j$.

6 Conclusion

Due to economic consequences and safety issues, it is necessary for the industry to perform systematic studies using reliability concepts. There exist plenty of scenarios where a statistical comparison of reliability measures is required in both reliability engineering and biomedical fields. In this paper, we have proposed a new stochastic order based on the MIT function called strong mean inactivity time (SMIT) order. The relationship of this new stochastic order with other well-known stochastic orders such as the RHR and the MIT orders are discussed. It was shown that the SMIT order lies in the frame work of the RHR and the MIT orders, and hence it enjoys several reliability properties which provide several applications in reliability and survival analysis. We discussed several characterization and preservation properties of this new order under reliability operations of monotone transformation, mixture, and shock models. To enhance the study, we proposed a new class of life distributions called strong increasing mean inactivity time (SIMIT) class. Several reliability properties of the new class as well as a number of applications in the context of reliability and survival analysis are included. Our results provide new concepts and applications in reliability, statistics, and risk theory. Further properties and applications of the new stochastic order and the new proposed class can be considered in the future of this research. In particular, the following topics are interesting, and still remain as open problems.

(i) Closure properties of the SMIT order and the SIMIT class under convolution, and coherent structures.

(ii) Discrete version of the SMIT orders, and enhances the obtained results related to the D-SIMIT class

(iii) Testing exponentially against the SIMIT class.

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Appendix

Proof of theorem 1:

(i) Note that $X \leq_{RHR} Y$ implies $F(x)/G(x)$ is non-increasing in x, or equivalently

$$\left[\frac{F(x)}{F(t)} - \frac{G(x)}{G(t)}\right] \geq 0, for\ all\ x \leq t$$

This simply gives

$$\sum_{x=0}^{t} x\left[\frac{F(x)}{F(t)} - \frac{G(x)}{G(t)}\right] \geq 0, for\ all\ t > 0$$

Which means $X \leq_{D-SMIT} Y$.

(ii) Suppose that X, and Y have the MIT functions m_X, and m_Y, respectively. Note that for all $t \geq 0$ we get

$$\begin{aligned} m_X(t) - m_Y(t) &= \sum_{x=0}^{t}\left[\frac{F(x)}{F(t)} - \frac{G(x)}{G(t)}\right]dx \\ &= \sum_{x=0}^{t}\frac{1}{x}d\left[\sum_{x=0}^{x}\left(\frac{uF(u)}{F(t)} - \frac{uG(u)}{G(t)}\right)du\right] \\ &= \sum_{x=0}^{\infty} h(x)dW_t(x). \end{aligned}$$

Where $h(x) = 1/x$, is a non-negative non-increasing function, $dW_t(x) = w_t(x)dx$, and

$$w_t(x) = \left[\frac{xF(x)}{F(t)} - \frac{xG(x)}{G(t)}\right]I[x \leq t],$$

Where $I[x \leq t]$ stands for the indicator function of the set$[x \leq t]$. For all $s > t > 0$, we have

$$\sum_{x=0}^{s} dW_t(x) = \sum_{x=0}^{t}\left[\frac{xF(x)}{F(t)} - \frac{xG(x)}{G(t)}\right]dx \geq 0.$$

Note also that $X \leq_{D-SMIT} Y$ implies that, for all $t \geq 0$,

$$\frac{\sum_{x=0}^{t} xF(x)dx}{\sum_{x=0}^{t} xG(x)dx} \geq \frac{F(t)}{G(t)}. \qquad (9)$$

In addition, from (1) it holds that, for all $t \geq s > 0$,

$$\frac{\sum_{x=0}^{s} xF(x)dx}{\sum_{x=0}^{s} xG(x)dx} \geq \frac{\sum_{x=0}^{t} xF(x)dx}{\sum_{x=0}^{t} xG(x)dx}. \qquad (10)$$

Combining (9) and (10), one gets, for all $t \geq s > 0$,

$$\frac{\sum_{x=0}^{s} xF(x)dx}{\sum_{x=0}^{s} xG(x)dx} \geq \frac{F(t)}{G(t)},$$

Which provides that, for all $t \geq s > 0$,

$$\sum_{x=0}^{s} dW_t(x) = \sum_{x=0}^{s}\left[\frac{xF(x)}{F(t)} - \frac{xG(x)}{G(t)}\right]dx \geq 0.$$

Therefore, $X \leq_{D-SMIT} Y$ implies that$\sum_{x=0}^{s} dW_t(x) \geq 0$, for all $s, t \geq 0$. Finally, appealing to Lema 7.1(b) of Barlow and Proschan [24], it is concluded that $\sum_{x=0}^{\infty} h(x)dW_t(x) \geq 0$, for all $t > 0$, and hence the proof is completed.

