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Accelerated Life Testing for Bivariate Distributions based on Progressive Censored Samples with Random Removal

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Abstract: The significance of the statistical inference problem in reliability theory for the bivariate models under accelerated life testing (ALT) is enormous. In practice, independent variables are assumed for the sake of convenience, which contradicts the nature of the problem. The constant stress accelerated life testing (CS-ALT) for the bivariate model based on copula function is introduced in this study. The copula is the method that describes the dependence structure between variables. The model parameters are evaluated using maximum likelihood and Bayesian estimation methods, taking into account that units fail due to only two dependent variables under continuous stress ALTs and a Type-II progressive censoring scheme. For the bivariate model, random removal has been referred to as binomial removal. The Bayesian estimation has been created using symmetric and asymmetric loss functions. The asymptotic confidence intervals are generated using approximated confidence intervals. Interval Bayesian estimators have been employed with credible confidence intervals. The set of simulated data is evaluated for demonstrative reasons, taking into account two and numerous stress levels. Different Monto Carlo simulations are built to compare estimating approaches.

Keywords: Copula; accelerated life testing; progressive censoring; binomial removal; Bayesian inference; Monto Carlo simulation.

1 Introduction

Field tracking studies and warranty databases are two traditional sources of lifetime data in reliability. Unit lifespan information such as warranty durations, unit safety, and unit reliability specifications are taken from this distribution. It goes without saying that poor estimation, particularly in the lower percentiles of the unit's lifetime distribution, can result in significant industry losses due to excessive warranty returns. Accelerated life testing (ALT) is required for industrial units with extended lifespans to expedite unit failure by stressing this highly reliable equipment beyond what they would typically endure in actual operation. The goal is to acquire lifetime data as quickly as possible and at a reasonable cost. ALT, on the other hand, has the following drawbacks:

– To begin, a decision must be taken about how to accelerate failure. Failure modes for some units are known in advance by means of physical/chemical theory or previous experience with similar tests. ALT, on the other hand, must only lead to failure modes that may occur under regular use settings and should not introduce any new failure modes in order to be valid. If a novel failure mode emerges, it must be discovered and taken into account in the lifespan data analysis that follows.
– A choice on how much to accelerate must be made. Excessive stress will obviously cause the unit to fail in a very short time, but failure time data may not provide relevant information regarding the unit's lifetime. Typically, test stress levels are chosen that are outside of the product's specification limitations but within its design limits.

The Arrhenius model, exponential model, and inverse power model are three examples of accelerated models that can be used in investigations. The sort of stress (voltage, pressure, or temperature) that the researcher intends to apply determines the acceleration model to use. For thermal and non-thermal stresses, the Arrhenius model and the inverse power model are

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commonly utilized. Constant-stress ALT (CSALT), step-stress ALT (SSALT), and progressive-stress ALT are the most prevalent types of ALTs (PSALT). In the CSALT, stress is constant throughout the experiment, whereas in the SSALT, stress is gradually raised at precise moments. In the PSALT, on the other hand, tension increases as time passes. ALTs are separated into two types based on the number of stress levels: simple and multiple level ALT. There are only two levels of stress in simple ALT, but multiple ALT has more than two levels of stress.

Several writers have looked into the CSALT, including El-Din et al. [1], who looked into the CSALT estimation problem for the extension of the exponential distribution under progressive Type-II censoring. Abdel Ghaly et al. [2] discussed different estimation methods for CSALT under exponentiated family distribution. The topic of obtaining the optimal CSALT designs for the Lindley distribution was studied by Mohie El-Din et al. [3]. Abd El-Raheem [4,5] discussed CSALT under Type-I and Type-II censoring for the extension of the exponential distribution, respectively. Abd El-Raheem et al. [6] obtained classical and Bayesian inference for modified Kies exponential lifetime distribution based on CSALT under progressively censored sample by using binomial removal. Sief et al. [7] derived likelihood inference for a CSALT model under progressive Type-I interval-censored data from the generalized half-normal distribution. Zhang et al. [8] reviewed on planning and analysis of multi-stress ALT for reliability assessment. For more information of ALT see Nelsen [10].

A copula approach discusses the joint distribution of random variables into individual variable marginal distributions and the copula parameter that is connecting the marginals. When working on a modeling topic, a researcher must strive for the greatest possible fit for the observed dependence structure. Constructing a new ideal copula that can describe the observed dependence is one option. Recently, different authors used the copula function in ALT in different styles to make a dependency of variables. Wang and Yan [11] used bivariate Clayton copula to construct dependence structure of lifetimes of CSALT dependent competing failure model under progressively Type-II censoring. Soliman et al. [12] considered Pareto (Clayton) copula as a tool for the dependency of the causes of failure under CSALT for the competing risks model. Abdel Ghaly et al. [13] introduced a new multi-dimensional Archimedean copula function that is characterized and used this copula to construct a model based on SSALT with dependent competing risks under Type-II censoring. Abdel Ghaly [14] discussed SSALT under Type-II censoring when the lifetime distribution of the different risks are dependent on using the copula approach.

In censored samples, Type-I and Type-II censoring are the most frequent censoring schemes. The Type-I censoring strategy is used when the experimenter reports the test time and the number of failures is random. However, if the number of failed units is provided and the test time is random, we are talking about a Type-II filtering scheme. When an experimenter reports that R_i units, where $i = 1,\ldots,m$, and m is a censored sample size, should be removed at any point during the experiment because they are better suited for another function, we are referring to a progressive censoring system. A progressive Type-II (PT-II) censoring is a Type-II censoring extension. Balakrishnan and Aggarwala [15], and Balakrishnan and Kundu [16] are good places to go for more information on this scheme. It's worth noting that $R_1, R_2, \ldots, R_m$ are all pre-fixed in this approach. In some cases, though, these numerals may appear at random. Assume that each unit eliminated from the life test is independent of the others but has the same probability p as the others. The amount of units removed at each failure time is then distributed according to a binomial distribution. For more information about fixed and random removal see Tse et al. [17], El-Sherpieny et al. [18], Dey et al. [19], Almongy et al. [20] and Ashour et al. [21].

In estimating the bivariate model parameters based on a censored sample, Angali et al. [25] discussed Bayesian estimation of bivariate exponential distributions based on LINEX and Quadratic loss functions. El-Sherpieny et al. [18] obtained parameter estimators by using likelihood and Bayesian estimation methods of bivariate generalized Rayleigh distribution based on Clayton copula under PT-II censored sample. Muhammed and Almetwally [22] discussed Bayesian and Non-Bayesian estimation for the bivariate inverse Weibull distribution under PT-II censored samples. Muhammed [23] drive bivariate inverted Topp-Leone distribution and estimated its parameters based on PT-II censored samples. El-Morshedy et al. [24] obtained a bivariate Burr X generator of distributions and estimated parameters based on Type-II censored samples.

The paper aims to analyze the bivariate model under the dependent exponential times when the failure times are accelerated with CSALT. And, the lifetime data is collected under the Type-II progressive censoring scheme. Also, binomial removal is used at each failure time. The copula approach is used to describe the dependence structure between variables. The parameters of the proposed model are estimated with the likelihood and Bayesian estimation method for point and interval estimators. Also, different loss functions are used to construct the Bayesian estimators. The developed results are used for the analysis of some simulated data sets. Also, estimators are assessed and compared with the Monto Carlo simulation study.

