Journal of Statistics Applications & Probability
An International Journal

http://dx.doi.org/10.18576/jsap/110225

A New Discrete Mean Residual Life Order and its Reliability Applications

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Received: 19 Dec. 2020, Revised: 2 Feb. 2021, Accepted: 16 Mar. 2021.
Published online: 1 May 2022.

Abstract: The purpose of this paper is to introduce study and analyze a new stochastic order that lies in the framework of the discrete mean residual life and the convexity orders. Several preservation properties of the new order under reliability operations of monotone transformation, mixture, weighted distributions and shock models are discussed. In addition, two characterization properties of the new order based on the concept of discrete residual life at random time and the concept of excess lifetime in discrete renewal processes are given. Finally, we introduce some new applications of this order in the context of reliability theory.

Keywords: Discrete Mean Residual Life; Discrete Hazard rate; Characterization; Preservation; Mixture; Shock Models; Excess Lifetime.

1 Preliminaries

In this section we introduce the topic considered in this paper, motivations, definitions and relevant facts.
Stochastic ordering is a fundamental guide for decision making under uncertainty. Two well-known stochastic orders that have been introduced and studied in reliability theory are the discrete hazard rate (D-HR) order and the discrete mean residual life (D-MRL) order, whose definitions are recalled here.

Let X and Y be two random variables having distribution functions F and G, respectively, and denote by $\overline{F}\,(f)$ and $\overline{G}\,(g)$ their respective survival (density) functions.

The lifetime random variable X is said to be smaller than Y

a) in the d-HR order (denoted as $X \leq_{D-HR} Y$) if $G(k)/F(K)$ is non-decreasing in $k \in \{1,2,3,..\}$.

b) in d-MRL order (denoted as $X \leq_{D-MRL} Y$) if

$$\frac{\sum_{u=k}^{\infty} \overline{G}(u)}{\sum_{u=k}^{\infty} \overline{F}(u)} \quad \text{is non-decreasing in } k \in \{1,2,3,\dots\}.$$

Recently, another stochastic ordering has evolved in reliability and life testing problems [1]. To the best of our knowledge this article introduces a discrete analog to this interesting concept. We begin with some terminology.

The lifetime random variable X is said to be smaller than Y in the discrete combination convexity order (denoted as $X \leq_{D-CCX} Y$) if

$$\frac{\sum_{u=k}^{\infty} uG(u)}{\sum_{u=k}^{\infty} uF(u)} \quad \text{is non-decreasing in k for all } k = 0,1,2,\dots$$

For the non-negative random variable X with $\mu = E(X) < \infty$, the random variable $\widetilde{X}$ is called the equilibrium version associated with X with survival function (cf. [2],[3] and [4]). A discrete analog is defined by

$$\overline{F}_{\tilde{X}}(t) = \frac{1}{\mu}\sum_{u=k}^{\infty} \overline{F}(u) \quad \text{for all } k = 0,1,2,\dots$$

It is not hard to prove that the MRL order can be characterized in terms of the D-HR order as given below

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$$X \leq_{D-MRL} Y \Leftrightarrow \tilde{X} \leq_{D-HR} \tilde{Y} \ , \qquad (1.1)$$

By mimicking the result in [5].
Let X have the density function f and let w be a non-negative function such that

$$E(w(X)) < \infty.$$

Then, in accordance with [6], define a random variable X_w with density

$$f_w(x) = \frac{w(x)f(x)}{E\big(w(X)\big)}, \qquad \text{for all } x \in \mathbb{R}^+.$$

The random variable X_w is the weighted version of X with weight function w. A special case of interest arises when the weight function is of the form $w(x) = x^{\alpha}$, for some $\alpha > 0$. Such distributions are known as size-biased distributions of order α. The most common cases of size-biased distributions occur when $\alpha = 1$or 2. These special cases are termed as length and area-biased, respectively. Suppose that $(\tilde{X})_w$ be the weighted versions of $\tilde{X}$, with weight function w, Then, the density functions of $(\tilde{X})_w$ is given by ([7],[8],[9],[10]).

$$f_{(\tilde{X})_w}(t) = \frac{w(t)\bar{F}(t)}{\sum_{u=0}^{\infty} w(u)\bar{F}(u)}$$

Similarly define $(\tilde{Y})_w$ and $g_{(\tilde{Y})_w}(t)$. To compare $\tilde{X}_w$ and $\tilde{Y}_w$ with respect to the D-HR order, we have $\left(\tilde{X}\right)_w \leq_{D-HR} \left(\tilde{Y}\right)_w$ if, and only if

$$\frac{\sum_{x=k}^{\infty} w(x)\bar{F}(x)}{\bar{F}(k)} \leq \frac{\sum_{x=k}^{\infty} w(x)\bar{G}(x)}{\bar{G}(k)} \text{ , for all k } 0,1,2,\ldots \qquad (1.2)$$

For weighted random variables X_w and Y_w associated with X and Y, with an increasing weight function w, an analogous result to Theorem 9(a) in [11] states that $X \leq_{D-HR} Y$ implies $X_w \leq_{D-HR} Y_w$. Here, if we assume that w is an increasing weight function, then

$$\tilde{X} \leq_{D-HR} \tilde{Y} \Rightarrow \left(\tilde{X}\right)_w \leq_{D-HR} \left(\tilde{Y}\right)_w \qquad (1.3)$$

In order to compare two lifetime variables based on the D-HR of the length-biased versions of their equilibrium distributions we consider the increasing weight function $w(x) = x$. In view of (1.1) and (1.3) if $X \leq_{D-MRL} Y$ then $\left(\tilde{X}\right)_w \leq_{D-HR} \left(\tilde{Y}\right)_w$ when w(x) = x. So, the D-HR order of length-biased versions of $\tilde{X}$and $\tilde{Y}$ is a weaker order than the original D-MRL order. Motivated by this, we propose the following stochastic order.