Proof of theorem 2:

we can write, for all $t > 0$,

$$\frac{\sum_{x=0}^{t} xF(x)}{F(t)} = \frac{\sum_{x=0}^{t}\sum_{u=0}^{x} F(x)}{F(t)}$$
$$= \frac{\sum_{u=0}^{t}\sum_{x=u}^{t} F(x)}{F(t)}$$
$$= tm_X(t)\frac{\sum_{u=0}^{t}\sum_{x=0}^{u} F(x)}{F(t)}$$

Similarly,

$$\frac{\sum_{x=0}^{t} xG(x)}{G(t)} = tm_Y(t)\frac{\sum_{u=0}^{t}\sum_{x=0}^{u} G(x)}{G(t)}$$

Therefore, using the assumptions, for all $t > 0$,

$$\frac{\sum_{x=0}^{t} xF(x)\,dx}{F(t)} = \frac{\sum_{x=0}^{t} xG(x)}{G(t)}$$
$$= t[m_X(t) - m_Y(t)] + \frac{\sum_{x=0}^{x}\sum_{u=0}^{x} G(u)}{G(t)}$$
$$-\frac{\sum_{x=0}^{t}\sum_{u=0}^{x} F(u)}{F(t)} \geq t[m_X(t) - m_Y(t)] \geq 0,$$

This completes the proof.

Proof of theorem 3:

First, Let $X_t \leq_{D-SMIT} X$ for all $t \geq 0$. It follows that, for all $s > 0$,

$$\sum_{x=0}^{s} x[F(t+x) - F(t)] \geq \frac{[F(t+s) - F(t)]}{F(s)}\sum_{x=0}^{s} xF(x)\,dx. \qquad (11)$$

Now, we have

$$\frac{\sum_{t=0}^{\infty}\sum_{x=0}^{s} x[F(t+x) - F(t)]\,g(t)}{\sum_{t=0}^{\infty}[F(t+s) - F(t)]g(t)}$$
$$\geq \frac{\sum_{t=0}^{\infty}\left[\frac{[F(t+s) - F(t)]}{F(s)}\sum_{x=0}^{\infty} xF(x)\right]g(t)}{\sum_{t=0}^{\infty}[F(t+s) - F(t)]g(t)} = \frac{\sum_{x=0}^{s} xF(x)\,dx}{F(s)}, for\ any\ s > 0.$$

In view of (2), this gives$X_Y \leq_{D-SMIT} X$. On the other hand, suppose that $X_Y \leq_{D-SMIT} X$ holds for any non-negative random variable Y. Then $X_t \leq_{D-SMIT} X$, for all $t \geq 0$ follows by taking Y as a degenerate variable.

Proof of theorem 4:
First note that for any $t \geq 0$ and $s > 0$, based upon (2), (3), (11) we have

$$\sum_{x=0}^{s} xP(\gamma(t) \leq x) = \sum_{x=0}^{s} [xF(t+x) - xF(t)]$$
$$+ \sum_{u=0}^{t} \sum_{x=0}^{s} [xF(t-u+x) - xF(t-u)]\, m(u)$$
$$\geq \sum_{x=0}^{s} x[F(t+x) - F(t)]\, dx$$
$$+ \sum_{u=0}^{t} \left[\frac{F(t-u+s) - F(t-u)}{F(s)} \sum_{x=0}^{s} xF(x)\right] m(u)$$
$$= \sum_{x=0}^{s} x[F(t+x) - F(t)]$$
$$+ \frac{\sum_{x=0}^{s} xF(x)}{F(s)} [P(\gamma(t) \leq s) - F(t+s) + F(t)]$$
$$\geq \frac{\sum_{x=0}^{s} xF(x)}{F(s)} [F(t+s) - F(t)]$$
$$+ \frac{\sum_{x=0}^{s} xF(x)}{F(s)} [P(\gamma(t) \leq s) - F(t+s) + F(t)] = \frac{\sum_{x=0}^{s} xF(x)}{F(s)} P(\gamma(t) \leq s).$$

Hence, it holds that, for all $t \geq 0$ and $s > 0$,

$$\frac{\sum_{x=0}^{s} xP(\gamma(t) \leq x)}{P(\gamma(t) \leq s)} \geq \frac{\sum_{x=0}^{s} xF(x)}{F(s)},$$

That is, $\gamma(t) \leq_{D-SMIT} \gamma(0)$ for all $t \geq 0$.