The paper is organized as follows. The lifetime distribution and test assumptions are given in Section 2. The maximum likelihood estimation (MLE) is presented in Section 3. The Bayesian estimation is presented in Section 4. In Section 5, a computational illustration is presented to investigate the proposed model and given data analysis. At last, some conclusions are given in Section 6.

2 Lifetime Distribution and Test Assumptions

In this section, we discussed bivariate exponential distribution based on FGM copula, multiple CSALT assumptions, concomitants of order statistics, binomial removal, PT-II censored sample, and introduced the likelihood function of bivariate model based on CSALT under PT-II censored sample with random removal.

2.1 Bivariate Exponential Distribution based on Copula

A copula is a useful tool for describing multivariate distributions in a dependence structure. Copulas were introduced by Nelsen [9] as a function that combines multivariate distribution functions with uniform [0, 1] margins. The pdf and cdf for the two-dimension copula were introduced by Sklar [26] as follows consider two random variables X and Y with distribution functions $F(x)$ and $F(y)$, respectively, then the joint cdf for bivariate copula is given as:

$$F(x,y) = C(F(x;\Delta_1),\ F(y;\Delta_2)), \tag{1}$$

and the joint probability density function (pdf) for bivariate copula as follows:

$$f(x,y) = f(x;\Delta_1)\ f(y;\Delta_2)\ c(F(x;\Delta_1),\ F(y;\Delta_2)). \tag{2}$$

Many couples have been defined based on Equations (1) and (2), we used Farlie-Gumbel-Morgenstern (FGM) as a simple example of this model. The FGM copula is one of the most well-known parametric copula families; it was initially introduced by Gumbel [27]. The following are the FGM copula's joint CDF and pdf:

$$C(F(x;\Delta_1),\ F(y;\Delta_2)) = F(x;\Delta_1)\ F(y;\Delta_2)\ \{1+\theta\ [(1-F(x;\Delta_1))\ (1-F(y;\Delta_2))]\}, \tag{3}$$

and

$$c(F(x;\Delta_1),\ F(y;\Delta_2)) = 1+\theta\,(1-2\,F(x;\Delta_1))\,(1-2\,F(y;\Delta_2)), \tag{4}$$

respectively, where θ is a copula parameter $-1<\theta<1$.

We used exponential distribution to make the bivariate distribution as a simple example of this copula. The following are the exponential distribution CDF and pdf:

$$F(x;\delta) = 1-e^{-\delta x}, x>0,\ \delta>0. \tag{5}$$

The corresponding pdf of (5) is given by

$$f(x;\delta) = \delta e^{-\delta x}, x>0,\ \delta>0. \tag{6}$$

According to Sklar theorem in Equations (1) and (2), FGM copula Equations (3) and (4), and exponential distribution Equations (5) and (6), we get the joint cdf and pdf of bivariate exponential distribution based on FGM copula function which can be denoted as BFGME distribution as follows:

$$F(x,y) = \left\{1-e^{-\delta_1 x}\right\}\left\{1-e^{-\delta_2 y}\right\}\left(1+\theta\left\{e^{-\delta_1 x-\delta_2 y}\right\}\right), \tag{7}$$

and

$$f(x,y) = \delta_1\delta_2 e^{-\delta_1 x-\delta_2 y}\left(1+\theta\left\{2\,e^{-\delta_1 x}-1\right\}\left\{2\,e^{-\delta_2 y}-1\right\}\right). \tag{8}$$

2.2 Multiple Constant-Stress Accelerated Life Testing Assumptions

The constant-stress accelerated life testing (CSALT) assumption for bivariate distributions is introduced in this subsection. Assume that an accelerated life testing (ALT) contains a number of stress levels $L\geq 2$, where the stress is arranged ascendingly where $\phi_1<\phi_2<...<\phi_L$, and identical n_l units are exposed to an accelerated condition within the level l, $l=1,2,...,L$ so that the number of units under the lifetime experiment is $\sum_{l=1}^{L} n_l = n$, where n is the total sample size in the test.

Multiple CSALT's assumptions in this situation are as follows:

1. The lifetime of the experimental units follows an exponential(σ_l) distribution at each stress level ϕ_l.

2.The following relationship connects the scale parameter in each stress level σ_l and the stress level ϕ_l:

$$\log(\sigma_l) = \zeta + \beta\eta_l,\ l = 0,1,...,L, \tag{9}$$

where $\zeta \in (-\infty,\infty)$ and $\beta > 0$ are the unknown model parameters and $\eta_l = \eta(\phi_l)$ is an increasing function of ϕ.
(a)If $\eta(\phi_l) = \ln(\phi_l)$, the model in (9) becomes the **inverse power model**.
(b)If $\eta(\phi_l) = \frac{1}{-\phi_l}$, the model in (9) becomes **Arrhenius model**.
(c)If $\eta(\phi_l) = \phi_l$, the model in (9) becomes **exponential model**.
We can turn to Nelson [10] book, notably Chapter 2, for further information on acceleration and its various models.

From equation (9), we have

$$\sigma_l = \sigma_0 \exp\{\beta(\eta_l - \eta_0)\} = \sigma_0\alpha^{h_l} > 0, \qquad l = 0,1,...,L, \tag{10}$$

where $\alpha = \exp\{\beta(\eta_1 - \eta_0)\} = \frac{\sigma_1}{\sigma_0} > 1$, $\sigma_0 > 0$ is the scale parameter of the marginal distribution under usage conditions ϕ_0, and

$$h_l = \frac{\phi_l - \phi_0}{\phi_1 - \phi_0}, \quad l = 1,2,...,L, \text{ satisfying } h_L > h_{L-1} > ... > h_1 = 1.$$

In general form, the transformation from the parameters σ_l to the new parameters $(\sigma_0\alpha^{h_l})$ is an one-to-one mapping. Thus, the unknown parameters for X-variable should be estimated are α_1, and σ_{01} and for Y-variable should be estimated are α_2, and σ_{02}.

2.3 Concomitants of Order Statistics

Assume (X,Y) is a random vector with joint CDF and pdf from the BFGME distribution, as shown in Equations (7) and (8). If the X-variates are arranged in ascending order as follows:

$$X_{1:m:n} \le X_{2:m:n} \le \cdots \le X_{m:m:n},$$

The Y-variates that are paired with these order statistics are then indicated as:

$$Y_{[1:m:n]} \le Y_{[2:m:n]} \le \cdots \le Y_{[m:m:n]},$$

and dubbed the order statistics concomitants. Yang [29] has addressed the general distribution theory for concomitants of order statistics. David and Galambos [30] explored the asymptotic theory of concomitant of order statistics under the assumption that (X,Y) has a bivariate normal distribution. The joint pdf of $(X_{i:m:n}, Y_{[i:m:n]})$ based concomitants of order statistics can be written as follows

$$f_{X_{i:m:n},Y_{[i:m:n]}}(x,y) = \frac{n!}{(i-1)!(n-i)!} f(x_i,y_i)\,[F(x_i)]^{i-1}\,[1-F(x_i)]^{n-i}$$

2.4 Progressive Censored Sample

Assume n independent units are placed through a life-test, with failure times $X_l,\ldots,X_n$ distributed uniformly using CDF $F(x)$ and PDF $f(x)$. Assume that there are m failures to be observed and that the progressive Type-II censoring scheme is $(R_l,\ldots,R_m)$. These progressive Type-II censoring failure times will be denoted by the notation $X_{i:m:n}, i = 1,2,\ldots,m$, but keep in mind that they are still dependent on the specific choice of $(R_l,R_2,\ldots,R_m)$. When the first failure occurs, $x_{1:m:n}$, then remove R_1 units at random from the remaining $(n-1)$ surviving units; when the second failure occurs, $x_{2:m:n}$, then remove R_2 units at random from the surviving $n-2-R_1$ units; and so on until the m^th failure occurs, at which time all remaining $n-m-R_1-R_2-\cdots-R_{(m-1)}$ items are withdrawn. For more details see Balakrishnan and Aggarwala [28].

