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Empirical Bayes Inference for Rayleigh Distribution

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Abstract: This paper proposes the empirical Bayes inference for Rayleigh distribution. The empirical Bayes point estimation of the parameter, reliability and hazard function under symmetric (square error) and asymmetric (linear exponential and general entropy) loss functions are obtained. Besides, the maximum likelihood and Bayes estimators are computed too. The Bayes and empirical Bayes prediction interval also derived, where one and two sample prediction are considered. Numerical computations are established by using the Monto Carlo simulation method where the mean square errors are derived to assess and make comparisons among different estimators. The study was constructed based on progressively type-II censoring scheme.

Keywords: Loss function, empirical Bayes, Bayes estimation, progressively type-II censoring

1 Introduction

The Rayleigh distribution is a continuous distribution introduced by [1]. Its origin and properties are discussed by [2]. Rayleigh distribution is considered as a special case of the two-parameter Weibull distribution. The probability density function (PDF) of Rayleigh distribution is given by

$$f(x|\theta) = \frac{x}{\theta}\exp\left(-\frac{x^2}{2\theta}\right) \; x>0, \theta>0, \tag{1}$$

and the cumulative distribution function (CDF) is

$$F(x|\theta) = 1-\exp\left(-\frac{x^2}{2\theta}\right), \quad x>0, \theta>0. \tag{2}$$

The reliability function $R(t)$, and the hazard function $h(t)$ at time t for Rayleigh distribution are respectively, given by

$$R(t) = \exp\left(-\frac{t^2}{2\theta}\right) \quad t>0, \tag{3}$$

$$h(t) = \frac{t}{\theta}, \quad t>0. \tag{4}$$

This distribution is a crucial life distribution. It is applied in several fields such as reliability theory, survival analysis, operations research, probability theory and, other fields. Bayesian estimation and prediction problems for Rayleigh model were discussed recently by some authors. For instance, [3], they derived Bayesian estimator and highest posterior density credible intervals for the unknown parameter, reliability and hazard functions. Their study was constructed based on hybrid censoring and different loss functions were used. Also, the Bayes predictive estimator and prediction interval for future observations were obtained. [4] applied the Bayesian approach for Rayleigh model to estimate the unknown parameter, reliability function and hazard function. Different loss function were used based on type-II censored data.

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A comparison between estimators was constructed via simulation study and real data example. Bayesian inference for Rayleigh distribution is also discussed by [5] they studied the Bayes estimator of the unknown parameter of Rayleigh distribution. Three kinds of loss functions were used, square error (SE), linear exponential (LINEX) and general entropy (GE). Monto Carlo simulation was applied to compare the performances of different estimators where the Bayesian risk for each estimator was computed. Also, they proved the best estimator by real data example was applied. Recently, [6] derived the empirical Bayes estimators of parameter, reliability and hazard function for Ku- maraswamy distribution under LINEX loss function and progressively type-II censored samples with binomial removal and type-II censored samples. These estimators are illustrated and compared via real data example.

But up to now, the empirical Bayes inference related to the Rayleigh distribution were not addressed under progressively censoring scheme. Therefore, the main aim of this paper to study the empirical inference for Rayleigh distribution based on progressively type-II censoring scheme using three types of loss functions, square error, linear exponential and general entropy loss function.

Under this scheme of censoring, n units are placed on a test at time zero and only m units are completely observed until failure. At the time of the first failure, r_1 units are randomly withdrawn (or removed) from the remaining $n-1$ surviving units. Similarly, at the second failure time, r_2 units are randomly withdrawn from the remaining $n-2-r_1$ surviving units. This process continues until the mth failure. At this time, all remaining $r_m = n-m-r_1-r_2-\cdots-r_{m-1}$ are removed from the test. In this censoring scheme, $r_1, r_2, \cdots, r_m$ and m are all prefixed. Hence the censoring times are random. For deep insight of this censoring scheme refer to the comprehensive reference [7]. Progressive type-II censoring schemes (PCS-II) includes complete sample (no censoring) and type-II censoring as spacial cases where, if $r_1 = r_2 = \cdots = r_{m-1} = r_m = 0$ it reduced to the case of complete sample, while if $r_1 = r_2 = \cdots = r_{m-1} = 0$ and $r_m = n-m$ it reduced to the case of type-II censoring [7].

The likelihood function of a lifetime distribution with PDF or PMF $f(x|\theta)$ and CDF $f(x|\theta)$ under PCS-II is given by

$$L(\underline{x}|\theta) \propto \prod_{i=1}^{m} f(x_{i:m:n}|\theta)[(1-F(x_{i:m:n}|\theta))]^{r_i}, \tag{5}$$

where $A = n(n-1-r_1)(n-2-r_1-r_2)\cdots(n-\sum_{i=1}^{m-1}(r_i+1))$.

The rest of the paper is organized as follows. In Section 2, the estimation of the parameter by different methods of estimation, maximum likelihood, Bayes, and empirical Bayes estimators under loss functions. Bayes and empirical Bayes prediction interval under progressive type-II censoring is presented in section 3. A comparison of the competitive estimators via a Monte Carlo simulation method is given in Section 4. Finally, in section 5, conclusions and discussion are presented.

2 Estimation of the parameter and the reliability performances

2.1 Maximum Likelihood Estimation

The likelihood function of Rayleigh distribution under PCS-II, $\underline{x} = (x_1, x_2, \cdots, x_m)$ is obtained by using (1) and (2) as follows

$$L(\underline{x}|\theta) = A\prod_{i=1}^{m} x_i \theta^{-m} \exp\left(-\frac{T}{\theta}\right).$$

Where, $T = \frac{1}{2}\sum_{i=1}^{m}(1+r_i)x_i{}^2$,

and $A = n(n-1-r_1)(n-2-r_1-r_2)\cdots(n-\sum_{i=1}^{m-1}(1+r_i))$.