Definition 1.1

The lifetime random variable X is said to be smaller than the lifetime variable Y in the combination discrete mean residual life order (denoted by $X \leq_{D-CMRL} Y$) if.

$$\frac{\sum_{x=k}^{\infty} x\bar{F}(x)}{\bar{F}(k)} \leq \frac{\sum_{x=k}^{\infty} x\bar{G}(x)}{\bar{G}(k)} \text{ , for all k} = 0,1,2,\ldots$$

As demonstrated in Theorem 2.1. (iii) of Section 2 below, the combination discrete mean residual life (D-CMRL) order lies in the framework of the D-MRL and the combination discrete convexity (D-CCX) orders. As a result, the study of the D-CMRL order is meaningful because it throws an important light on the understanding of the properties of the D-MRL and the D-CCX orders, and of the relationships among these two orders and other related stochastic orders. Furthermore, the D-CMRL order enjoys several reliability properties which provide some applications in reliability and survival analysis.

Definition 1.2

A non-negative function $\beta(x, y)$ is said to be totally positive of order 2 (TP_2) in $(x, y) \in \chi \times \gamma$ if

$$\begin{vmatrix} \beta(x_1, y_1) & \beta(x_1, y_2) \\ \beta(x_2, y_1) & \beta(x_2, y_2) \end{vmatrix} \geq 0$$

for all $x_1 \leq x_2 \in \chi$ and $y_1 \leq y_2 \in \gamma$ in which χ and γ are two real subsets of $\mathbb{R}$.

Definition 1.3

Probability vector $\underline{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_i > 0$ for $i = 1, 2, \dots, n$ is said to be smaller than another probability vector $\underline{\beta} = (\beta_1, \beta_2, \dots, \beta_n)$ in the sense of discrete likelihood ratio order (denoted as $\underline{\alpha} \leq_{DLR} \underline{\beta}$) if

$$\frac{\beta_i}{\alpha_i} \leq \frac{\beta_j}{\alpha_j}, \text{ for all } 1 \leq i \leq j \leq n$$

The remainder of this paper is organized as follows. In Section 2, some characterizations and implications regarding the D-CMRL order are provided. Preservation properties under some reliability operations such as monotonic transformation and mixture are discussed in Section 3. In that section, we discuss preservation of the D-CMRL order under general weighted distributions. In Section 4, to illustrate the concepts, some applications in the context of reliability theory are included. Finally, in Section 5, we give a brief conclusion, and some remarks of the current and future of this research.

2 Characterizations and Implications

The objective of this section is to build relations between the D-CMRL order with other well-known stochastic orders. Some relevant characterization results are also discussed.

In the following theorem, some equivalent conditions to the D-CMRL order are given.

Theorem 2.1

Let X and Y be two lifetime random variables. The following assertions are equivalent:

i) $X \leq_{D-CMRL} Y$.

ii) $\frac{\sum_{x=t}^{\infty} x\bar{G}(x)}{\sum_{x=t}^{\infty} x\bar{F}(x)}$ is non − decreasing in $t \in \mathbb{R}^+$.

iii) $E\big((X^2|X > t)\big) \leq E\big((Y^2|Y > t)\big)$, for all $t \in \mathbb{R}^+$.

Proof

We first prove that (i) and (ii) are equivalent. We have

$$\frac{\sum_{x=t}^{\infty} x\bar{G}(x)}{\sum_{x=t}^{\infty} x\bar{F}(x)} = \frac{t\bar{G}(t) + \sum_{x=t+1}^{\infty} x\bar{G}(x)}{t\bar{F}(t) + \sum_{x=t+1}^{\infty} x\bar{F}(x)}.$$

The above Equation can be rewritten as

$$\sum_{x=t}^{\infty} x\bar{G}(x) \sum_{x=t+1}^{\infty} x\bar{F}(x) + t\bar{F}(t) \sum_{x=t}^{\infty} x\bar{G}(x) - \sum_{x=t}^{\infty} x\bar{F}(x) \sum_{x=t+1}^{\infty} x\bar{G}(x) - t\bar{G}(t) \sum_{x=t}^{\infty} x\bar{F}(x) = 0.$$

Then

$$\sum_{x=t}^{\infty} x\bar{G}(x) \sum_{x=t+1}^{\infty} x\bar{F}(x) - \sum_{x=t}^{\infty} x\bar{F}(x) \sum_{x=t+1}^{\infty} x\bar{G}(x) + \left[t\bar{F}(t) \sum_{x=t}^{\infty} x\bar{G}(x) - t\bar{G}(t) \sum_{x=t}^{\infty} x\bar{F}(x)\right] = 0$$

By definition, the last term is nonnegative if and only if $X \leq_{D-CMRL} Y$ where

$$\frac{\sum_{x=t}^{\infty} x\bar{F}(x)}{\bar{F}(t)} \leq \frac{\sum_{x=t}^{\infty} x\bar{G}(x)}{\bar{G}(t)}.$$

Or equivalently

$$t\bar{F}(t) \sum_{x=t}^{\infty} x\bar{G}(x) \geq t\bar{G}(t) \sum_{x=t}^{\infty} x\bar{F}(x).$$

Then

$$\frac{\sum_{x=t}^{\infty} x\bar{G}(x)}{\sum_{x=t}^{\infty} x\bar{F}(x)} \text{ is non − decreasing in } t \in \{1,2,3, \dots\}.$$