Proof of theorem 5:
Without loss of generality, assume that ϕ is differentiable, and denote its first derivative by ϕ. $X \leq_{D-SMIT} Y$implies that, for all $t > 0$,

$$\sum_{x=0}^{\phi^{-1}(t)} \left[\frac{xF(x)}{F(\phi^{-1}(t))} - \frac{xG(x)}{G(\phi^{-1}(t))}\right] \geq 0.$$

On the other hand, $\phi(X) \leq_{D-SMIT} \phi(Y)$ iff, for all $t \geq 0$,

$$\frac{\sum_{x=0}^{t} xP(\phi(X) \leq x)}{P(\phi(X) \leq t)} \geq \frac{\sum_{x=0}^{t} xP(\phi(Y) \leq x)}{P(\phi(Y) \leq t)},$$

Which is equivalent to, for all $t \geq 0$,

$$\sum_{x=0}^{\phi^{-1}(t)} v(x) \left[\frac{xF(x)}{F(\phi^{-1}(t))} - \frac{xG(x)}{G(\phi^{-1}(t))}\right] \geq 0,$$

Where $v(x) = \phi(x)/x$. Its well-known that, if ϕ is non-negative and concave with $\phi(0) = 0$, then $\phi(x)/x$ is non-increasing. Thus, due to the assumption, $v(x)$ is the product of two non-negative non-increasing functions, and hence $v(x)$ is non-increasing. Finally, Lemma 7.1(b) of Barlow and Proschan [24] can be used to conclude the proof.

Proof of theorem 6:
Select θ and θ'in the support of Θ. Let $F(.|\theta), G(.|\theta), F(.|\theta'), and\ G(.|\theta')$ be the distribution functions of $(X|\Theta = \theta), (Y|\Theta = \theta), (X|\Theta = \theta'), and\ (Y|\Theta = \theta')$, respectively. The proof is similar to that of theorem 1.B.8 in Shaked and Shanthikumar [13]. It is sufficient to show that, for each $a \in (0,1)$, and for all $t > 0$, we have

$$\frac{a\sum_{u=0}^{t} uF(u|\theta) + (1-\alpha)\sum_{u=0}^{t} uF(u|\theta')}{aF(t|\theta) + (1-\alpha)F(t|\theta')}$$
$$\geq \frac{a\sum_{u=0}^{t} uG(u|\theta) + (1-\alpha)\sum_{u=0}^{t} uG(u|\theta')}{aG(t|\theta) + (1-\alpha)G(t|\theta')}$$

This is an inequality of the form

$$\frac{a+b}{c+d} \geq \frac{w+x}{y+z}$$

Where all eight variables are non-negative, and by the assumptions of the theorem they satisfy

$$\frac{a}{c} \geq \frac{w}{y}, \frac{a}{c} \geq \frac{x}{z}, \frac{b}{d} \geq \frac{w}{y}, and\ \frac{b}{d} \geq \frac{x}{z}.$$

It is easy to verify that the latter four inequalities imply the former one, completing the proof of the theorem.

Proof of theorem 7:

Because of (1), we need to establish that, for all $0 < x < y$,

$$\frac{\sum_{u=0}^{\infty}(x-u)\sum_{i=1}^{n}\beta_i F_i(x-u)}{\sum_{u=0}^{\infty}(x-u)\sum_{i=1}^{n}\alpha_i F_i(x-u)} \leq \frac{\sum_{v=0}^{\infty}(y-v)\sum_{i=1}^{n}\beta_i F_i(y-v)}{\sum_{v=0}^{\infty}(y-v)\sum_{i=1}^{n}\alpha_i F_i(y-v)} \quad (12)$$