The ML estimate of θ is obtained as

$$\hat{\theta}_{MLE} = \frac{T}{m}. \tag{6}$$

The ML estimates of $R(t)$ and $h(t)$ can be derived by replacing $\hat{\theta}_{MLE}$ given in equation (6) as follow

$$\hat{R}(t)_{MLE} = \exp\left(-\frac{t^2}{2\hat{\theta}_{MLE}}\right). \tag{7}$$

$$h(t) = \frac{t}{\hat{\theta}_{MLE}}. \tag{8}$$

2.2 *Bayesian Estimation*

For Bayesian estimation, we assume that θ follows inverse gamma conjugate prior density of the form

$$f(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \frac{1}{\theta^{\alpha+1}} \exp\left(-\frac{\beta}{\theta}\right), \theta > 0, \alpha, \beta > 0. \tag{9}$$

Combining the likelihood function (6) with the prior PDF (9), via Bayes theorem, results in the posterior density function of θ, that is given by

$$f(\theta|x) \propto \theta^{-(\alpha+m+1)} \exp\left(-\frac{(\beta+T)}{\theta}\right). \tag{10}$$

Which is inverse gamma distribution; $IG(\alpha+m, \beta+T)$.
The estimation of the unknown parameter θ depends on the Bayesian approach where different loss functions are considered, due to loss function shows the loss related with an error in estimation. The loss functions typically increased as the distance between $\hat{\theta}$ (estimated value) and θ (actual value) increases. In addition, it is nonnegative function [8].
The Bayes estimator of the parameter θ under square error loss function represents the posterior mean and given by

$$\hat{\theta}_{BS} = E(\theta|x). \tag{11}$$

The second loss function is Linear exponential loss function (LINEX) which was introduced by [9]. This loss function is asymmetric and defined as

$$L(\hat{\theta}, \theta)_{BL} = c\left(e^{c(\hat{\theta}-\theta)} - c(\hat{\theta}-\theta) - 1\right) \qquad c \neq 0. \tag{12}$$

The direction and degree of symmetry determined by the sign and magnitude of c respectively. Hence, there is overestimation when $c > 0$, on other hand the underestimation appears as result of $c < 0$. In addition, if c close to zero, the LINEX loss is approximately square error loss and thus almost symmetric.[10],[11].
The Bayes estimator of θ under LINEX loss function $\hat{\theta}_{BL}$ is the value $\hat{\theta}$ which minimizes the following equation

$$E_{post}(L(\hat{\theta}, \theta)) = e^{c\hat{\theta}} E_{post}\left(e^{-c\theta}\right) - c\left(\hat{\theta} - E_{post}(\theta)\right) - 1. \tag{13}$$

The previous equation provides the posterior expectation of the LINEX loss function, where $E_{post}(.)$ indicates the posterior expectation with respect to the posterior density of θ. The Bayes estimator of θ under LINEX loss function $\hat{\theta}_{BL}$ is given by

$$\hat{\theta}_{BL} = -\frac{1}{c} \ln(E(e^{-c\theta})), \tag{14}$$

given that $E(e^{-c\theta})$ is finite and exist [12].
The general entropy loss function is asymmetric loss function and proposed by [12]. This loss function is useful in situations where appropriate to express the loss in terms of the ratio $\hat{\theta}/\theta$, it has form

$$L(\theta, \hat{\theta}) \propto \left[\left(\frac{\hat{\theta}}{\theta}\right)^{c} - c\ln\left(\frac{\hat{\theta}}{\theta}\right) - 1\right] \qquad c \neq 0, \tag{15}$$

whose minimum occurs at $\hat{\theta} = \theta$. From the equation (15), we can get the LINEX loss function given in (12) if we assume, $\left(\frac{\hat{\theta}}{\theta}\right) = e^{c(\hat{\theta}-\theta)}$ and $\ln\left(\frac{\hat{\theta}}{\theta}\right) = (\hat{\theta}-\theta)$.
The Bayes estimator of the parameter θ under general entropy loss function has form

$$\hat{\theta}_{BGE} = \left[E(\theta^{-c}|x)\right]^{-\frac{1}{c}}, \tag{16}$$

provided that, $E(\theta^{-c})$ exist and finite. From the equation (16), we can get the SE loss function when $c = -1$. [12],[13].

The Bayes estimators of the parameter and the reliability performances under loss functions

The Bayes estimators under SE loss function (BS)

Under SE loss function, the Bayes estimator $\hat{\theta}_{BS}$ of θ represents the posterior mean, which is given by

$$\hat{\theta}_{BS} = E(\theta|x) = \frac{\beta+T}{\alpha+m-1}. \tag{17}$$

From (11), (3), and (10), the Bayes estimator of the reliability function $R(t)$ is given by

$$\begin{aligned}\hat{R}(t)_{BS} &= \frac{(\beta+T)^{\alpha+m}}{\Gamma(\alpha+m)} \int_0^\infty \exp\left(-\frac{t^2}{2\theta}\right) \theta^{-(\alpha+m+1)} \exp\left(-\frac{(\beta+T)}{\theta}\right) d\theta \\ &= \left[1+\frac{t^2/2}{(\beta+T)}\right]^{-(\alpha+m)}.\end{aligned} \tag{18}$$

Similarly, by making use of (11), (4) and (10), the Bayes estimator of the hazard function $h(t)$ is given by

$$\begin{aligned}\hat{h}(t)_{BS} &= \frac{(\beta+T)^{\alpha+m}}{\Gamma(\alpha+m)} \int_0^\infty \left(\frac{t}{\theta}\right) \theta^{-(\alpha+m+1)} \exp\left(-\frac{(\beta+T)}{\theta}\right) d\theta \\ &= \frac{t(\alpha+m)}{(\beta+T)}.\end{aligned} \tag{19}$$

The Bayes estimators under LINEX loss function (BL)

From, (14) and (10) the Bayes estimator of θ under LINEX loss function is given by

$$\begin{aligned}\hat{\theta}_{BL} &= \frac{-1}{c} \log\left[\frac{(\beta+T)^{\alpha+m}}{\Gamma(\alpha+m)} \int_0^\infty e^{-c\theta} \theta^{-(\alpha+m+1)} \exp\left(-\frac{(\beta+T)}{\theta}\right) d\theta\right] \\ &= \frac{-1}{c} \log\left[\frac{2c^{\frac{\alpha+m}{2}} (\beta+T)^{\frac{1}{2}(\alpha+m)}}{\Gamma(\alpha+m)} K_{(m+\alpha)}\left(2\sqrt{c}\sqrt{\beta+T}\right)\right].\end{aligned} \tag{20}$$

Where, $K_{(m+\alpha)}\left(2\sqrt{c}\sqrt{\beta+T}\right)$ is the modified Bessel function of the second kind.