2.5 Binomial Removal

Assume that each unit that is eliminated from the test is unrelated to the others but has the same probability of being removed (p). After that, a binomial distribution is used to distribute the number of units withdrawn at each failure time. That is $R_1 \sim binomial(n-m,P)$, $R_j \sim binomial(n-m-\sum_{l=1}^{m-1} R_j,P)$; $j=2,\ldots,m-1$, and $R_m = n-m-\sum_{j=1}^{m-1} R_j$. The probability mass function and the number of units removed at each failure time are assumed to have a binomial distribution:

$$Pr(R_1=r_1)=\binom{n-m}{r_1}P^{r_1}(1-P)^{n-m-r_1}$$

while, for $i=2,3,...,m-1$:

$$Pr(R_i=r_i|R_{i-1})=\binom{n-m-\sum_{j=1}^{m-1} r_j}{r_i}p^{r_i}(1-p)^{n-m-\sum_{j=1}^{m} r_j}$$

where $0 \le r_j \le n-m_l-\sum_{j=1}^{l-1} r_j$. For more details see Tse et al. [17].

2.6 Likelihood Function

Assume we set $(X_{1:m_l:n_l}, Y_{[1:m_l:n_l]})$, $(X_{2:m_l:n_l}, Y_{[2:m_l:n_l]}),\ldots,$ $(X_{m_l:m_l:n_l}, Y_{[m_l:m_l:n_l]})$ be a random sample from a bivariate distribution with joint cdf $F(x,y)$ and joint pdf $f(x,y)$, then the likelihood function of bivariate distribution based on progressive Type-II censoring using binomial removal is given as following:

$$L^*(\Theta)=\prod_{l=1}^{L} c_l \prod_{i=1}^{m_l} f(x_{li:m_l:n_l}, y_{[li:m_l:n_l]};\Theta)\left(1-F(x_{li:m_l:n_l};\Theta_1)\right)^{R_{li}} Pr(\boldsymbol{R}_{li}), \tag{11}$$

where $\Theta=(\Theta_1,\Theta_2,\theta)$ is a vector of parameters for joint pdf, $\Theta_j;\ j=1,2$ is a vector parameter of each marginal pdf, c_l is a constant and doesn't depend on parameters Θ. For more details of bivariate data under simple and multiple ramp-stress of CSALT based on progressive Type-II censoring see Tables 1 and 2, receptively. Furthermore, R_{li} are independent of $X_{li:m_l:n_l}$, and so the MLE of p can be derived by maximizing $Pr(R_{li})$ directly. Hence the likelihood function of p is given by

$$Pr(\boldsymbol{R}_{li}=r_{li})=\frac{(n_l-m_l)!}{\left(n_l-m_l-\sum_{i=1}^{m_l-1} r_{li}\right)!\prod_{i=1}^{m_l-1} r_{li}!}p^{\sum_{i=1}^{m_l-1} r_{li}}(1-p)^{(m_l-1)(n_l-m_l)-\sum_{i=1}^{m_l-1}(m_l-i)r_{li}}$$

Hence the MLE of p is given by

$$\hat{p}=\frac{\sum_{i=1}^{m_l-1} r_{li}}{(m_l-1)(n_l-m_l)-\sum_{i=1}^{m_l-1}(m_l-i-1)r_{li}} \tag{12}$$

We can simplify $\prod_{i=1}^{m} f(x_{i:m:n}, y_{i:m:n};\Theta)\left(1-F(x_{i:m:n};\Theta_1)\right)^{R_i}$ by this notation $L(\Theta)$. For more information of likelihood function of bivariate distribution based on progressive Type-II censoring see Balakrishnan and Kim [31,32].

Table 1: Simple Ramp-stress

tress level	X-variable	Y-variable	R-removal
S	$x_{1:m_1:n_1}$	$y_{[1:m_1:n_1]}$	R_{11}
h_1	$x_{2:m_1:n_1}$	$y_{[2:m_1:n_1]}$	R_{12}
	$\vdots$	$\vdots$	$\vdots$
	$x_{m_1:m_1:n_1}$	$y_{[m_1:m_1:n_1]}$	R_{1m_1}
h_2	$x_{1:m_2:n_2}$	$y_{[1:m_2:n_2]}$	R_{21}
	$x_{2:m_2:n_2}$	$y_{[2:m_2:n_2]}$	R_{22}
	$\vdots$	$\vdots$	$\vdots$
	$x_{m_2:m_2:n_2}$	$y_{[m_2:m_2:n_2]}$	R_{2m_2}

Table 2: Multiple Ramp-stress

tress level	X-variable	Y-variable	R-removal
S h_1	$x_{1:m_1:n_1}$	$y_{[1:m_1:n_1]}$	R_{11}
	$x_{2:m_1:n_1}$	$y_{[2:m_1:n_1]}$	R_{12}
	$\vdots$	$\vdots$	$\vdots$
	$x_{m_1:m_1:n_1}$	$y_{[m_1:m_1:n_1]}$	R_{1m_1}
h_2	$x_{1:m_2:n_2}$	$y_{[1:m_2:n_2]}$	R_{21}
	$x_{2:m_2:n_2}$	$y_{[2:m_2:n_2]}$	R_{22}
	$\vdots$	$\vdots$	$\vdots$
	$x_{m_2:m_2:n_2}$	$y_{[m_2:m_2:n_2]}$	R_{2m_2}
$\vdots$	$\vdots$	$\vdots$	$\dots$
h_L	$x_{1:m_L:n_L}$	$y_{[1:m_L:n_L]}$	R_{L1}
	$x_{2:m_L:n_L}$	$y_{[2:m_L:n_L]}$	R_{L2}
	$\vdots$	$\vdots$	$\vdots$
	$x_{m_L:m_L:n_L}$	$y_{[m_L:m_L:n_L]}$	R_{Lm_L}

3 Estimation via Maximum Likelihood Method

In this section, the classical estimates of the parameters of BFGME distribution under PT-II censored sample with binomial removal are obtained, based on the model assumption of CSALT, then the joint CDF and pdf of BFGME distribution based on CSALT is becomes,

$$F_l(x,y)=\left\{1-e^{-\sigma_{01}\alpha_1^{h_l}x}\right\}\left\{1-e^{-\sigma_{02}\alpha_2^{h_l}y}\right\}\left(1+\theta\left\{e^{-\sigma_{01}\alpha_1^{h_l}x-\sigma_{02}\alpha_2^{h_l}y}\right\}\right), \tag{13}$$

and

$$f_l(x,y)=\sigma_{01}\alpha_1^{h_l}\sigma_{02}\alpha_2^{h_l}e^{-\sigma_{01}\alpha_1^{h_l}x-\sigma_{02}\alpha_2^{h_l}y}\left(1+\theta\left\{2\,e^{-\sigma_{01}\alpha_1^{h_l}x}-1\right\}\left\{2\,e^{-\sigma_{02}\alpha_2^{h_l}y}-1\right\}\right). \tag{14}$$