After simple calculations, (12) can be written in the form

$$\sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\left\{\beta_j\alpha_i\sum_{u=0}^{\infty}(x-u)F_j(x-u)\times\sum_{u=0}^{\infty}(y-v)F_i(y-v)\right.$$

$$\leq \sum_{\substack{i=1\\ i\neq j}}^{n}\sum_{j=1}^{n}\left\{\beta_j\alpha_i\sum_{u=0}^{\infty}(y-u)F_j(y-u)\times\sum_{u=0}^{\infty}(x-v)F_i(x-v)\right.$$

$$\sum_{\substack{i=1\\ i<j}}^{n}\sum_{j=1}^{n}\left\{\beta_j\alpha_i\sum_{u=0}^{\infty}(x-u)F_j(x-u)\times\sum_{u=0}^{\infty}(y-v)F_i(y-v)\right.$$

$$+\beta_i\alpha_i\sum_{u=0}^{\infty}(x-u)F_i(x-u)\times\sum_{u=0}^{\infty}(y-v)F_j(y-v)$$

$$\leq \sum_{\substack{i=1\\ i<j}}^{n}\sum_{j=1}^{n}\left\{\beta_j\alpha_i\sum_{u=0}^{\infty}(y-v)F_j(y-v)\times\sum_{u=0}^{\infty}(x-u)F_i(x-u)\right.$$

$$+\beta_i\alpha_i\sum_{u=0}^{\infty}(y-v)F_i(y-v)\times\sum_{u=0}^{\infty}(x-u)F_j(x-u)$$

Now, for each fixed pair (i,j) with $i < j$, we have

$$\left\{\beta_j\alpha_i\sum_{u=0}^{\infty}(y-v)F_j(y-v)\times\sum_{u=0}^{\infty}(x-u)F_i(x-u)\right.$$

$$+\beta_i\alpha_i\sum_{u=0}^{\infty}(y-v)F_i(y-v)\times\sum_{u=0}^{\infty}(x-u)F_j(x-u)$$

$$-\beta_j\alpha_i\sum_{u=0}^{\infty}(x-u)F_j(x-u)\times\sum_{u=0}^{\infty}(y-v)F_i(y-v)$$

$$+\beta_i\alpha_i\sum_{u=0}^{\infty}(x-u)F_i(x-u)\times\sum_{u=0}^{\infty}(y-v)F_j(y-v)$$

$$=\left(\beta_j\alpha_i-\beta_i\alpha_i\right)\sum_{u=0}^{\infty}(y-v)F_j(y-v)\times\sum_{u=0}^{\infty}(x-u)F_i(x-u)$$

$$-\sum_{u=0}^{\infty}(x-u)F_j(x-u)\times\sum_{u=0}^{\infty}(y-v)F_i(y-v)$$

Which is non-negative because both terms are non-negative by the assumptions. This completes the proof.

Proof of theorem 8:

It suffices to verify that $\sum_{x=0}^{t}F(x)\,dx/tF(t)$ is non-decreasing in $t > 0$. Note that

$$F(t) = \frac{1}{t}\sum_{x=0}^{t}\left(F(x)+xf(x)\right)dx; t > 0.$$

Define now

$$K(i,t)=\sum_{u=0}^{\infty}\phi(i,x)\psi(x,t)dx, i=1,2,$$

Where $\psi(x,t)=I[x\leq t]$, and

$$\phi(i,x)=\begin{cases}xf(x)+F(x), & if\ i=1\\ F(x), & if\ i=2,\end{cases}$$

Due to the assumption, $\phi(i,x)$ is TP_2 in(i,x), for $i=1,2$, and $x>0$. Also, it is easy to observe that $\psi(x,t)$ is TP_2 in(x,t), for $x>0$, and $t>0$. From the general composition theorem of Karlin [16] it is deduced that $K(i,t)$ is TP_2 in(i,t), for $i=1,2$,and $t>0$; and thus the proof is completed.