The Bayes estimator of $R(t)$ under LINEX loss function $\hat{R}(t)_{BL}$ is obtained by using (14),(3) and (10) as follow

$$\hat{R}(t)_{BL} = -\frac{1}{c} \log\left[\frac{(\beta+T)^{\alpha+m}}{\Gamma(\alpha+m)} \int_0^\infty e^{-ce^{-t^2/2\theta}} \theta^{-(\alpha+m+1)} \exp\left(-\frac{(\beta+T)}{\theta}\right) d\theta\right]. \tag{21}$$

The Bayes estimator of $R(t)$ at time t under LINEX loss function is obtained numerically by using R package.

The Bayes estimator of $h(t)$ at time t under LINEX loss function can be derived as follow

$$\begin{aligned}\hat{h}(t)_{BL} &= \frac{-1}{c} \log\left[\frac{(\beta+T)^{\alpha+m}}{\Gamma(\alpha+m)} \int_0^\infty e^{-ct/\theta} \theta^{-(\alpha+m+1)} \exp\left(-\frac{(\beta+T)}{\theta}\right) d\theta\right] \\ &= \frac{\alpha+m}{c} \log\left[1+\frac{ct}{(\beta+T)}\right].\end{aligned} \tag{22}$$

The Bayes estimators under GE loss function (BG)

Under GE loss function the Bayes estimator of θ is obtained as

$$\begin{aligned}\hat{\theta}_{BG} &= \left[\frac{(\beta+T)^{\alpha+m}}{\Gamma(\alpha+m)} \int_0^\infty \theta^{-c} \theta^{-(\alpha+m+1)} \exp\left(-\frac{(\beta+T)}{\theta}\right) d\theta\right]^{-1/c} \\ &= \left[\frac{\Gamma(\alpha+m)}{\Gamma(\alpha+m+c)}\right]^{1/c} (\beta+T).\end{aligned} \tag{23}$$

The Bayes estimators of $R(t)$ and $h(t)$ at time t under GE loss function are given, respectively, by

$$\begin{aligned}\hat{R}(t)_{BG} &= \left[\frac{(\beta+T)^{\alpha+m}}{\Gamma(\alpha+m)}\int_0^\infty e^{\left(\frac{ct^2}{2\theta}\right)}\theta^{-(\alpha+m+1)}\exp\left(-\frac{(\beta+T)}{\theta}\right)d\theta\right]^{-1/c} \\ &= \left[1-\frac{ct^2/2}{(\beta+T)}\right]^{((\alpha+m)/c)}. \end{aligned} \tag{24}$$

$$\begin{aligned}\hat{h}(t)_{BG} &= \left[\frac{(\beta+T)^{\alpha+m}}{\Gamma(\alpha+m)}\int_0^\infty \left(\frac{t}{\theta}\right)^{-c}\theta^{-(\alpha+m+1)}\exp\left(-\frac{(\beta+T)}{\theta}\right)d\theta\right]^{-1/c} \\ &= \left[\frac{\Gamma(\alpha+m)}{\Gamma(\alpha+m-c)}\right]^{1/c}\frac{t}{(\beta+T)}. \end{aligned} \tag{25}$$

2.3 Empirical Bayes Estimation

The Bayes estimator of the unknown parameter θ can be computed directly as shown before. In case of the unknown hyperparameters it can't be computed so, we have to obtain the empirical Bayes estimator by using the marginal distribution and then use classical estimation method specifically the ML estimation to estimate the unknown hyperparameters. The empirical Bayes (EB) method was first introduced and named by robbins [14]. [15] was discussed the parametric empirical Bayes (PEB) where the prior distribution is viewed as $f(\theta)$ with finite-dimensional parameter $\theta \in \Theta$.

The marginal distribution is derived by using the PDF of Rayleigh distribution (1) and the prior (9) as the following

$$\begin{aligned} f(x) &= \int_0^\infty f(x|\theta)f(\theta)d\theta \\ &= x\alpha\beta^\alpha\left(\beta+\frac{x^2}{2}\right)^{-(\alpha+1)} \quad x>0,\alpha,\beta>0. \end{aligned} \tag{26}$$

The PDF of the marginal distribution is given in equation (26) and hence, the CDF is obtained as

$$F(x) = 1-\beta^\alpha\left(\beta+\frac{x^2}{2}\right)^{-\alpha}, x>0,\alpha,\beta>0. \tag{27}$$

By using PDF and CDF which are respectively, given in (26),(27) with the formula of the PCS-II given in (5), the likelihood function of the marginal distribution under PCS-II, $\underline{x}=(x_1,x_2,\cdots,x_m)$ can be derived as follow

$$L(\underline{x}|\alpha,\beta) = A\alpha^m\beta^{\alpha m}\prod_{i=1}^m x_i\left(\beta+\frac{x_i^2}{2}\right)^{-(\alpha+1)}\beta^{\alpha r_i}\left(\beta+\frac{x_i^2}{2}\right)^{-\alpha r_i}. \tag{28}$$

The log-likelihood function is given by

$$\begin{aligned}\mathscr{L} = \log L(\underline{x}|\alpha,\beta) =& \log A + m\log\alpha + \alpha m\log\beta + \sum_{i=1}^m \log x_i - (\alpha+1)\sum_{i=1}^m \log(\beta+x_i^2/2) \\ &+\alpha\sum_{i=1}^m r_i\log\beta - \alpha\sum_{i=1}^m r_i\log(\beta+x_i^2/2). \end{aligned} \tag{29}$$

Upon differentiating equation (29) w.r.t α and β as follows

$$\frac{\partial\mathscr{L}}{\partial\alpha} = \frac{m}{\alpha}+m\log\beta-\sum_{i=1}^m\log(\beta+x_i^2/2)+\sum_{i=1}^m r_i\log\beta-\sum_{i=1}^m r_i\log(\beta+x_i^2/2)=0 \tag{30}$$

$$\frac{\partial\mathscr{L}}{\partial\beta} = \frac{\alpha m}{\beta}-\frac{(\alpha+1)}{\sum_{i=1}^m(\beta+x_i^2/2)}+\alpha\frac{\sum_{i=1}^m r_i}{\beta}-\alpha\sum_{i=1}^m\frac{r_i}{(\beta+x_i^2/2)}=0 \tag{31}$$

By solving the above equations numerically by Newton-Raphson and we get the estimated values $\hat{\alpha}$ and $\hat{\beta}$.