By using Equation (11), the likelihood function under PT-II censoring based on CSALT after estimating P by Equation (12), can have the following form

$$\begin{aligned}L(\Theta)=&\sigma_{01}^{\sum_{l=1}^{L}m_l}\alpha_1^{\sum_{l=1}^{L}m_lh_l}e^{-\sum_{l=1}^{L}\sigma_{01}\alpha_1^{h_l}\sum_{i=1}^{m_l}(R_{li}+1)x_{li:m_l:n_l}-\sum_{l=1}^{L}\sigma_{02}\alpha_2^{h_l}\sum_{i=1}^{m_l}y_{[li:m_l:n_l]}}\\&\sigma_{02}^{\sum_{l=1}^{L}m_l}\alpha_2^{\sum_{l=1}^{L}m_lh_l}\prod_{l=1}^{L}C_l\prod_{i=1}^{m_l}\left[1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l}x_{li:m_l:n_l}}-1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l}y_{[li:m_l:n_l]}}-1\right)\right],\end{aligned} \tag{15}$$

where Θ is a vector of parameters $(\alpha_1,\sigma_{01},\alpha_2,\sigma_{02},\theta)$, and C_l is a normalizing constant with knowledge m_1 and n_1 doses not depend on Θ.
The log-likelihood function after removing the normalizing constant, can be formed as in Equation (16),

$$\begin{aligned}\ell(\Theta)=&[\ln(\alpha_1)+\ln(\alpha_2)]\sum_{l=1}^{L}m_lh_l-\sigma_{01}\sum_{l=1}^{L}\alpha_1^{h_l}\sum_{i=1}^{m_l}(R_{li}+1)x_{li:m_l:n_l}-\sigma_{02}\sum_{l=1}^{L}\alpha_2^{h_l}\sum_{i=1}^{m_l}y_{[li:m_l:n_l]}\\&[\ln(\sigma_{01})+\ln(\sigma_{02})]\sum_{l=1}^{L}m_l+\sum_{l=1}^{L}\sum_{i=1}^{m_l}\ln\left[1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l}x_{li:m_l:n_l}}-1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l}y_{[li:m_l:n_l]}}-1\right)\right],\end{aligned} \tag{16}$$

By finding partial derivatives of ℓ with respect to the distribution parameters, we have

$$\begin{aligned}\frac{\partial\ell(\Theta)}{\partial\alpha_1}=&\frac{\sum_{l=1}^{L}m_lh_l}{\alpha_1}-\sigma_{01}\sum_{l=1}^{L}h_l\alpha_1^{h_l-1}\sum_{i=1}^{m_l}(R_{li}+1)x_{li:m_l:n_l}-\\&2\sigma_{01}\theta\sum_{l=1}^{L}h_l\alpha_1^{h_l-1}\sum_{i=1}^{m_l}\frac{x_{li:m_l:n_l}e^{-\sigma_{01}\alpha_1^{h_l}x_{li:m_l:n_l}}\left(2e^{-\sigma_{02}\alpha_2^{h_l}y_{[li:m_l:n_l]}}-1\right)}{1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l}x_{li:m_l:n_l}}-1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l}y_{[li:m_l:n_l]}}-1\right)},\end{aligned} \tag{17}$$

$$\frac{\partial \ell(\Theta)}{\partial \sigma_{01}} = -\sum_{l=1}^{L} \alpha_1^{h_l} \sum_{i=1}^{m_l} (R_{li}+1) x_{li:m_l:n_l} + \frac{\sum_{l=1}^{L} m_l}{\sigma_{01}} - 2\theta \sum_{l=1}^{L} \alpha_1^{h_l} \sum_{i=1}^{m_l} \frac{x_{li:m_l:n_l} e^{-\sigma_{01}\alpha_1^{h_l} x_{li:m_l:n_l}} \left(2e^{-\sigma_{02}\alpha_2^{h_l} y_{[li:m_l:n_l]}} - 1\right)}{1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l} x_{li:m_l:n_l}} - 1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l} y_{[li:m_l:n_l]}} - 1\right)}, \tag{18}$$

$$\frac{\partial \ell(\Theta)}{\partial \alpha_2} = \frac{\sum_{l=1}^{L} m_l h_l}{\alpha_2} - \sigma_{02} \sum_{l=1}^{L} h_l \alpha_2^{h_l-1} \sum_{i=1}^{m_l} y_{[li:m_l:n_l]} - 2\sigma_{02}\theta \sum_{l=1}^{L} h_l \alpha_2^{h_l-1} \sum_{i=1}^{m_l} \frac{y_{[li:m_l:n_l]} e^{-\sigma_{02}\alpha_2^{h_l} y_{[li:m_l:n_l]}} \left(2e^{-\sigma_{01}\alpha_1^{h_l} x_{li:m_l:n_l}} - 1\right)}{1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l} x_{li:m_l:n_l}} - 1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l} y_{[li:m_l:n_l]}} - 1\right)}, \tag{19}$$

$$\frac{\partial \ell(\Theta)}{\partial \sigma_{02}} = -\sum_{l=1}^{L} \alpha_2^{h_l} \sum_{i=1}^{m_l} y_{[li:m_l:n_l]} + \frac{\sum_{l=1}^{L} m_l}{\sigma_{02}} - 2\theta \sum_{l=1}^{L} \alpha_2^{h_l} \sum_{i=1}^{m_l} \frac{y_{[li:m_l:n_l]} e^{-\sigma_{02}\alpha_2^{h_l} y_{[li:m_l:n_l]}} \left(2e^{-\sigma_{01}\alpha_1^{h_l} x_{li:m_l:n_l}} - 1\right)}{1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l} x_{li:m_l:n_l}} - 1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l} y_{[li:m_l:n_l]}} - 1\right)}, \tag{20}$$

and

$$\frac{\partial \ell(\Theta)}{\partial \theta} = \sum_{l=1}^{L} \sum_{i=1}^{m_l} \frac{\left(2e^{-\sigma_{01}\alpha_1^{h_l} x_{li:m_l:n_l}} - 1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l} y_{[li:m_l:n_l]}} - 1\right)}{1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l} x_{li:m_l:n_l}} - 1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l} y_{[li:m_l:n_l]}} - 1\right)}. \tag{21}$$

The MLE of Θ is $\hat{\Theta} = (\hat{\alpha}_1, \hat{\sigma}_{01}, \hat{\alpha}_2, \hat{\sigma}_{02}, \hat{\theta})$, which can be obtained by solving Equations (17)-(21) at the same time. Unfortunately, solving these equations will be quite difficult, therefore we will have to resort to numerical methods such as the Newton-Raphson method. The normal approximation of the MLE of Θ can be used for constructing approximate confidence intervals and for testing $\alpha_1, \sigma_{01}, \alpha_2, \sigma_{02}$, and θ, as will be seen in section 5.