Next we prove that (i) and (iii) are equivalent. Note that

$$m_X(t) = tE(X_t) + \tfrac{1}{2}E(X_t)^2 = \tfrac{1}{2}E[X^2 - t^2|X > t], \text{for all } t = 0,1,2, \dots$$

Similarly,

$$m_Y(t) = \frac{1}{2}E[Y^2 - t^2|Y > t]$$

Hence we have

$$X \leq_{D-CMRL} Y \Leftrightarrow m_X(t) \leq m_Y(t) \quad \text{for all } t = 0,1,2,\ldots$$
$$\Leftrightarrow \frac{1}{2}E[Y^2 - t^2|Y > t] \leq \frac{1}{2}E[X^2 - t^2|X > t], \text{for all } t = 0,1,2,\ldots$$
$$\Leftrightarrow E[Y^2|Y > t] \leq E[X^2|X > t], \quad \text{for all } t = 0,1,2,\ldots,$$

This completes the proof.

In general, the CMRL order does not imply the MRL order. However, the next theorem provides a sufficient condition under which

$$X \leq_{D-CMRL} Y \text{ implies } X \leq_{D-MRL} Y$$

Theorem 2.2

$$\frac{\sum_{x=t}^{\infty}\sum_{u=x}^{\infty} u\bar{G}(u)}{\bar{G}(t)} \leq \frac{\sum_{x=t}^{\infty}\sum_{u=x}^{\infty} u\bar{F}(u)}{\bar{F}(t)}, \text{for all } t = 0,1,2,\ldots$$

Then, $X \leq_{D-CMRL} Y$ implies $X \leq_{D-MRL} Y$.

Proof

First, let $\mu_X(t) = \frac{\sum_{x=t}^{\infty} x\bar{F}(x)}{\bar{F}(t)}$, for all $t = 0,1,2,\ldots$, which is the MRL of the discrete random variable X. We can write, for all $t = 0,1,2,\ldots$

$$\frac{\sum_{x=t}^{\infty}\sum_{u=x}^{\infty} \bar{F}(u)}{\bar{F}(t)} = \frac{\sum_{u=x}^{\infty}(u-t)\bar{F}(u)}{\bar{F}(t)} = m_X(t) - t\mu_X(t).$$

Similarly, we can get

$$\frac{\sum_{x=t}^{\infty}\sum_{u=x}^{\infty} \bar{G}(u)}{\bar{G}(t)} = m_Y(t) - t\mu_Y(t).$$

Therefore, by assumptions, for all $t = 0,1,2,\ldots$, $m_Y(t) - m_X(t) + \left[\frac{\sum_{x=t}^{\infty}\sum_{u=x}^{\infty}\bar{F}(u)}{\bar{F}(t)} - \frac{\sum_{x=t}^{\infty}\sum_{u=x}^{\infty}\bar{G}(u)}{\bar{G}(t)}\right] \geq 0$, which completes the proof.

The following definition is needed to define a family of stochastic orders arising from HR comparison with size-biased equilibrium distributions.

Definition 2.1

For each fixed $\alpha \in [0,\infty)$, the lifetime random variable X is said to be smaller than Y in the discrete combination mean residual life order of order α, denoted by

$$X \leq_{D-CMRL}^{(\alpha)} Y, \text{if}$$

$$\frac{\sum_{x=t}^{\infty} x^{\alpha}\bar{F}(x)}{\bar{F}(t)} \leq \frac{\sum_{x=t}^{\infty} x^{\alpha}\bar{G}(x)}{\bar{G}(t)}, \text{for all } t = 0,1,2,\ldots, \tag{2.1}$$

or equivalently if

$$\frac{\sum_{x=t}^{\infty} x^{\alpha}\bar{F}(x)}{\sum_{x=t}^{\infty} x^{\alpha}\bar{G}(x)} \text{ is non-decreasing in } t \in \mathbb{R}^{+}. \tag{2.2}$$

It is observed that when $\alpha = 0$ and $\alpha = 1$, the $X \leq_{D-CMRL}^{(\alpha)} Y$ correspond to $X \leq_{D-MRL} Y$ and $X \leq_{D-CMRL} Y$, respectively. Below we give some useful implications.

Theorem 2.3

Let $\alpha \leq \beta \in [0,\infty)$ be fixed. Then

i) $X \leq_{D-CMRL}^{(\alpha)} Y$ implies $X \leq_{D-CMRL}^{(\beta)} Y$.

ii) $X \leq_{D-CMRL}^{(\alpha)} Y$ implies $\sum_{x=t}^{\infty} x^{\alpha}[\bar{G}(x) - \bar{F}(x)] \geq 0$, for all $t = 0,1,2,\ldots,$

Proof

For each fixed $\alpha \leq \beta \in [0, \infty)$ and for all $t = 0,1,2, \ldots$, one has

$$\sum_{x=t}^{\infty} x^{\alpha}\left[\frac{\bar{G}(x)}{\bar{G}(t)} - \frac{\bar{F}(x)}{\bar{F}(t)}\right] = \sum_{x=t}^{\infty} x^{\beta-\alpha}\ \Delta\left[\sum_{u=0}^{x}\left[\frac{u^{\alpha}\bar{G}(u)}{\bar{G}(t)} - \frac{u^{\alpha}\bar{F}(u)}{\bar{F}(t)}\right]\right] = \sum_{x=0}^{\infty} h(x) w_t(x).$$