EB estimators of the parameter and the reliability performances under loss functions

EB estimators under SE loss function (EBS)

The EB estimator of $\hat{\theta}$ based on SE loss function is given by

$$\hat{\theta}_{EBS} = \frac{\hat{\beta}+T}{\hat{\alpha}+m-1}. \tag{32}$$

where α and β in (17) are replaced by $\hat{\alpha}$ and $\hat{\beta}$ respectively.
Substituting $\hat{\alpha}$ and $\hat{\beta}$ into (18) and (19) , the EB estimators of $R(t), h(t)$ are respectively obtained as

$$\hat{R}(t)_{EBS} = \left(1+\frac{t^2/2}{(\hat{\beta}+T)}\right)^{-(\hat{\alpha}+m)}. \tag{33}$$

$$\hat{h}(t)_{EBS} = \frac{t(\hat{\alpha}+m)}{(\hat{\beta}+T)}. \tag{34}$$

EB estimators under LINEX loss function (EBL)

The EB estimator of θ under LINEX loss function is given by

$$\hat{\theta}_{EBL} = \frac{-1}{c}\log\left[\frac{2c^{\frac{\hat{\alpha}+m}{2}}(\hat{\beta}+T)^{\frac{1}{2}(\hat{\alpha}+m)}}{\Gamma(\hat{\alpha}+m)}K_{(m+\hat{\alpha})}\left(2\sqrt{c}\sqrt{\hat{\beta}+T}\right)\right]. \tag{35}$$

The EB estimator of $R(t)$ is obtained numerically from the next equation

$$\hat{R}(t)_{EBL} = -\frac{1}{c}\log\left[\frac{(\hat{\beta}+T)^{\hat{\alpha}+m}}{\Gamma(\hat{\alpha}+m)}\int_0^\infty e^{-ce^{-t^2/2\theta}}\theta^{-(\hat{\alpha}+m+1)}\exp\left(-\frac{(\hat{\beta}+T)}{\theta}\right)d\theta\right]. \tag{36}$$

The EB estimator of $h(t)$ at time t under LINEX loss function is given by

$$\hat{h}(t)_{EBL} = \frac{\hat{\alpha}+m}{c}\log\left[1+\frac{ct}{(\hat{\beta}+T)}\right]. \tag{37}$$

EB estimators under GE loss function (EBG)

EB estimation of θ under GE loss function is given by

$$\hat{\theta}_{EBG} = \left[\frac{\Gamma(\hat{\alpha}+m)}{\Gamma(\hat{\alpha}+m+c)}\right]^{1/c}\left(\hat{\beta}+T\right). \tag{38}$$

EB estimators of $R(t)$ and $h(t)$ at time t under GE loss function are respectively, given by

$$\hat{R}(t)_{EBG} = \left[1-\frac{ct^2/2}{(\hat{\beta}+T)}\right]^{((\hat{\alpha}+m)/c)}. \tag{39}$$

$$\hat{h}(t)_{EBG} = \left(\frac{\Gamma(\hat{\alpha}+m)}{\Gamma(\hat{\alpha}+m-c)}\right)^{1/c}\frac{t}{(\hat{\beta}+T)}. \tag{40}$$

Complete data

By setting $r_1 = r_2 = \cdots = r_{m-1} = r_m = 0$, in equation (6) and (29), and proceed to above subsequent equations, we can get the Bayes and empirical estimators under complete data as follows. The likelihood function of Rayleigh distribution and the posterior density function of θ, based on complete data, $\underline{x} = (x_1, x_2, \cdots, x_n)$ are given, respectively, by

$$L(\underline{x}|\theta) = \frac{\prod_{i=1}^{n} x_i}{\theta^n} \exp\left(-\frac{\sum_{i=1}^{n} {x_i}^2}{2\theta}\right). \tag{41}$$

$$f(\theta|x) \propto \theta^{-(\alpha+n+1)} \exp\left(-\frac{(\beta+H)}{\theta}\right). \tag{42}$$

Where, H=$\sum_{i=1}^{n} x_i^2/2$.

The ML, Bayes, and EB estimators of parameter, reliability and hazard function for Rayleigh distribution based on complete data are summarized in the table below

Table 1: Estimation for Rayleigh distribution under complete data

		θ	$R(t)$	$h(t)$
MLE		$\frac{H}{n}$	$\exp\left(-\frac{t^2}{2\hat{\theta}_{MLE}}\right)$	$\frac{t}{\hat{\theta}_{MLE}}$
Bayes	SE	$\frac{(\beta+H)}{\alpha+n-1}$	$\left[1+\frac{t^2/2}{(\beta+H)}\right]^{-(\alpha+n)}$	$\frac{t(\alpha+n)}{(\beta+H)}$
	LINEX	$\frac{-1}{c}\log\left[\frac{2(c(\hat{\beta}+H))^{\frac{(\hat{\alpha}+r)}{2}} K_{(n+\hat{\alpha})}\left(2\sqrt{c(\hat{\beta}+H)}\right)}{\Gamma(\hat{\alpha}+n)}\right]$	has no closed-form	$\frac{\alpha+n}{c}\log\left[1+\frac{ct}{(\beta+H)}\right]$
	GE	$\left(\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n+c)}\right)^{1/c}(\beta+H)$	$\left[1-\frac{ct^2/2}{(\beta+H)}\right]^{((\alpha+n)/c)}$	$\left[\frac{\Gamma(\alpha+n)}{\Gamma(\alpha+n-c)}\right]^{1/c}\frac{t}{(\beta+H)}$
EB	SE	$\frac{(\hat{\beta}+H)}{\hat{\alpha}+n-1}$	$\left[1+\frac{t^2/2}{(\hat{\beta}+\sum_{i=1}^{n} t_i^2/2)}\right]^{-(\hat{\alpha}+1)}$	$\frac{t(\hat{\alpha}+n)}{(\hat{\beta}+H)}$
	LINEX	$\frac{-1}{c}\log\left[\frac{2(c(\hat{\beta}+H))^{\frac{(\hat{\alpha}+r)}{2}} K_{(n+\hat{\alpha})}\left(2\sqrt{c(\hat{\beta}+H)}\right)}{\Gamma(\hat{\alpha}+n)}\right]$	has no closed-form	$\frac{\hat{\alpha}+n}{c}\log\left[1+\frac{ct}{(\hat{\beta}+H)}\right]$
	GE	$\left(\frac{\Gamma(\hat{\alpha}+n)}{\Gamma(\hat{\alpha}+n+c)}\right)^{1/c}(\hat{\beta}+H)$	$\left[1-\frac{ct^2/2}{(\hat{\beta}+H)}\right]^{((\hat{\alpha}+n)/c)}$	$\left[\frac{\Gamma(\hat{\alpha}+n)}{\Gamma(\hat{\alpha}+n-c)}\right]^{1/c}\frac{t}{(\hat{\beta}+H)}$