4 Bayesian Estimation

This section includes the Bayesian estimation (BEs) of α_1, σ_{01}, α_2, σ_{02} and θ. We assume that α_1, σ_{01}, α_2, σ_{02} and θ are independent and have gamma priors. DeGroot and Goel [33] were the first to examine a gamma prior for the acceleration parameters $\alpha_1, \alpha_2 > 1$. They claimed that the accelerating parameter α_1, α_2 will be greater than 1 in most accelerated life testing problems. To ensure that the acceleration model is applicable to all positive values of α_1, α_2, we will use prior distributions for α_1, α_2 that assigns positive density to all positive values of α_1, α_2. If the experimenter is almost certain that $\alpha_1, \alpha_2 > 1$, he can choose a gamma prior distribution that gives the interval $0 < \alpha_1, \alpha_2 < 1$ a small probability. See Section 3 of DeGroot and Goel [33] for further information on this subject. The following are the gamma priors for distribution parameters:

$$\Pi_j(\alpha_j) \propto \alpha_j^{\omega_j - 1} e^{-\kappa_j \alpha_j}, \quad \alpha_j > 1,\ \omega_j, \kappa_j > 0, j = 1,3, \tag{22}$$

$$\Pi_j(\sigma_{0j}) \propto \sigma_{0j}^{\omega_j - 1} e^{-\kappa_j \sigma_{0j}}, \quad \sigma_{0j} > 0,\ \omega_j, \kappa_j > 0, j = 2,4, \tag{23}$$

and

$$\Pi_5(\theta) \propto (1-\theta)^{\omega_5 - 1}(1+\theta)^{\kappa_5 - 1}, \quad 0 < \frac{1-\theta}{2} < 1,\ \omega_5, \kappa_5 > 0. \tag{24}$$

According to Equation (24), $\frac{1-\theta}{2}$ follows a $Beta(\omega_5, \kappa_5)$ distribution where $\theta \in (-1,1)$.

The joint prior of $\alpha_1, \sigma_{01}, \alpha_2, \sigma_{02}$ and θ is obtained as

$$\Pi(\Theta) \propto \alpha_1^{\omega_1 - 1} \alpha_2^{\omega_3 - 1} \sigma_{01}^{\kappa_2 - 1} \sigma_{02}^{\kappa_4 - 1} \exp\{-(\alpha_1\kappa_1 + \sigma_{01}\kappa_2 + \alpha_2\kappa_3 + +\sigma_{02}\kappa_4)\}(1-\theta)^{\omega_5 - 1}(1+\theta)^{\kappa_5 - 1}. \tag{25}$$

We can use the estimates and variance-covariance matrix of the MLE approach to find suitable and superior values for the hyper-parameters (elective hyper-parameters) of the independent joint prior.

The joint posterior distribution of Θ is created by multiplying Equation (15) by (25) and applying some simplifications.

$$\begin{aligned}\Pi(\Theta|x,y) \propto & \sigma_{01}^{\kappa_2-1+\sum_{l=1}^{L} m_l} \sigma_{02}^{\kappa_4-1+\sum_{l=1}^{L} m_l} e^{-\sigma_{01}\left(\kappa_2+\sum_{l=1}^{L}\alpha_1^{h_l}\sum_{i=1}^{m_l}(R_{li}+1)x_{li:m_l:n_l}\right)-\sigma_{02}\left(\kappa_4+\sum_{l=1}^{L}\alpha_2^{h_l}\sum_{i=1}^{m_l}y_{[li:m_l:n_l]}\right)} \\ & \alpha_1^{\kappa_1-1+\sum_{l=1}^{L} m_l h_l} \alpha_2^{\kappa_3-1+\sum_{l=1}^{L} m_l h_l} \prod_{l=1}^{L}\prod_{i=1}^{m_l}\left[1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l}x_{li:m_l:n_l}}-1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l}y_{[li:m_l:n_l]}}-1\right)\right] \\ & e^{-\alpha_2\kappa_3-\alpha_1\kappa_1}(1-\theta)^{\omega_5-1}(1+\theta)^{\kappa_5-1}.\end{aligned} \tag{26}$$

Using squared error (SE) and Linear-Exponential (LINEX) loss functions, the BEs of $u(\Theta)=u(\alpha_1,\sigma_{01},\alpha_2,\sigma_{02},\theta)$ are as an Algorithm 1. The Bayesian estimator based on squared-error loss function (SELF) is,

$$\tilde{u}_{SE}(\Theta)=E(u(\Theta)). \tag{27}$$

The symmetric loss function is the SELF, which is defined by,

$$L_S(\tilde{\Theta},\Theta)\propto\left(\tilde{\Theta}-\Theta\right)^2. \tag{28}$$

The Bayesian estimator based on SELF is

$$\tilde{u}_{LINEX}(\Theta)=-\frac{1}{c}\log[E(e^{-c\,u(\Theta)})],\;\; c\neq 0. \tag{29}$$

The LINEX loss function can be stated as follows, assuming that the minimal loss occurs at $\tilde{\Theta}=\Theta$:

$$L_c(\tilde{\Theta},\Theta)\propto e^{c\left(\tilde{\Theta}^L-\Theta\right)}-c\left(\tilde{\Theta}^L-\Theta\right)-1;\;\; c\neq 0, \tag{30}$$

Both BEs of $u(\alpha_1,\sigma_{01},\alpha_2,\sigma_{02},\theta)$ in (27) and (29) are obviously considered to be the division of more than one integration over each other. Multiple integrals, as we all know, are extremely difficult to solve analytically or even mathematically by hand. To get an approximate value of integrals, we must utilize the Markov Chain Monte Carlo (MCMC) technique. The Metropolis-Hastings (MH) algorithm, also known as the random walk algorithm, is a key component of the MCMC approach. The MH method considers for each iteration of the process, a candidate value can be generated from normal proposal distribution, similar to acceptance-rejection sampling.

The MH algorithm produces a series of draws from BFGME distribution of CSALT based on progressive Type-II censoring as follows:

Then, the BEs of $u(\alpha_1,\sigma_{01},\alpha_2,\sigma_{02},\theta)$ using MCMC under SE, and LINEX loss functions are respectively

$$\tilde{u}_{SE}=\frac{1}{I-M}\sum_{i=M+1}^{I}u\left(\tilde{\alpha}_1^{(i)},\tilde{\sigma}_{01}^{(i)},\tilde{\alpha}_2^{(i)},\tilde{\sigma}_{02}^{(i)},\tilde{\theta}^{(i)}\right),$$

$$\tilde{u}_{LINEX}=-\frac{1}{c}\log\left[\frac{1}{I-M}\sum_{i=M+1}^{N}\exp\left\{-c\,u\left(\tilde{\alpha}_1^{(i)},\tilde{\sigma}_{01}^{(i)},\tilde{\alpha}_2^{(i)},\tilde{\sigma}_{02}^{(i)},\tilde{\theta}^{(i)}\right)\right\}\right],$$

where M is the burn-in period.

The conditional posterior distributions used in the MH algorithm are as follows:

$$\begin{aligned}\mathbb{C}^*(\alpha_1|\sigma_{01},\alpha_2,\sigma_{02},\theta)\propto & \alpha_1^{\kappa_1-1+\sum_{l=1}^{L}m_l h_l}e^{-\alpha_1\kappa_1}e^{-\sigma_{01}\sum_{l=1}^{L}\alpha_1^{h_l}\sum_{i=1}^{m_l}(R_{li}+1)x_{li:m_l:n_l}} \\ & \prod_{l=1}^{L}\prod_{i=1}^{m_l}\left[1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l}x_{li:m_l:n_l}}-1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l}y_{[li:m_l:n_l]}}-1\right)\right],\end{aligned} \tag{31}$$

$$\begin{aligned}\mathbb{C}^*(\sigma_{01}|\alpha_1,\alpha_2,\sigma_{02},\theta)\propto & \sigma_{01}^{\kappa_2-1+\sum_{l=1}^{L}m_l}e^{-\sigma_{01}\left(\kappa_2+\sum_{l=1}^{L}\alpha_1^{h_l}\sum_{i=1}^{m_l}(R_{li}+1)x_{li:m_l:n_l}\right)} \\ & \prod_{l=1}^{L}\prod_{i=1}^{m_l}\left[1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l}x_{li:m_l:n_l}}-1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l}y_{[li:m_l:n_l]}}-1\right)\right],\end{aligned} \tag{32}$$