Proof of theorem 9:

To prove the "if" part, note that $m_{ZX}(t)=zm_X(t/z)$, for each $z\in(0,1]$, and any $t>0$. Then, put $Z=z$, implying $ZX\leq_{D-MIT}X$, for all $z\in(0,1]$, which means X is SIMIT. For the "only if" part, assume that Z has distribution function G. from the assumption, and the well-known Fubini's theorem, for all $x>0$, it follows that

$$\begin{aligned}m_{ZX}(x)&=\frac{\sum_{u=0}^{x}P(ZX\leq u)}{P(ZX\leq u)}\\&=\frac{\sum_{u=0}^{x}\sum_{z=0}^{1}F\left(\frac{u}{z}\right)g(z)du}{\sum_{z=0}^{1}F\left(\frac{x}{z}\right)g(z)}\\&=\frac{\sum_{z=0}^{1}\sum_{u=0}^{x}F\left(\frac{u}{z}\right)g(z)}{\sum_{z=0}^{1}F\left(\frac{x}{z}\right)g(z)}\\&=\frac{\sum_{z=0}^{1}zm_X\left(\frac{x}{z}\right)F\left(\frac{x}{z}\right)g(z)}{\sum_{z=0}^{1}F\left(\frac{x}{z}\right)g(z)}\\&\geq\frac{\sum_{z=0}^{1}m_X(x)F\left(\frac{x}{z}\right)g(z)}{\sum_{z=0}^{1}F\left(\frac{x}{z}\right)g(z)}=m_X(x)\end{aligned}$$

That is $ZX\leq_{D-MIT}X$.

Proof of theorem 10:

It suffices to prove that

$$\begin{aligned}\frac{\sum_{y=0}^{t}F\left(\phi^{-1}(y)\right)}{tF\left(\phi^{-1}(y)\right)}&=\frac{\sum_{y=0}^{\phi^{-1}t}\phi'(y)F(y)}{tF\left(\phi^{-1}(t)\right)}\\&=\frac{\sum_{y=0}^{x}\phi'(y)F(y)}{\phi(x)F(x)}, x=\phi'(t)\geq 0\end{aligned}$$

is non-decreasing in $t>0$. We know that

$$\phi(x)F(x)=\sum_{y=0}^{x}[\phi'(y)F(y)+\phi(y)F(y)]\,dy.$$

Define

$$K(i,x)=\sum_{y=0}^{\infty}\varphi(i,y)\psi(y,x)\,dy, \qquad (13)$$

for$i=1,2$, and $x>0$, where

$$\varphi(i,y)=\begin{cases}\phi'(y)F(y)+\phi(y)F(y), & if\ i=1\\ \phi'(y)F(y), & if\ i=2,\end{cases}$$

and $\psi(y,x)=I[y\leq x]$. Easily, $\psi(y,x)$ is TP_2 in(y,x).By assumption, f/F and ϕ/ϕ' are non-increasing and non-negative. Hence $\varphi(i,y)$ is TP_2 in(i,y). On applying the general composition theorem of Karlin [16] to the identity given in (13), $K(i,t)$ is TP_2 in $(i,t)\in\{1,2\}\times(0,\infty)$. This completes the proof.

Proof of theorem 11:
It is enough to show that $\sum_{x=0}^{t} H(x)dx/tH(t)$ is non-decreasing in t. To this end, first we get, for all $t>0$, that

$$\sum_{x=0}^{t} H(x) = \sum_{x=0}^{t}\sum_{j=0}^{\infty}\frac{e^{-\lambda x}(\lambda x)^j}{j!}P_j$$
$$= \sum_{j=0}^{\infty} P_j \sum_{x=0}^{t}\frac{e^{-\lambda x}(\lambda x)^j}{j!}$$
$$= \sum_{j=0}^{\infty}\frac{P_j}{\lambda}\left[1-\sum_{k=0}^{j}\frac{e^{-\lambda t}(\lambda t)^k}{k!}\right]$$
$$= \frac{1}{\lambda}\sum_{j=0}^{\infty}\sum_{k=j+1}^{\infty}\frac{e^{-\lambda t}(\lambda x)^k}{k!}P_j$$
$$= \frac{1}{\lambda}\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\frac{e^{-\lambda t}(\lambda x)^k}{k!}P_j$$
$$= \frac{1}{\lambda}\sum_{k=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\sum_{j=0}^{k-1} P_j$$

Note then that

$$tH(t) = \frac{1}{\lambda}\sum_{k=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}kP_{k-1},$$

For each positive integer k. Hence, T is SIMIT if

$\frac{\sum_{k=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}\sum_{j=0}^{k-1} P_j}{\sum_{k=1}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!}kP_{k-1}}$ is non-decreasing in t, or equivalently if

$$\Psi(i,t) = \sum_{k=1}^{\infty}\Phi(i,k)\frac{(\lambda t)^k}{k!}e^{-\lambda t}\ is\ TP_2\ in\ (i,t),$$

for $i \in \{1,2\}$, and $t \in (0,\infty)$, where

$$\Phi(i,k) = \begin{cases} kP_{k-1}, & if\ i=1 \\ \sum_{j=0}^{k-1} P_j, & if\ i=2, \end{cases}$$