Type-II Censoring Scheme

By setting $r_1 = r_2 = \cdots = r_{m-1} = 0$ and $r_m = n - m$ in equation (6) and (29), and proceed to above subsequent equations, we can get the Bayes and empirical estimators under type-II censoring scheme as follows.
The likelihood function of Rayleigh distribution and the posterior density function of θ, under type-II censoring, $\underline{x} = (x_1, x_2, \cdots, x_r)$ are given, respectively, by

$$L(\underline{x}|\theta) \propto \theta^{-r} \exp\left(-\frac{1}{2\theta}\left(\sum_{i=1}^{r} x_i^2 + (n-r)x_r^2\right)\right). \tag{43}$$

$$f(\theta|x) \propto \theta^{-(\alpha+r+1)} \exp\left(-\frac{1}{\theta}(\beta+R)\right). \tag{44}$$

Where, $R = \frac{1}{2}(\sum_{i=1}^{r} x_i^2 + (n-r)x_r^2)$.

The ML, Bayes, and EB estimators of parameter, reliability and hazard function for Rayleigh distribution based on type-II censoring scheme are summarized in the table below

Table 2: Estimation for Rayleigh distribution under type-II censoring scheme

		θ	R(t)	$h(t)$
MLE		$\frac{R}{r}$	$\exp\left(-\frac{t^2}{2\hat{\theta}_{MLE}}\right)$	$\frac{t}{\hat{\theta}_{MLE}}$
Bayes	SE	$\frac{\beta+R}{\alpha+r-1}$	$\left[1+\frac{t^2/2}{(\beta+R)}\right]^{-(\alpha+r)}$	$\frac{t(\alpha+r)}{(\beta+R)}$
	LINEX	$\frac{-1}{c}\log\left[\frac{2(c(\beta+R))^{\frac{(\alpha+r)}{2}}K_{(r+\alpha)}\left(2\sqrt{c(\beta+R)}\right)}{\Gamma(\alpha+r)}\right]$	has no closed-form	$\frac{\alpha+r}{c}\log\left[1+\frac{ct}{(\beta+R)}\right]$
	GE	$\left[\frac{\Gamma(\alpha+r)}{\Gamma(\alpha+r+c)}\right]^{1/c}(\beta+R)$	$\left(1-\frac{ct^2/2}{(\beta+R)}\right)^{((\alpha+r)/c)}$	$\left(\frac{\Gamma(\alpha+r)}{\Gamma(\alpha+r-c)}\right)^{1/c}\frac{t}{(\beta+R)}$
EB	SE	$\frac{\hat{\beta}+R}{\hat{\alpha}+r-1}$	$\left[1+\frac{t^2/2}{(\hat{\beta}+R)}\right]^{-(\hat{\alpha}+r)}$	$\frac{t(\hat{\alpha}+r)}{(\hat{\beta}+R)}$
	LINEX	$\frac{-1}{c}\log\left[\frac{2(c(\hat{\beta}+R))^{\frac{(\hat{\alpha}+r)}{2}}K_{(r+\hat{\alpha})}\left(2\sqrt{c(\hat{\beta}+R)}\right)}{\Gamma(\hat{\alpha}+r)}\right]$	has no closed-form	$\frac{\hat{\alpha}+r}{c}\log\left(1+\frac{ct}{(\hat{\beta}+R)}\right)$
	GE	$\left[\frac{\Gamma(\hat{\alpha}+r)}{\Gamma(\hat{\alpha}+r+c)}\right]^{1/c}\left(\hat{\beta}+R\right)$	$\left(1-\frac{ct^2/2}{(\hat{\beta}+R)}\right)^{((\hat{\alpha}+r)/c)}$	$\left[\frac{\Gamma(\hat{\alpha}+r)}{\Gamma(\hat{\alpha}+r-c)}\right]^{1/c}\frac{t}{(\hat{\beta}+R)}$

3 Prediction Interval Based on PCS-II

Prediction is the procedure where the values of unknown observation are obtained depend on known observation. Prediction is very important in statistical inference and includes two methods point and interval prediction. Moreover, there are two types of prediction one sample prediction and two sample prediction. This study discussed prediction interval and two types are considered. Prediction interval is an interval that use the result of a preceding sample to get the result from a future sample from the same distribution with a specific probability.

3.1 One sample Bayesian prediction

The Bayesian prediction interval is obtained for Rayleigh distribution under PCS-II, $\underline{x}=(x_1,x_2,\cdots,x_m)$ as fallows. By using the equations (1), (2), PDF and CDF of Rayleigh distribution, the conditional density function is given by

$$f_m(x_s|\underline{x})=\frac{(n-m)!}{(s-m-1)!(n-s)!}[e^{-\frac{x_m^2}{2\theta}}-e^{-\frac{x_s^2}{2\theta}}]^{s-m-1}[e^{-\frac{x_s^2}{2\theta}}]^{n-s}[e^{-\frac{x_m^2}{2\theta}}]^{-(n-m)}\frac{x_s}{\theta}e^{-\frac{x_s^2}{2\theta}}. \tag{45}$$

Expanding, $[e^{-\frac{x_m^2}{2\theta}}-e^{-\frac{x_s^2}{2\theta}}]^{s-m-1}$ binomially, the conditional density function will be,

$$f_m(x_s|\underline{x})=\sum_{w_1=0}^{s-m-1}C_{w_1}\frac{x_s}{\theta}e^{\frac{-1}{\theta}[(n-m-w_1)(x_s^2-x_m^2)/2]}. \tag{46}$$

Where,

$$C_{w_1}=\frac{(-1)^{s-m-w_1-1}(n-m)!}{w_1!(s-m-w_1-1)!(n-s)!}. \tag{47}$$

The predictive density function is obtained by using the posterior density under PCS-II given in (10), as follow

$$\begin{aligned}f_m^*(x_s|\underline{x}) &= \int_0^\infty f_m(x_s|\underline{x})f(\theta|x)d\theta\\ &= I\sum_{w_1=0}^{s-m-1}C_{w_1}x_s\int_0^\infty \theta^{-(\alpha+m+2)}e^{\frac{-1}{\theta}[(n-m-w_1)(x_s^2-x_m^2)/2+(\beta+T)]}d\theta.\end{aligned}$$