Algorithm 1: MH algorithm

```
Input: α1^(0) = α̂1, σ01^(0) = σ̂01, α2^(0) = α̂2, σ02^(0) = σ̂02, θ^(0) = θ̂.
Input: I is a length of MCMC vector result for each parameter.
for each parameter do
    Set: i = 1.
    for each, i ∈ I do
        Assume: α1* has proposal distribution (Normal) N(α1^(i−1), var(α1^(i−1))). Calculate the acceptance probability
        A(α1^(i−1)|α1*) = min[1, P*(α1*|σ01^(i−1), α2*, σ02^(i−1), θ^(i−1)) / P*(α1^(i−1)|σ01^(i−1), α2*, σ02^(i−1), θ^(i−1))].
        Generate: U ~ U(0,1).
        if U ≤ A(α1^(i−1)|α1*), then
            α1^(i) = α1*
        end
        else
            α1^(i) = α1^(i−1)
        end
    end
    α̃1 = α1^(1) ... α1^(I)
end
return (α̃1, σ̃01, α̃2, σ̃02, θ̃)
```

$$\mathbb{C}^*(\alpha_2|\alpha_1,\sigma_{01},\sigma_{02},\theta) \propto \alpha_2^{\kappa_3-1+\sum_{l=1}^{L} m_l h_l} e^{-\alpha_2\kappa_3} e^{-\sigma_{02}\sum_{l=1}^{L}\alpha_2^{h_l}\sum_{i=1}^{m_l} y_{[li:m_l:n_l]}} \prod_{l=1}^{L}\prod_{i=1}^{m_l}\left[1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l}x_{li:m_l:n_l}}-1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l}y_{[li:m_l:n_l]}}-1\right)\right], \tag{33}$$

$$\mathbb{C}^*(\sigma_{02}|\alpha_1,\alpha_2,\sigma_{01},\theta) \propto \sigma_{02}^{\kappa_4-1+\sum_{l=1}^{L} m_l} e^{-\sigma_{02}\left(\kappa_4+\sum_{l=1}^{L}\alpha_2^{h_l}\sum_{i=1}^{m_l} y_{[li:m_l:n_l]}\right)} \prod_{l=1}^{L}\prod_{i=1}^{m_l}\left[1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l}x_{li:m_l:n_l}}-1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l}y_{[li:m_l:n_l]}}-1\right)\right], \tag{34}$$

and

$$\mathbb{C}^*(\theta|\alpha,\sigma_0) \propto (1-\theta)^{\omega_5-1}(1+\theta)^{\kappa_5-1}\prod_{l=1}^{L}\prod_{i=1}^{m_l}\left[1+\theta\left(2e^{-\sigma_{01}\alpha_1^{h_l}x_{li:m_l:n_l}}-1\right)\left(2e^{-\sigma_{02}\alpha_2^{h_l}y_{[li:m_l:n_l]}}-1\right)\right], \tag{35}$$

Bayesian estimates contain CIs that are called credible intervals or are often called highest posterior density (HPD) intervals. They employed a technique that has been widely used to create HPD ranges for unknown distribution parameters. The proposed MH algorithm is utilized to create estimates in this method; for further information on the HPD algorithm, see Chen and Shao [34].

5 Computational Illustration

5.1 Simulation Study

We need to simulate PT-II censored samples with binomial removals in order to verify the efficiency of the parameters estimators of the BFGME distribution based on accelerated life testing derived in previous sections. The R statistical programming language was used to code all of the relevant computational techniques. We used two important packages to implement the computations: the ”CODA” package, which includes methods for summarising and displaying the output of MCMC simulations, and the ”maxLik” package, which employs the Newton-Raphson method of maximization in the computations. In the simulation study, we consider different sample sizes n_l, different number of failures, m_l, and different true values of the parameters.

In addition, probability of binomial removal P is considered to be 0.35, and 0.85 for each stress level, ϕ_l, $l = 1,2,...,L$. In the PT-II censored sample, at the time of the first failure $(x_{l1:m_l:n_l}, y_{[l1:m_l:n_l]})$, R_{l1} of the $n_l - 1$ remaining units are randomly excluded from the test. In the same manner R_{l2} of the surviving units, $n_l - R_{l1} - 1$, are randomly excluded from the test after the second failure $(x_{l2:m_l:n_l}, y_{[l2:m_l:n_l]})$ is detected. This mechanism continues until the failure of m_l^{th} occurs. The remaining surviving units $R_{lm_l} = n_l - m_l - \sum_{j=1}^{m_l-1} R_{lj}$ are excluded from the test after the m_l^{th} occurs, and the test is terminated. Suppose that the elimination of an individual unit from the test is independent of the others but with the same probability of removal P. Then, the number of units withdrawn at each failure time has a binomial distribution. That is $R_1 \sim binomial(n_l - m_l, P)$, $R_{lj} \sim binomial(n_l - m_l - \sum_{l=1}^{m_l-1} R_{lj}, P), l = 2,...,m_l$ and $R_{lm_l} = n_l - m_l - \sum_{j=1}^{m_l-1} R_{lj}$.

The simulation study is done using N=10,000 iterations. The following criteria are used to assess the estimating method's performance: average bias, mean squared error (MSE), and length of confidence interval (L.CI), all of which are derived by,

$$Bias(\Theta_j) = \frac{1}{N}\sum_{1}^{N}\left(\hat{\Theta}_j - \Theta_j\right),\ MSE(\Theta_j) = \frac{1}{N}\sum_{1}^{N}\left(\hat{\Theta}_j - \Theta_j\right)^2,\ \text{and}\ L.CI(\Theta_j) = Upper(\Theta_j) - Lower(\Theta_j).$$

Case 1: $\alpha_1 = 0.5, \sigma_1 = 1.6, \alpha_2 = 0.8, \sigma_2 = 1.7, \theta = 0.85$ in Tables 3 and 4.
Case 2: $\alpha_1 = 2.5, \sigma_{01} = 1.6, \alpha_2 = 2.8, \sigma_{02} = 1.7, \theta = 0.5$ in Tables 5 and 7.
Case 3: $\alpha_1 = 1.5, \sigma_{01} = 2.6, \alpha_2 = 1.8, \sigma_{02} = 2.7, \theta = -0.6$ in Tables 6 and 8.
The Bias indicates how far the average of the estimated values deviates from the genuine parameter, whereas the MSE indicates how far the average of the errors' squares deviates from the true parameter. Estimators that produce Biases and MSEs that are near to zero are often recommended. The frequency of intervals covering the true values of Θ_j should be closer to 95 percent for a 95 percent confidence level.

5.2 Example Simulated Data

Let $X_{i:m:n}, Y_{[i:m:n]} \sim$ accelerated $BFGME(\alpha_1, \sigma_{01}, \alpha_2, \sigma_{02}, \theta)$ with h_i. We can generate samples based on PT-II censoring by using binomial removal of elements to be censored. We will generate $n_1 = 30, n_2 = 35, n_3 = 35, n_4 = 20$ observations from the BFGME distribution, set the number of censoring $m_1 = 20, m_2 = 22, m_3 = 18, m_4 = 15$, and the stress of accelerated level is $\phi_0 = 30, \phi_1 = 50, \phi_2 = 80, \phi_3 = 100, \phi_4 = 130$. An example of the data obtained when $\alpha_1 = 1.5, \sigma_{01} = 2.6, \alpha_2 = 1.8, \sigma_{02} = 2.7, \theta = -0.6$ in Table 9. In the first, second, third, and fourth levels, we can see that 17, 19, 17, and 13 respectively, observations are censored and receive the largest lifetime among the observed lifetimes of X with Y observation as a concomitant of order statistics of X. Also, the vector R sample of binomial removal with probability 0.85 of each element, where the value 0 is attributed to censored data. Table 10 discussed the MLE and Bayesian estimation with point and interval estimation and we conclude that Bayesian estimation is the best estimation method of this model. Table 11 comparison between different loss function by loss value in Equations (28) and (30), respectively. By results in Table 11, we conclude that the LINEX loss function is better than SELF.