Where $h(x) = x^{\beta-\alpha}$ which is nonnegative and increasing in $x > 0$, and $\Delta W_t(x) = w_t(x), W_t(x) = \sum_{u=0}^{x}\left[\frac{u^{\alpha}\bar{G}(u)}{\bar{G}(t)} - \frac{u^{\alpha}\bar{F}(u)}{\bar{F}(t)}\right]$ with

$$w_t(x) = \left[\frac{x^{\alpha}\bar{G}(x)}{\bar{G}(t)} - \frac{x^{\alpha}\bar{F}x}{\bar{F}(t)}\right]\ \forall\, x > t, \text{and 0 otherwise.}$$

Because of (2.1), for all $0 \leq s \leq t$, we have

$$W_s(x) = W_t(x) = \sum_{x=t}^{\infty}\left[\frac{x^{\alpha}\bar{G}(x)}{\bar{G}(t)} - \frac{x^{\alpha}\bar{F}(x)}{\bar{F}(t)}\right]$$

In addition, because of (2.2), for all $0 \leq t \leq s$, it stands that

$$\frac{\sum_{x=t}^{\infty} x^{\alpha}\bar{G}(x)}{\sum_{x=t}^{\infty} x^{\alpha}\,\bar{F}(x)} \leq \frac{\sum_{x=s}^{\infty} x^{\alpha}\bar{G}(x)}{\sum_{x=s}^{\infty} x^{\alpha}\,\bar{F}(x)} \qquad (2.3)$$

and in view of (2.1) we can write, for all $t \in [0,1,2, \ldots\}$,

$$\frac{\sum_{x=t}^{\infty} x^{\alpha}\bar{G}(x)}{\sum_{x=t}^{\infty} x^{\alpha}\,\bar{F}(x)} \geq \frac{\bar{G}(t)}{\bar{F}(t)} \qquad (2.4)$$

Combining (2.3) and (2.4), for all $0 \leq t \leq s$, we arrive at

$$\frac{\sum_{x=s}^{\infty} x^{\alpha}\bar{G}(x)}{\sum_{x=s}^{\infty} x^{\alpha}\,\bar{F}(x)} \geq \frac{\bar{G}(t)}{\bar{F}(t)}$$

Therefore, for all $0 \leq t \leq s$, we have

$$\sum_{x=s}^{\infty}\left[\frac{x^{\alpha}\bar{G}(x)}{\bar{G}(t)} - \frac{x^{\alpha}\bar{F}(x)}{\bar{F}(t)}\right] \geq 0.$$

Thus, for all $s, t \geq 0$ it holds that

$$\sum_{x=s}^{\infty}\ \Delta\left[\sum_{u=0}^{x}\left[\frac{u^{\alpha}\bar{G}(u)}{\bar{G}(t)} - \frac{u^{\alpha}\bar{F}(u)}{\bar{F}(t)}\right]\right] \geq 0.\ \text{for all } s, t \geq 0.$$

Applying a discrete version of Lemma 7.1(a) in [11], gives that

$$\sum_{x=0}^{\infty} h(x) w_t(x) \geq 0.\ for\ all\ t \geq 0.,$$ and hence the result in (i).

(ii)In the light of (2.3) and (2.4), and substituting t=0, for all $s \geq 0$ we have

$$\frac{\sum_{x=s}^{\infty} x^{\alpha}\bar{G}(x)}{\sum_{x=s}^{\infty} x^{\alpha}\,\bar{F}(x)} \geq \frac{\sum_{x=0}^{\infty} x^{\alpha}\bar{G}(x)}{\sum_{x=0}^{\infty} x^{\alpha}\,\bar{F}(x)} \geq \frac{\bar{G}(0)}{\bar{F}(0)}$$

which proves the result.

The following corollary follow easily form Theorem 2.3.

Corollary 2.1

Let X and Y be two lifetime random variables. Then

i) $X \leq_{D-MRL} Y$ implies $X \leq_{D-CMRL}^{\alpha} Y$, for all $\beta \geq 0$.

ii) $X \leq_{D-CMRL} Y$ implies $X \leq_{D-CMRL}^{\beta} Y$, for all $\beta \geq 1$.

iii) $X \leq_{D-CMRL} Y$ implies $X \leq_{D-CCX} Y$, for all $s \geq 0$.

In many reliability problems, it is interesting to study $X_Y = [X - Y | X > Y]$, the residual life of X with a random age Y. The residual life at random time (RLRT) represents the actual working time of the standby unit if X is regarded as the total random life of a warm standby unit with its age Y. Suppose that X and Y are independent. Then, the survival function of X_Y, for any $x \geq 0$, is given by

$$P(X_Y > x) = \frac{\sum_{x=0}^{\infty} \bar{F}(x+y)g(y)}{\sum_{x=0}^{\infty} \bar{F}(y)g(y)} \qquad (2.5)$$

Theorem 2.4

$X_Y \leq_{D-CMRL} X$ for any Y which is independent of X, if and only if, $X_t \leq_{D-CMRL} X$ for all $t = 0,1,2,\ldots.$

Proof

First, let

$X_t \leq_{D-CMRL} X$ for all $t = 0,1,2,\ldots.$ It then follows that, for all s = 1,2, ...

$$\sum_{x=s}^{\infty} x\bar{F}(t+x) \leq \frac{\bar{F}(t+x)}{\bar{F}(s)} \sum_{x=s}^{\infty} x\bar{F}(x) \qquad (2.6)$$

By summing both sides of (2.6) over all nonnegative integral values of t and the pdf g, we have

$$\sum_{t=0}^{\infty}\sum_{x=s}^{\infty} \bar{F}(t+x)g(t) \leq \sum_{t=0}^{\infty}\left[\frac{\bar{F}(t+x)}{\bar{F}(s)} \sum_{x=s}^{\infty} x\,\bar{F}(x)g(t)\right]$$

$$\sum_{t=0}^{\infty} \bar{F}(t+x)g(t) \leq \frac{\sum_{x=s}^{\infty} x\,\bar{F}(x)}{\bar{F}(s)}$$

which is equivalent to saying that $X_Y \leq_{D-CMRL} X$, for all Y that are independent from X. On the other hand, suppose that $X_Y \leq_{D-CMRL} X$ holds for any nonnegative random variable Y. Then $X_t \leq_{D-CMRL} X$, for all $t = 0,1,\ldots$ follows by taking Y as a degenerate variable.