Where, $I=\frac{(\beta+T)^{\alpha+m}}{\Gamma(\alpha+m)}$ therefor it can be derived as

$$f_m^*(x_s|\underline{x})=\sum_{w_1=0}^{s-m-1}C_{w_1}\frac{(\alpha+m)x_s(\beta+T)^{\alpha+m}}{[(n-m-w_1)(x_s^2-x_m^2)/2+(\beta+T)]^{\alpha+m+1}}. \tag{48}$$

From predictive density function (48), we simply obtain the predictive survival function of x_s as follow

$$\begin{aligned}\overline{F}^*_{X_s}(t|\underline{x}) &= \int_t^\infty f^*_m(x_s|\underline{x})dx_s \\ &= \sum_{w_1=0}^{s-m-1} \frac{C_{w_1}}{(n-m-w_1)}\left[1+\frac{(n-m-w_1)(t^2-x_m^2)/2}{(\beta+T)}\right]^{-(\alpha+m)} \end{aligned} \tag{49}$$

A $(1-\tau)100\%$ Bayesian prediction interval for x_s can obtained by solving the following nonlinear equations for lower and upper bounds L,U,

$$\sum_{w_1=0}^{s-m-1} \frac{C_{w_1}}{(n-m-w_1)}\left[1+\frac{(n-m-w_1)(L^2-x_m^2)/2}{(\beta+T)}\right]^{-(\alpha+m)} = 1-\frac{\tau}{2}, \tag{50}$$

$$\sum_{w_1=0}^{s-m-1} \frac{C_{w_1}}{(n-m-w_1)}\left[1+\frac{(n-m-w_1)(U^2-x_m^2)/2}{(\beta+T)}\right]^{-(\alpha+m)} = \frac{\tau}{2}. \tag{51}$$

A $(1-\tau)100\%$ EB prediction interval for x_s can obtained by solving the following nonlinear equations for lower and upper bounds L,U,

$$\sum_{w_1=0}^{s-m-1} \frac{C_{w_1}}{(n-m-w_1)}\left[1+\frac{(n-m-w_1)(L^2-x_m^2)/2}{(\hat\beta+T)}\right]^{-(\hat\alpha+m)} = 1-\frac{\tau}{2}, \tag{52}$$

$$\sum_{w_1=0}^{s-m-1} \frac{C_{w_1}}{(n-m-w_1)}\left[1+\frac{(n-m-w_1)(U^2-x_m^2)/2}{(\hat\beta+T)}\right]^{-(\hat\alpha+m)} = \frac{\tau}{2}, \tag{53}$$

where, $\hat\alpha$ and $\hat\beta$ are the estimated values of unknown hyperparameters under PCS-II which are obtained by solving the equations (30) and (31).

3.2 Two sample Bayesian prediction

$$f_m(y_s|\underline{x}) = \frac{v!}{(s-1)!(v-s)!}[1-e^{-\frac{y_s^2}{2\theta}}]^{s-1}[e^{-\frac{y_s^2}{2\theta}}]^{v-s}\frac{y_s}{\theta}e^{-\frac{y_s^2}{2\theta}} \tag{54}$$

By expanding $[1-e^{-\frac{y_s^2}{2\theta}}]^{s-1}$ therefore,

$$f_m(y_s|\underline{x}) = \sum_{w_2=0}^{s-1} C_{w_2}\frac{y_s}{\theta}e^{\frac{-1}{\theta}[(v-w_2)y_s^2/2]} \tag{55}$$

Where,

$$C_{w_2} = \frac{(-1)^{s-v-w_2-1}(v)!}{w_2!(s-w_2-1)!(v-s)!}. \tag{56}$$

The predictive density function is given by

$$\begin{aligned} f^*_m(y_s|\underline{x}) &= \int_0^\infty f_m(y_s|\underline{x})f(\theta|x)d\theta \\ &= I\sum_{w=0}^{s-1} C_{w_2}y_s\int_0^\infty \theta^{-(\alpha+m+2)}e^{\frac{-1}{\theta}[(v-w_2)y_s^2/2+(\beta+T)]}d\theta \end{aligned}$$

Where I=$\frac{(\beta+T)^{\alpha+m}}{\Gamma(\alpha+m)}$ and hence,

$$f^*_m(y_s|\underline{x}) = (\alpha+m)y_s\sum_{w_2=0}^{s-1} C_{w_2}\frac{(\beta+T)^{\alpha+m}}{[(v-w_2)y_s^2/2+(\beta+T)]^{\alpha+m+1}} \tag{57}$$

From predictive density function (57), we simply obtain the predictive survival function of y_s as follow

$$\begin{aligned}\overline{F}^*_{Y_s}(t|\underline{x}) &= \int_t^\infty f^*_m(y_s|\underline{x})dy_s \\ &= \sum_{w_2=0}^{s-1} \frac{C_{w_2}}{(v-w_2)}\left[1+\frac{(v-w_2)(t^2)/2}{(\beta+T)}\right]^{-(\alpha+m)}\end{aligned}$$

A $(1-\tau)100\%$ Bayesian prediction interval for y_s can obtained by solving the following nonlinear equations for lower and upper bounds L,U,

$$\sum_{w_2=0}^{s-1} \frac{C_{w_2}}{(v-w_2)}\left[1+\frac{(v-w_1)L^2/2}{(\beta+T)}\right]^{-(\alpha+m)} = 1-\frac{\tau}{2}. \tag{58}$$

$$\sum_{w_2=0}^{s-1} \frac{C_{w_2}}{(v-w_2)}\left[1+\frac{(v-w_1)U^2/2}{(\beta+T)}\right]^{-(\alpha+m)} = \frac{\tau}{2}. \tag{59}$$

A $(1-\tau)100\%$ EB prediction interval for y_s can obtained by solving the following nonlinear equations for lower and upper bounds L,U.

$$\sum_{w_2=0}^{s-1} \frac{C_{w_2}}{(v-w_2)}\left[1+\frac{(v-w_1)L^2/2}{(\hat{\beta}+T)}\right]^{-(\hat{\alpha}+m)} = 1-\frac{\tau}{2}, \tag{60}$$

$$\sum_{w_2=0}^{s-1} \frac{C_{w_2}}{(v-w_2)}\left[1+\frac{(v-w_1)U^2/2}{(\hat{\beta}+T)}\right]^{-(\hat{\alpha}+m)} = \frac{\tau}{2}, \tag{61}$$

where, $\hat{\alpha}$ and $\hat{\beta}$ are the estimated values of unknown hyperparameters under PCS-II, which are obtained from solving the equations (30) and (31).