Reliability functions under different stress levels have been illustrated in Figures 1. The MCMC iterations and the kernel histograms of the posterior samples of the parameters of BFGME distribution are plotted in Figures 2, 3, and 4.

Table 3: MLE and Bayesian of the Parameters of BFGME distribution under Accelerated Life Testing based on progressive Type-II censored sample: Case 1, simple ramp-stress.

$\alpha_1 = 0.5, \sigma_1 = 1.6, \alpha_2 = 0.8, \sigma_2 = 1.7, \theta = 0.85$															
$\Phi_0 = 30, \Phi_1 = 50, \Phi_2 = 100$				MLE			SELF			Linex (c=1.5)			Linex (c=-0.5)		
P	n_1, n_2	m_1, m_2		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
0.35	40,35	20,15	α_1	0.0065	0.0075	0.3389	-0.0014	0.0019	0.1237	-0.0038	0.0019	0.1223	0.0010	0.0019	0.1248
			σ_1	0.0948	0.4834	2.7018	0.1430	0.3738	1.0915	-0.0245	0.0832	0.8571	0.3236	0.3637	1.3395
			α_2	0.0190	0.0209	0.5617	0.0016	0.0185	0.1977	-0.0047	0.0180	0.1937	0.0082	0.0195	0.2019
			σ_2	0.0943	0.7085	3.2808	0.0729	0.3629	1.1451	-0.0236	0.1021	0.9294	0.0999	0.4282	1.5873
			θ	0.2030	0.9082	3.7316	-0.1083	0.2417	1.9146	-0.2628	0.2031	1.9007	0.0402	0.2330	1.9117
		25,30	α_1	0.0051	0.0030	0.2144	0.0003	0.0006	0.0900	-0.0012	0.0005	0.0892	0.0018	0.0006	0.0907
			σ_1	0.0720	0.2843	2.0724	0.0699	0.0650	0.8290	-0.0318	0.0423	0.7032	0.1876	0.1269	0.9970
			α_2	0.0047	0.0084	0.3598	0.0070	0.0044	0.1544	-0.0006	0.0043	0.1526	0.0123	0.0061	0.1560
			σ_2	0.1123	0.3427	2.2536	0.0980	0.0927	0.9359	-0.0246	0.0495	0.7829	0.2414	0.1951	1.1993
			θ	0.1893	0.8649	3.5716	-0.0911	0.1670	1.5605	-0.2010	0.1621	1.5930	0.0125	0.1531	1.5163
	50,60	30,40	α_1	0.0045	0.0025	0.1957	0.0016	0.0004	0.0814	0.0004	0.0004	0.0808	0.0020	0.0005	0.0815
			σ_1	0.0706	0.2193	1.8154	0.0562	0.0475	0.7458	-0.0284	0.0333	0.6436	0.0871	0.0574	0.7870
			α_2	0.0026	0.0056	0.2938	-0.0010	0.0010	0.1271	-0.0040	0.0010	0.1259	0.0000	0.0010	0.1279
			σ_2	0.0981	0.2495	1.9208	0.0909	0.0679	0.8731	-0.0108	0.0421	0.7385	0.1289	0.0850	0.9449
			θ	0.0888	0.2721	2.0158	-0.0889	0.1268	1.3558	-0.1740	0.1156	1.4080	-0.0611	0.1210	1.3460
		45,50	α_1	0.0039	0.0016	0.1558	0.0004	0.0003	0.0666	-0.0005	0.0003	0.0660	0.0007	0.0003	0.0667
			σ_1	0.0333	0.1182	1.3421	0.0437	0.0301	0.6327	-0.0163	0.0226	0.5747	0.0651	0.0349	0.6600
			α_2	0.0030	0.0045	0.2633	-0.0007	0.0008	0.1101	-0.0029	0.0008	0.1095	0.0000	0.0008	0.1103
			σ_2	0.0626	0.1585	1.5418	0.0573	0.0369	0.6829	-0.0110	0.0262	0.5947	0.0819	0.0438	0.7181
			θ	0.0299	0.1046	1.2633	-0.0905	0.0916	1.1389	-0.1589	0.0812	1.2019	-0.0683	0.0859	1.1328
0.85	40,35	20,15	α_1	0.0093	0.0061	0.3035	0.0015	0.0014	0.1212	-0.0009	0.0013	0.1206	0.0023	0.0014	0.1215
			σ_1	0.0941	0.4092	2.4815	0.1032	0.1063	1.0821	-0.0322	0.0639	0.8630	0.1566	0.1388	1.1743
			α_2	0.0183	0.0174	0.5129	0.0044	0.0049	0.1929	-0.0018	0.0046	0.1890	0.0065	0.0050	0.1936
			σ_2	0.0774	0.4163	2.5122	0.1063	0.1220	1.1013	-0.0455	0.0690	0.9179	0.1659	0.1622	1.1831
			θ	0.1806	0.9005	3.6211	-0.0972	0.2621	2.0602	-0.2494	0.2315	1.9410	-0.0464	0.2572	2.0407
		25,30	α_1	0.0025	0.0035	0.2318	-0.0006	0.0006	0.0937	-0.0021	0.0005	0.0933	-0.0001	0.0006	0.0939
			σ_1	0.1101	0.3233	2.1879	0.0851	0.0667	0.8737	-0.0219	0.0395	0.7370	0.1255	0.0860	0.9412
			α_2	0.0006	0.0078	0.3474	-0.0022	0.0013	0.1400	-0.0059	0.0013	0.1395	-0.0010	0.0013	0.1401
			σ_2	0.1165	0.3127	2.1450	0.0909	0.0748	0.8974	-0.0277	0.0440	0.7502	0.1364	0.0988	0.9870
			θ	0.1432	0.8575	3.5882	-0.1058	0.1629	1.5513	-0.2162	0.1521	1.6041	-0.0694	0.1537	1.5565
	50,60	30,40	α_1	-0.0006	0.0023	0.1878	-0.0015	0.0004	0.0815	-0.0026	0.0004	0.0812	-0.0011	0.0004	0.0815
			σ_1	0.0989	0.2282	1.8329	0.0728	0.0534	0.7874	-0.0141	0.0350	0.6886	0.1047	0.0652	0.8313
			α_2	-0.0001	0.0060	0.3028	-0.0015	0.0011	0.1322	-0.0044	0.0011	0.1311	-0.0005	0.0011	0.1324
			σ_2	0.1014	0.2547	1.9390	0.0775	0.0566	0.8223	-0.0212	0.0367	0.7135	0.1139	0.0705	0.8686
			θ	0.1244	0.3504	2.2699	-0.0668	0.1314	1.4147	-0.1561	0.1261	1.4665	-0.0377	0.1260	1.4046
		45,50	α_1	0.0035	0.0016	0.1578	0.0006	0.0003	0.0672	-0.0003	0.0003	0.0668	0.0009	0.0003	0.0673
			σ_1	0.0392	0.1276	1.3924	0.0399	0.0277	0.5831	-0.0189	0.0213	0.5247	0.0609	0.0321	0.6100
			α_2	0.0022	0.0044	0.2595	-0.0006	0.0008	0.1031	-0.0028	0.0008	0.1025	0.0002	0.0008	0.1035
			σ_2	0.0636	0.1526	1.5118	0.0523	0.0304	0.6211	-0.0149	0.0214	0.5464	0.0765	0.0365	0.6511
			θ	0.0429	0.1096	1.2875	-0.0832	0.0908	1.1247	-0.1523	0.0901	1.1962	-0.0607	0.0857	1.1111

Table 4: MLE and Bayesian of the Parameters of BFGME distribution under Accelerated Life Testing based on progressive Type-II censored sample: Case 1, multiple ramp-stress.