3 Preservation Properties

In this section, we develop some preservation properties of the CMRL order under some reliability operations such as monotone transformation and mixture. The next result shows that the CMRL order is preserved under monotone increasing convex transformations.

Theorem 3.1

Let ϕ be strictly increasing and convex such that $\phi(0) = 0$, Then, $X \leq_{D-CMRL} Y$ implies $\phi(X) \leq_{D-CMRL} \phi(Y)$.

Proof

Without loss of generality, assume that ϕ is differentiable and denote its first derivative by ϕ' .

Notice that $X \leq_{D-CMRL} Y$ implies that, for all $t = 0,1,\ldots$

$$\sum_{x=\phi^{-1}(t)}^{\infty}\left[\frac{x\bar{G}(x)}{\bar{G}(\phi^{-1}(t))} - \frac{x\bar{F}(x)}{\bar{F}(\phi^{-1}(t))}\right] \geq 0$$

On the other hand, $X \leq_{D-CMRL} Y$ if and only if, for all $t = 0,1,\ldots$

$$\frac{\sum_{x=t}^{\infty} xP(\phi(Y) > x)}{P(\phi(Y) > x)} \geq \frac{\sum_{x=t}^{\infty} xP(\phi(X) > x)}{P(\phi(X) > x)},$$

which is equivalent to, for all $t = 0,1,\ldots$

$$\sum_{x=\phi^{-1}(t)}^{\infty} \gamma(x)\left[\frac{x\bar{G}(x)}{\bar{G}(\phi^{-1}(t))} - \frac{x\bar{F}(x)}{\bar{F}(\phi^{-1}(t))}\right] \geq 0$$

where $\gamma(x) = \frac{\phi(x)\phi'(x)}{x}$. Its well-known that if $\phi(x)$ is nonnegative and convex with $\phi(0) = 0$, then $\phi(x)/x$ is non-decreasing. Thus, due to the assumption, $\gamma(x)$ is the product of two nonnegative non-decreasing functions and hence $\gamma(x)$ is non-decreasing. Finally, Lemma 7.1(a) of [12] concludes the proof.

Theorem 3.2

Let X, Y, and Θ be random variables such that $[X|\Theta = \theta] \leq_{D-CMRL} [Y|\Theta = \theta']$ for all θ and θ'in the support of Θ. Then $X \leq_{D-CMRL} Y$.

Proof

Select θ and θ' in the support of Θ. Let $\overline{F}(\cdot|\theta)$, $\overline{G}(\cdot|\theta)$, $\overline{F}(\cdot|\theta')$, and $\overline{G}(\cdot|\theta')$ be the survival functions of $[X|\Theta = \theta]$, $[Y|\Theta = \theta]$, $[X|\Theta = \theta']$ and $[Y|\Theta = \theta']$, respectively. The proof is similar to that of Theorem 1.B.8 in [13]. It is sufficient to show that for each $\upsilon \in (0.1)$ and for all $t = 0,1, \ldots$ we have

$$\frac{\upsilon \sum_{u=t}^{\infty} \overline{F}(u|\theta) + (1-\upsilon)\sum_{u=t}^{\infty} \overline{F}(u|\theta')}{\upsilon\overline{F}(t|\theta) + (1-\upsilon)\overline{F}(u|\theta')} \leq \frac{\upsilon \sum_{u=t}^{\infty} \overline{G}(u|\theta) + (1-\upsilon)\sum_{u=t}^{\infty} \overline{G}(u|\theta')}{\upsilon\overline{G}(t|\theta) + (1-\upsilon)\overline{G}(u|\theta')}$$

This is an inequality of the form

$$\frac{a+b}{c+d} \geq \frac{w+x}{y+z}$$

where all eight variables are non-negative and by the assumptions of the theorem they satisfy

$$\frac{a}{c} \geq \frac{w}{y}, \frac{a}{c} \geq \frac{x}{z}, \frac{b}{d} \geq \frac{w}{y}, \text{and } \frac{b}{d} \geq \frac{x}{z}$$

It is easy to verify that the latter four inequalities imply the former one, completing the proof of the theorem.

In Theorem 3.3, we discuss the preservation property of the CMRL order under finite mixture. Let $X_i, i = 1, \ldots, n$ be a collection of independent random variables. Suppose that F_i is the distribution function of X_i. Let $\underline{\alpha} = (\alpha_1, \alpha_1, \ldots, \alpha_n)$ and $\underline{\beta} = (\beta_1, \beta_1, \ldots, \beta_n)$ be two probability vectors and let X and Y be two random variables having the respective survival functions $\overline{F}$ and $\overline{G}$ defined by

$$\overline{F}(x) = \sum_{i=1}^{n} \alpha_1 \overline{F}_i(x) \text{ and } \overline{G}(x) = \sum_{i=1}^{n} \alpha_1 \overline{G}_i(x) \qquad (3.1)$$

Next result gives conditions under which X and Y are comparable with respect to the CMRL order. The similar property for MRL order was obtained in [14].