4 Simulation Study

4.1 Estimation based on PCS-II

In this section Monte Carlo simulation study is used by R software, to compare the ML, Bayes, and EB estimates of the parameter θ and functions of θ. Applying the algorithm of [16] as the following steps:

1.Generate m independent uniform, $U(0,1)$ random variables $W_1,W_2,\cdots,W_m$.

2.For given values of the PCS-II $r_1,r_2,\cdots,r_m$, set

$$V_i = W^{1/(i+r_m+r_{m-1}+\cdots+r_{m-i+1})}, i = 1,2,\cdots,m.$$

3.The PCS-II samples of size m from U(0,1) are,

$$U_i = 1-V_mV_{m-1}\cdots V_{m-i+1}, i = 1,2,\cdots,m.$$

4.Finally, we set $X_i = \sqrt{-2\theta\log(1-U)}$ for $i=1,2,\cdots,m$ which is the inverse CDF of the Rayleigh distribution. Hence, $X_1,X_2,\cdots,X_m$ are the required progressive type-II censored sample of size m from Rayleigh distribution.

The ML, Bayes and EB estimators of θ, $R(t)$ and $h(t)$ are computed from the previous mathematical results. The MSE of θ, $R(t)$, and $h(t)$ estimates under loss functions are shown in next tables and computed from

$$MSE_q = \frac{\sum_{i=1}^N(\hat{q}-q_i)^2}{N}. \tag{62}$$

Where N is the number of iterations and equal 10000, $q=(\theta,R(t),h(t))$ and $\hat{q}=(\hat{\theta},\hat{R}(t),\hat{h}(t))$.
The estimation study under PCS-II is illustrated where $(n=30,m=20)$ and three types of censoring schemes are considered as shown in table (3). According to [17], complete data and type-II censored data are special cases of PCS-II. From table (3) scheme[1] is complete data, scheme[2] type-II censored data and scheme[3] a kind of PCS-II. For Bayesian computations $\alpha=3,\beta=1$. Moreover, for EB estimation the estimated values of unknown hyperparameters based on different types of data are given in table (3).

4.2 Prediction interval based on PCS-II

Table (9) and (10) provide 95% Bayesian and EB prediction interval based on PCS-II where one and two sample prediction are considered. We generate PCS-II sample from Rayleigh distribution with $(m = 15, n = 20)$ via using the steps of [3] algorithms with censoring scheme, $r = (0,0,0,1,0,0,1,0,0,2,1,0,0,0,0)$.
The values of hyperparameters related to Bayesian are $\alpha = 3$, $\beta = 1$ and the estimated values of hyperparameters are $\hat{\alpha} = 4.423041$, $\hat{\beta} = 0.4409566$.

Table 3: Censoring schemes for Rayleigh distribution.

Number	Scheme	$\hat{\alpha}, \hat{\beta}$
[1]	(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)	$\hat{\alpha} = 2.288605$ $\hat{\beta} = 0.7229978$
[2]	(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,10)	$\hat{\alpha} = 2.326446$ $\hat{\beta} = 0.6309468$
[3]	(0,0,0,0,0,0,0,3,0,0,0,0,0,1,0,2,1,1,1,1)	$\hat{\alpha} = 4.214621$ $\hat{\beta} = 0.2441586$

Table 4: MSE of θ estimators for Rayleigh distribution under PCS-II.

Estimate			Censoring Scheme		
			[1]	[2]	[3]
MLE			6.242e-04	3.601e-02	2.065e-02
Bayes	SE		5.941e-05	2.149e-03	1.706e-02
	LINEX	c=0.5	5.991e-05	4.687e-04	6.006e-04
		c=1.5	5.932e-05	5.026e-03	9.079e-04
	GE	c=0.5	5.584e-05	2.505e-04	2.040e-03
		c=1.5	5.024e-05	2.031e-04	1.596e-03
EB	SE		3.563e-05	2.670e-04	3.212e-04
	LINEX	c=0.5	5.971e-05	4.030e-03	6.183e-04
		c=1.5	5.950e-05	7.017e-03	6.096e-04
	GE	c=0.5	5.600e-05	2.008e-04	2.465e-04
		c=1.5	2.954e-05	2.924e-04	3.497e-04

Table 5: MSE of $R(t)$ estimators for Rayleigh distribution under PCS-II. $t = 2$

Estimate			Censoring Scheme		
			[1]	[2]	[3]
MLE			0.00025	0.00049	0.00053
Bayes	SE		2.702e-06	2.922e-06	1.434e-05
	LINEX	c=0.5	3.123e-04	4.821e-03	7.490e-03
		c=-1.5	3.007e-04	3.478e-03	6.754e-03
	GE	c=0.5	3.067e-06	2.735e-05	2.99e-05
		c=-1.5	2.665e-07	2.031e-06	1.047e-05
EB	SE		1.669e-07	2.771e-06	1.390e-05
	LINEX	c=0.5	3.490e-04	4.222e-03	5.490e-03
		c=1.5	2.090e-05	2.314e-03	2.754e-03
	GE	c=0.5	8.422e-07	3.993e-06	5.344e-06
		c=-1.5	6.432e-06	2.924e-04	1.772e-03

Table 6: MSE of $R(t)$ estimators for Rayleigh distribution under PCS-II.$t = 4$

Estimate			Censoring Scheme		
			[1]	[2]	[3]
MLE			0.00016	0.00045	0.00075
Bayes	SE		1.265e-07	5.269e-06	9.013e-06
	LINEX	c=0.5	7.123e-07	4.888e-04	5.799e-04
		c=-1.5	9.003e-07	3.056e-04	8.006e-04
	GE	c=0.5	1.266e-06	1.288e-05	1.766e-05
		c=-1.5	4.407e-06	1.239e-05	1.218e-05
EB	SE		1.222e-07	1.792e-05	2.629e-07
	LINEX	c=0.5	3.490e-07	3.160e-05	3.978e-05
		c=-1.5	2.090e-06	3.078e-04	3.999e-04
	GE	c=0.5	1.194e-06	2.013e-05	1.237e-05
		c=-1.5	1.205e-07	3.088e-05	5.050e-07