By the assumption, $\Phi(i,k)$ is TP$_2$ in(i,k), for $i \in \{1,2\}$, and $k \in \mathbb{N}$. It is also evident that $e^{-\lambda t}(\lambda t)^k/k!$ is TP$_2$ in(k,t), for $k \in \mathbb{N}$, and $t \in (0,\infty)$. The result now follows from general composition theorem of Karlin [16].

Proof of theorem 12:
Let $T_1 \leq_{D-MIT} T_2$ hold. Then, we have

$$\sum_{x=0}^{t} x\{F^n(x)G^n(t) - F^n(t)G^n(x)\} \geq 0, for\ all\ t>0.$$

Due to fact that
$F(x)G(t) - F(t)G(x) = [F^n(x)G^n(t) - F^n(t)G^n(x)]h(x).$
where $h(x) = \left[\sum_{i=1}^{n}\left(F(x)G(t)\right)^{n-i}\left(F(t)G(x)\right)^{i-1}\right]^{-1}$, and applying Lema 7.1(b) of Barlow and Proschan [24], it obtains that, for all $t>0$,

$$\sum_{x=0}^{t} x\{F(x)G(t) - F(t)G(x)\} \geq 0,$$

which means that $X_i \leq_{D-SMIT} Y_i$, $i = 1,2,\dots,n$.

Proof of theorem 13:
First, notice that $N_1 \leq_{D-HR} N_2$ indicates the hazard rate order between N_1 and N_2. Denote by $H_{N_i:N_i}$ the distribution function of$X_{N_i:N_i}$.

$$H_{N_i:N_i}(x) = \sum_{k=1}^{\infty} F^k(x) p_k^{[i]}, for\ all\ x > 0,$$

where F is the common distribution of the X_i and $p_k^{[i]} = P(N_i = k)$, for each $k \in \mathbb{N}$, is the pmf of $N_i, i = 1,2$, that

$$\psi(t, i) = \sum_{x=0}^{t} xH_{N_i:N_i}(x)dx$$
$$= \sum_{k=1}^{\infty} \phi(t, k)\tau(k, i),$$

where $\phi(t, k) = \sum_{k=1}^{\infty} xF^k(x)dx$, and $\tau(k, i) = p_k^{[i]}$. Denote $g(k, i) = \sum_{j=k}^{\infty} p_k^{[i]}$, for each $k \in \mathbb{N}$ and $i = 1,2$. It can easily be checked that (cf. Shaked and Shanthikumar [13]) if $N_1 \leq_{D-HR} N_2$ then $g(k, i)$ is TP2 in$(k, i) \in \mathbb{N} \times \{1,2\}$. On the other hand, $\phi(t, k)$ is TP2 in$(t, k) \in \mathbb{R}^+\mathbb{N}$. Appealing to Lemma 2.1 in Ortega [11] gives $\psi(t, i)$ is TP2 in$(t, i) \in \mathbb{R}^+ \times \{1,2\}$, which is equivalent to $H_{N_1:N_1} \leq_{D-SMIT} H_{N_2:N_2}$.

Proof of theorem 14:

In view of (5) and the proof of theorem 11, we have for all $t > 0$

$$\sum_{x=0}^{t} xH_i(x)dx = \lambda^{-1} \sum_{j=2}^{\infty} \frac{e^{-\lambda t}(\lambda\ t)^j}{j!} \left(\sum_{k=1}^{j-1} kP_{k-1}^{[i]} \right)$$

Because $e^{-\lambda t}(\lambda\ t)^j / j!$ is TP2 in(t, j), and by assuming

$$\frac{\sum_{k=1}^{j-1} kP_{k-1}^{[2]}}{\sum_{k=1}^{j-1} kP_{k-1}^{[1]}}, \text{ is non-decreasing in } j \in N,$$

the general composition theorem of Karlin [16] provides that $\sum_{x=0}^{t} xH_i(x)dx$ is TP2 in $(i, t) \in \{1,2\} \times R^+$, which implies that $T_1 \leq_{D-SMIT} T_2$. which helped to improve the paper.