Theorem 3.3

Let $X_1, X_2, \ldots, X_n$ be a collection of independent random variables with corresponding survival functions $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_n$ such that $X_1 \leq_{D-CMRL} X_2 \leq_{D-CMRL} \ldots \leq_{D-CMRL} Y$ and Let $\underline{\alpha} = (\alpha_1, \alpha_1, \ldots, \alpha_n)$ and $\underline{\beta} = (\beta_1, \beta_1, \ldots, \beta_n)$ be such that $\underline{\alpha} \leq_{DLR} \underline{\beta}$ Let X and Y have distribution functions $\overline{F}$ and $\overline{G}$ defined in (3.1). Then $X \leq_{D-CMRL} Y$

Proof

Because of Theorem 2.1(ii), we need to establish that, for all $0 < x < y$

$$\frac{\sum_{u=0}^{\infty}(x+u)\sum_{i=1}^{n}\beta_i\overline{F}_i(x+u)}{\sum_{u=0}^{\infty}(x+u)\sum_{i=1}^{n}\alpha_i\overline{F}_i(x+u)} \geq \frac{\sum_{u=0}^{\infty}(y+u)\sum_{i=1}^{n}\beta_i\overline{F}_i(y+u)}{\sum_{u=0}^{\infty}(y+u)\sum_{i=1}^{n}\alpha_i\overline{F}_i(y+u)} \qquad (3.2)$$

By simple calculations, (3.2) can be written in the following form:

$$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ i\leq j}}^{n} \left[\beta_j\alpha_i \sum_{u=0}^{\infty}(x+u)\overline{F}_j(x+u) \times \sum_{v=0}^{\infty}(y+v)\overline{F}_i(y+v)\right]$$

$$\geq \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i\leq j}}^{n} \left[\beta_j\alpha_i \sum_{v=0}^{\infty}(y+v)\bar{F}_j(y+v) \times \sum_{u=0}^{\infty}(x+u)\bar{F}_i(x+u) + \beta_i\alpha_j \sum_{v=0}^{\infty}(y+v)\bar{F}_i(y+v) \times \sum_{u=0}^{\infty}(x+u)\bar{F}_j(x+u)\right]$$

Now, for each fixed pair (i, j) with i < j we have

$$\left[\beta_j\alpha_i \sum_{v=0}^{\infty}(y+v)\bar{F}_j(y+v) \times \sum_{u=0}^{\infty}(x+u)\bar{F}_i(x+u) + \beta_i\alpha_j \sum_{v=0}^{\infty}(y+v)\bar{F}_i(y+v) \times \sum_{u=0}^{\infty}(x+u)\bar{F}_j(x+u)\right]$$
$$-\left[\beta_j\alpha_i \sum_{u=0}^{\infty}(x+u)\bar{F}_j(x+u) \times \sum_{v=0}^{\infty}(y+v)\bar{F}_i(y+v) + \beta_i\alpha_j \sum_{u=0}^{\infty}(x+u)\bar{F}_i(x+u) \times \sum_{v=0}^{\infty}(y+v)\bar{F}_j(y+v)\right]$$
$$= (\beta_j\alpha_i - \beta_i\alpha_j)\left[\sum_{v=0}^{\infty}(y+v)\bar{F}_i(y+v) \times \sum_{u=0}^{\infty}(x+u)\bar{F}_j(x+u) - \sum_{u=0}^{\infty}(x+u)\bar{F}_j(x+u) \times \sum_{v=0}^{\infty}(y+v)\bar{F}_i(y+v)\right]$$

which is nonnegative because both terms are nonnegative by the assumptions. This completes the proof.

Consider a family of survival functions $\{\bar{F}_\theta,\ \theta \in \chi\}$ where χ is a subset of the real line $\mathbb{R}$. Let $X(\theta) = [X|\Theta = \theta]$ denote a random variable with survival function $\bar{F}_\theta$. For any random variable Θ_i with support in χ, and with distribution function Λ_i we denote by $X(\Theta_i)$ the random variable that has survival function $\bar{F}_i$ given by

$$\bar{F}_i(x) = \sum_{\chi} \bar{F}_\theta(x)\, \lambda_i(\theta)$$

In this case, $X(\Theta_i)$ is called a mixture of $X(\theta)$ or of the family $\{\bar{F}_\theta,\ \theta \in \chi\}$ with respect to Θ_i for each i = 1,2. Below we make CMRL order between $X(\Theta_1)$ and $X(\Theta_2)$ under some suitable assumptions. Theorem 3.4 below provides another preservation property under mixture.

Theorem 3.4

Let $X(\Theta_1)$ and $X(\Theta_2)$ be as described above. If

$$X(\theta_1) \leq_{D-CMRL} X(\theta_2), \qquad \text{for all } \theta_1 \leq \theta_2 \in \chi \qquad (3.3)$$

and if

$$\Theta_1 \leq_{D-HR} \Theta_2$$

then

$$X(\Theta_1) \leq_{D-CMRL} X(\Theta_2)$$

Proof

We must prove that $H_i(t) = \sum_{x=t}^{\infty} x\bar{F}_i(x)$ is TP_2 in $(i,t) \in \{1,2\} \times [0,\infty)$ for all $t = 0,1,\ldots$ and $i = 1,2$, we can get

$$H_i(t) = \sum_{x=t}^{\infty}\sum_{\lambda=0}^{\infty} x\, \bar{F}_\theta(x)\lambda_i(\theta) = \sum_{\lambda=0}^{\infty} \phi(t,\theta)\lambda_i(\theta).$$

Because of Theorem 2.1 (iii), (3.3) implies that $\phi(t,\theta) = \sum_{x=t}^{\infty} x\bar{F}_\theta(x)$ is increasing in θ, and because of Theorem 2.1(ii), (3.3) is equivalent to saying that $\phi(t,\theta)$ is TP_2 in$(t,\theta) \in [0,\infty) \times \chi$.