Table 7: MSE of $h(t)$ estimators for Rayleigh distribution under PCS-II.$t = 2$

Estimate			Censoring Scheme		
			[1]	[2]	[3]
MLE			0.0127	0.0471	0.0523
Bayes	SE		0.0015	0.0044	0.0075
	LINEX	c=0.5	0.0011	0.0024	0.0079
		c=-1.5	0.00065	0.00120	0.00939
	GE	c=0.5	0.00135	0.00297	0.00805
		c=-1.5	0.00162	0.00392	0.00964
EB	SE		6.818e-05	7.792e-5	8.518e-05
	LINEX	c=0.5	3.327e-05	9.059e-05	10.040e-04
		c=-1.5	0.00035	0.00045	0.00055
	GE	c=0.5	4.384e-05	9.046e-05	10.448e-04
		c=-1.5	7.741e-05	6.658e-04	7.923e-04

Table 8: MSE of $h(t)$ estimators for Rayleigh distribution under PCS-II.$t = 4$

Estimate			Censoring Scheme		
			[1]	[2]	[3]
MLE			0.00523	0.0149	0.0278
Bayes	SE		0.00062	0.00116	0.00140
	LINEX	c=0.5	0.000311	0.000640	0.000750
		c=-1.5	0.00810	0.01147	0.025775
	GE	c=0.5	0.00541	0.0083	0.00087
		c=-1.5	0.00647	0.00692	0.00859
EB	SE		0.00027	0.00028	0.00034
	LINEX	c=0.5	5.171e-05	0.000437	0.000493
		c=-1.5	0.00661	0.00851	0.00899
	GE	c=0.5	0.00018	0.00036	0.00041
		c=-1.5	0.00031	0.000663	0.000695

Table 9: One sample Bayesian and EB Prediction Interval for Rayleigh distribution.

Bayesian prediction interval				EB prediction interval			
s	$L_{x_{s:m}}$	$U_{x_{s:m}}$	Length	s	$L_{x_{s:m}}$	$U_{x_{s:m}}$	Length
16	2.2414	2.7520	0.5106	16	1.6607	2.1275	0.4667
17	2.2742	3.0832	0.8089	17	1.6919	2.4200	0.7281
18	2.3407	3.4603	1.1195	18	1.7550	2.7483	0.9933
19	2.4464	3.9720	1.5256	19	1.8543	3.1891	1.3347
20	2.6261	4.9260	2.2999	20	2.0212	4.0035	1.9823

Table 10: Two sample Bayesian and EB Prediction Interval for Rayleigh distribution.

Bayesian prediction interval				EB prediction interval			
s	$L_{y_{s:v}}$	$U_{y_{s:v}}$	Length	s	$L_{y_{s:v}}$	$U_{y_{s:v}}$	Length
1	0.0524	0.6716	0.6191	1	0.0620	0.7887	0.7266
5	0.4378	1.2567	0.8188	5	0.5199	1.4697	0.9497
10	0.8411	1.9260	1.0848	10	1.000	2.2482	1.2472
13	1.1429	2.5376	1.3946	13	1.3608	2.9617	1.6009
15	1.4873	3.5575	2.0702	15	1.7703	4.1587	2.3884

5 Discussions and Conclusions

5.1 Estimation based on PCS-II

Parameter θ

Table (4) shows the MSE of the parameter estimators, where MLE, Bayes, and EB estimators are all have been obtained. For various censoring schemes the non-Bayesian estimator (MLE) yields the worst performance. On the other hand, the EB estimators are superior to the Bayes estimators under loss functions for all schemes except LINEX(c=1.5)in scheme[1], LINEX,GE(c=-1.5) in scheme[2] and LINEX(c=0.5) in scheme[3]. According to scheme[1], the Bayes and EB estimator under GE(c=-1.5) are the best among all its competitors. Among Bayes estimators that are related to scheme[2] the Bayes estimator under GE (c=-1.5) is better than other estimators. Beside this for EB estimators GE(c=0.5) loss functions provides the best estimation. While in case of scheme[3], the Bayes estimator under LINEX(c=0.5) is doing better as compared other loss functions. Furthermore, the EB estimator under GE(c=0.5) shows better performance than other EB estimators.

Reliability function

The MSE of $R(t)$ estimators at time ($t = 2$), are shown in table (5). The EB estimators are shown better performance than the Bayes estimators under loss functions for all schemes except LINEX(c=0.5), GE(c=-1.5) in scheme[1] and GE(c=-1.5) in scheme[2] and scheme[3]. For three types of scheme and among the Bayes estimators the GE(c=-1.5) presents the best Bayes estimator. On the other hand the SE provides the best EB estimator for scheme[1] and scheme[2]. While, in case of scheme[3], the EB estimator under GE(c=0.5) represents the best estimator as compared with the other loss functions. At time $t = 4$, the MSE of $R(t)$ estimators are given in table (6). According to scheme[1] and scheme[3], the EB estimators provide better estimators than the Bayes estimators for all loss functions except LINEX(c=-1.5), GE(c=-1.5) in scheme[1] and SE in scheme[3]. Moreover, in scheme[2], it can be noted that the Bayes estimators are superior to the EB estimators under loss functions except SE and LINEX(c=0.5). It can be deduced from table (5) the SE loss function providers the best Bayes and EB estimators where all three schemes are considered.

Hazard function

Table (6), provides the MSE of $h(t)$ estimators under loss functions at time $t = 2$. The EB estimators are better performance than Bayes estimators under loss functions for all types of schemes that are considered. According to scheme[1] and scheme[2] and among the Bayesian estimation LINEX(c=-1.5) gives the best Bayes estimator as compared other loss functions. While in case of scheme[3] the SE gives the best result. Between the EB estimation in scheme[2] and scheme[3] it can be observed that SE provides the best EB estimator. However in scheme[2] the EB estimator under LINEX (c=0.5) is the best. The calculated results for the MSE of hazard function estimates at time $t = 4$ are given in table (8). It can be noted that the EB estimators are performed better than the Bayes estimators for all schemes. According to various types of schemes, LINEX(c=0.5) shows the best performance as compared with other Bayesian estimators. Beside this, in scheme[2] and scheme[3] SE gives the best EB estimator between all its competitors. While in scheme[1] LINEX(c=0.5) gives the best EB estimator.

5.2 Prediction interval based on PCS-II

Table (9) shows that 95% Bayesian and EB one sample prediction interval, it's clear to observe that the EB performers better results as compared with Bayesian since the length of Bayesian interval is longer than the EB interval. Moreover, the length of Bayesian and EB prediction interval increases as s increases. However, the Bayesian prediction interval provides better results than EB in case of two sample prediction interval as shown table (10).

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

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