Moreover, (3.4) means that $\bar{\Lambda}_i(\theta)$ is TP_2 in $(i,\theta)) \in \{1,2\} \times [0,\infty)$ Now, Lemma 4.2 of [9] is applicable and completes the proof.

For two weight functions w_1 and w_2 assume that the notations X_{w1}, and X_{w2} are used to denote two random variables with respective density functions.

$$f_1(x) = \frac{w_1(x)f(x)}{\mu_1} \text{ for } x \geq 0 \text{ and } g_1(x) = \frac{w_2(x)g(x)}{\mu_2} \text{ for } x \geq 0,$$

where $0 < \mu_1 = E(w_1(X)) < \infty$ and $0 < \mu_2 = E(w_2(X)) < \infty$ The random variables X_{w1}, and X_{w2} are weighted versions of X and Y respectively. Let $\beta_1 = E(w_1(X)|X > x)$ and $\beta_2 = E(w_2(Y)|Y > x)$. Then the corresponding survival functions are given by

$$\bar{F}_1(x) = \frac{\beta_1(x)\bar{F}(x)}{\mu_1} \text{ for } x \geq 0 \text{ and } \bar{G}_1(x) = \frac{\beta_2(x)\bar{G}(x)}{\mu_2} \text{ for } x \geq 0 \qquad (3.5)$$

4 Reliability Applications

In this section, some relevant applications of the CMRL order in reliability theory are presented. Suppose that $\boldsymbol{X_1, X_2, \dots, X_n}$ are i.i.d. lifetime random variables from F and that $\boldsymbol{Y_1, Y_2, \dots, Y_n}$ are also i.i.d. lifetime random variables from G. Denote by

$$\boldsymbol{T_1 = min\{X_1, X_2, \dots, X_n\}}$$

and

$$\boldsymbol{T_2 = min\{Y_1, Y_2, \dots, Y_n\}}$$

The lifetimes of the two associated series systems. In the following result, we show that if the lifetimes of two series systems with i.i.d. components are CMRL ordered then their components are CMRL ordered.

Theorem 4.1

If $T_1 \leq_{CMRL} T_2$, then $X_i \leq_{CMRL} Y_i$, for all i = 1, 2,… , n.

Proof

Let $T_1 \leq_{CMRL} T_2$, then, we have

$$\int_t^\infty x\{\bar{G}^n(x)\bar{F}^n(t) - \bar{G}^n(t)\,\bar{F}^n(x)\}\,dx \geq 0, \text{for all } t \geq 0$$

Due to fact that

$$\bar{G}(x)\bar{F}(t) - \bar{G}(t)\bar{F}(x) = [\bar{G}^n(x)\bar{F}^n(t) - \bar{G}^n(t)\,\bar{F}^n(x)]h(x),$$

Where

$$h(x) = \left[\sum_{i=1}^{n} (\bar{G}(x)\bar{F}(t))^{n-i}(\,\bar{G}(t)\bar{F}(x)^{i-1})\right]^{-1}$$

is a non-decreasing function. On applying Lemma 7.1(a) of [12], it obtains that, for all t ≥ 0.

$$\int_t^\infty x\{\bar{G}(x)\bar{F}(t) - \bar{G}(t)\bar{F}(x)\}\,dx \geq 0$$

which means that $X_i \leq_{CMRL} Y_i$, for all i = 1, 2,… , n.

Shock models are of great interest in the context of reliability theory. The system is assumed to have an ability to withstand a random number of these shocks, and it is commonly assumed that the number of shocks and the inter-arrival times of shocks are independent. Let N denote the number of shocks survived by the system, and let Xj denote the random inter-arrival time between the (j — l)-th and j-th shocks. Then the lifetime T of the system is given by $T = \sum_{j=1}^{N} X_j$. In particular, if the inter arrivals are assumed to be independent and exponentially distributed with common parameter λ, then the survival function of T can be written as

$$\bar{H}(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!}\bar{P}_k\,, t \geq 0\,,$$

where $\bar{P}_k = P\,[N > k]$ for all k∊ N (and $\bar{P}_0 = 1$). Shock models of this kind, called Poisson shock models, have been studied extensively. For more details, we refer to [11],[15],[16],[17] and [18].

Consider now two devices subjected to shocks occurring as events of a Poisson counting process, as above; and let and $\bar{P}_k^{[1]}$, and $\bar{P}_k^{[2]}$ be the survival functions of the random number of shocks related to the two devices, respectively, which are the probabilities of surviving the first k shocks. Let T_i , denote the lifetime of the device, and let

$$\bar{H}_i(t) = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!}\bar{P}_k^{[i]}\,, t \geq 0\,, \qquad (4.1)$$

be its survival function. In the following, we give sufficient conditions under which the lifetimes $\mathbf{T_1}$ and $\mathbf{T_2}$ are ordered according to the CMRL order.

Theorem 4.2

If

$$\frac{\sum_{k=j+1}^{\infty} k\,\bar{P}_{k-1}^{[2]}}{\sum_{k=j+1}^{\infty} k\,\bar{P}_{k-1}^{[1]}} \text{ non}-\text{decreasing in } j \in N,$$

Then $\mathbf{T_1} \leq_{\mathbf{CMRL}} \mathbf{T_2}$

Proof

In view of some routine calculation, we have from (4.1) that, for all t > 0

$$\int_{t}^{\infty} x\,\bar{H}_i(x)\,dx = \lambda^{-2} \sum_{j=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \left(\sum_{k=j+1}^{\infty} \bar{P}_{k-1}^{[i]} \right)$$

Because $\mathbf{e^{-\lambda t}(\lambda t)^j/j!}$ is TP2 in (t, j) and by assuming

$$\frac{\sum_{k=j+1}^{\infty} k\,\bar{P}_{k-1}^{[2]}}{\sum_{k=j+1}^{\infty} k\,\bar{P}_{k-1}^{[1]}} \text{ is non}-\text{decreasing in } j \in N,$$

The general composition theorem of [19] provides that $\int_t^{\infty} x\,\bar{H}_i(x)\,dx$ is TP2 in (i,t) $\in$ {1,2} × R+, which implies that $\mathbf{T_1} \leq_{\mathbf{CMRL}} \mathbf{T_2}$.

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of mutually independent and identically distributed (i.i.d.) non-negative random variables with common distribution function F. For $n \geq 1$, denote $S_n = \sum_{i=1}^{n} X_i$ the time of the n-th arrival and $S_0 = 0$, let $N(t) = Sup\{n: S_n \leq t\}$ represent the number of arrivals during the interval [0, t]. Then, N = {N(t), t ≥ 0} is a renewal process with underlying distribution F (see [20]). Let γ (t) be the excess lifetime at time t ≥ 0. that is, $\gamma(t) = S_{N(t)+1} - t$.

In this context we denote the renewal function by M(t)= E[N(t)] which satisfies the following well known fundamental renewal equation:

$$M(t) = F(t) + \int_0^t F(t-y)dM(y), \quad t \geq 0$$

According to [12], it holds that, for all t ≥ 0 and x ≥ 0

$$P(\gamma(t) > x) = \bar{F}(t+x) + \int_0^t \bar{F}(t+x-u)dM(u). \quad (4.2)$$

In the literature, several results have been given to characterize the stochastic orders .Readers are referred to [21],[22] and [5]. Next, we will investigate the behavior of the excess lifetime of a renewal process with respect to the CMRL order.

Theorem 4.3

If $\mathbf{X_t} \leq_{\mathbf{CMRL}} \mathbf{X}$, for all t ≥ 0, then $\gamma(t) \leq_{\mathbf{CMRL}} \gamma(0)$ for all t ≥ 0.

Proof

First note that. $\mathbf{X_t} \leq_{\mathbf{CMRL}} \mathbf{X}$, for all t ≥ 0, if and only if for any t ≥ 0 and s > 0,

$$\int_s^{\infty} x\bar{F}(t+x)dx \leq \bar{F}(t+s)\frac{\int_s^{\infty} x\bar{F}(t)dx}{\bar{F}(s)} \quad (4.3)$$

In view of the identity of (4.2) and the inequality in (4.3) we can get

$$\int_s^\infty \boldsymbol{x\, P\,(\gamma(t) > x)dx} = \int_s^\infty \boldsymbol{x\bar{F}(t+x)dx} + \int_0^t \int_s^\infty \boldsymbol{x\bar{F}(t-u+x)dxdM(u)}.$$

$$\leq \int_s^\infty x\bar{F}(t+x)dx + \int_0^t \left[\frac{\bar{F}(t-u+s)}{\bar{F}(s)} \int_s^\infty x\bar{F}(x)dx\right] dM(u)$$

$$\leq \frac{\int_s^\infty x\bar{F}(x)dx}{\bar{F}(s)}\,[\bar{F}\,(t+s)] + \frac{\int_s^\infty x\bar{F}(x)dx}{\bar{F}(s)}\,[P\,(\gamma(t) > s) - \bar{F}\,(t+s)]$$

$$= \frac{\int_s^\infty x\bar{F}(x)dx}{\bar{F}(s)}\, P\,(\gamma(t) > s)$$

Hence, it holds that, for all $t \geq 0$ and $s > 0$,

$$\frac{\int_s^\infty x\, P\,(\gamma(t) > x)\, dx}{P\,(\gamma(t) > s)} \leq \frac{\int_s^\infty x\bar{F}(x)dx}{\bar{F}(s)}$$

Which means $\boldsymbol{\gamma(t)} \leq_{\mathbf{CMRL}} \boldsymbol{\gamma(0)}$for all $t \geq 0$.

5 Conclusion

The concept of length-biased distributions plays an important role in statistics, reliability and survival analysis. The relationship of CMRL order with other well-known stochastic orders is discussed. It was shown that the CMRL order enjoys from several reliability properties which provide several applications in reliability and survival analysis. Several characterization and preservation properties of the CMRL order under reliability operations of monotone transformation, mixture, and weighted distributions have been discussed. To illustrate the concepts, some applications in the context of statistics and reliability theory are included. In addition, our results provide new applications in reliability, statistics, and risk theory. Further properties and applications of the CMRL order can be considered in the future of this research. The closure properties of the CMRL under reliability operations of convolution and coherent systems are interesting topics, and still remain as open problems.

Acknowledgement

The authors would like to thank the Editorial Board and the Referees for their careful reading of the manuscript and for their constructive remarks and comments, which have enhanced the presentation.

Conflicts of Interest: The authors declare that there is no conflict of interest regarding the publication of this article.

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