$\alpha_1=0.5,\sigma_1=1.6,\alpha_2=0.8,\sigma_2=1.7,\theta=0.85$															
$\Phi_0=30,\Phi_1=50,\Phi_2=80,\Phi_3=100,\Phi_4=130$															
				MLE			SELF			Linex (c=1.5)			Linex (c=-0.5)		
P	n_1,n_2,n_3,n_4	m_1,m_2,m_3,m_4		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
0.35	20,20,15,20	10,12,10,13	α_1	0.0024	0.0028	0.2091	0.0006	0.0020	0.0901	-0.0008	0.0018	0.0895	0.0010	0.0021	0.0901
			σ_1	0.1639	0.5083	2.7213	0.1252	0.1235	1.1733	-0.0213	0.0668	0.9347	0.1818	0.1635	1.2719
			α_2	0.0032	0.0078	0.3455	-0.0029	0.0017	0.1528	-0.0062	0.0017	0.1510	-0.0018	0.0017	0.1536
			σ_2	0.1882	0.6139	2.9831	0.1569	0.1684	1.2678	-0.0150	0.0864	1.0452	0.2228	0.2232	1.3785
			θ	0.0699	0.7191	3.3145	-0.0467	0.1291	1.4003	-0.1094	0.1247	1.4084	-0.0256	0.1255	1.3900
		15,18,12,16	α_1	0.0028	0.0021	0.1792	-0.0005	0.0004	0.0742	-0.0015	0.0004	0.0739	-0.0002	0.0004	0.0744
			σ_1	0.1005	0.2954	2.0949	0.0865	0.0691	0.8629	-0.0180	0.0423	0.7448	0.1258	0.0879	0.9477
			α_2	0.0046	0.0051	0.2801	0.0001	0.0009	0.1198	-0.0023	0.0009	0.1182	0.0010	0.0010	0.1200
			σ_2	0.1092	0.3543	2.2949	0.0887	0.0790	0.9424	-0.0286	0.0497	0.8114	0.1323	0.0999	1.0082
			θ	0.0451	0.1916	1.7076	-0.0294	0.1030	1.2109	-0.0788	0.1014	1.2353	-0.0127	0.1006	1.1902
	30,35,25,40	20,22,18,30	α_1	0.0017	0.0013	0.1403	-0.0002	0.0002	0.0608	-0.0008	0.0002	0.0606	0.0000	0.0002	0.0608
			σ_1	0.0647	0.1811	1.6494	0.0546	0.0415	0.7274	-0.0200	0.0302	0.5953	0.0817	0.0494	0.7639
			α_2	0.0032	0.0033	0.2256	-0.0006	0.0006	0.0962	-0.0022	0.0006	0.0956	-0.0001	0.0006	0.0960
			σ_2	0.0730	0.2259	1.8419	0.0665	0.0519	0.7423	-0.0178	0.0359	0.6563	0.0972	0.0623	0.7733
			θ	0.0077	0.1025	1.2555	-0.0278	0.0655	0.9601	-0.0637	0.0637	0.9842	-0.0157	0.0641	0.9535
		25,30,22,35	α_1	0.0016	0.0010	0.1235	0.0000	0.0002	0.0549	-0.0005	0.0002	0.0551	0.0002	0.0002	0.0549
			σ_1	0.0527	0.1438	1.4728	0.0439	0.0320	0.6171	-0.0144	0.0243	0.5557	0.0646	0.0369	0.6369
			α_2	0.0012	0.0026	0.2017	-0.0013	0.0005	0.0884	-0.0025	0.0005	0.0883	-0.0008	0.0005	0.0886
			σ_2	0.0614	0.1663	1.5810	0.0532	0.0404	0.6836	-0.0138	0.0294	0.6051	0.0770	0.0471	0.7174
			θ	0.0006	0.0772	1.0894	-0.0354	0.0611	0.9436	-0.0664	0.0670	0.9439	-0.0250	0.0597	0.9429
0.85	20,20,15,20	10,12,10,13	α_1	0.0040	0.0027	0.2025	-0.0005	0.0005	0.0902	-0.0017	0.0005	0.0896	0.0000	0.0005	0.0903
			σ_1	0.1417	0.4559	2.5892	0.1271	0.1207	1.0884	-0.0194	0.0641	0.8747	0.1839	0.1614	1.1946
			α_2	0.0029	0.0075	0.3392	-0.0042	0.0014	0.1474	-0.0074	0.0013	0.1464	-0.0031	0.0014	0.1478
			σ_2	0.1776	0.5911	2.9338	0.1574	0.1565	1.2506	-0.0138	0.0774	0.9776	0.2230	0.2103	1.3618
			θ	0.0576	0.4204	2.5328	-0.0255	0.1194	1.3271	-0.0865	0.1313	1.3468	-0.0049	0.1175	1.3267
		15,18,12,16	α_1	0.0029	0.0020	0.1770	-0.0016	0.0004	0.0746	-0.0025	0.0004	0.0742	-0.0013	0.0004	0.0749
			σ_1	0.0937	0.3168	2.1768	0.0931	0.0843	0.9424	-0.0111	0.0517	0.7768	0.1322	0.1048	1.0032
			α_2	0.0037	0.0051	0.2793	-0.0014	0.0010	0.1226	-0.0038	0.0010	0.1217	-0.0006	0.0010	0.1229
			σ_2	0.1052	0.2998	2.1075	0.0971	0.0793	0.9453	-0.0191	0.0478	0.7984	0.1404	0.1009	1.0141
			θ	0.0149	0.2340	1.8963	-0.0396	0.0956	1.1960	-0.0914	0.1091	1.2426	-0.0222	0.0925	1.1775
	30,35,25,40	20,22,18,30	α_1	0.0015	0.0012	0.1364	-0.0012	0.0002	0.0604	-0.0018	0.0002	0.0603	-0.0010	0.0002	0.0604
			σ_1	0.0639	0.1799	1.6444	0.0696	0.0479	0.7326	-0.0070	0.0320	0.6464	0.0973	0.0578	0.7660
			α_2	0.0000	0.0031	0.2183	-0.0017	0.0006	0.0977	-0.0032	0.0006	0.0967	-0.0012	0.0006	0.0978
			σ_2	0.0977	0.2290	1.8375	0.0791	0.0582	0.8239	-0.0069	0.0386	0.7237	0.1104	0.0706	0.8544
			θ	0.0149	0.1071	1.2822	-0.0254	0.0698	0.9841	-0.0609	0.0662	1.0080	-0.0135	0.0682	0.9797
		25,30,22,35	α_1	0.0008	0.0010	0.1267	-0.0006	0.0002	0.0547	-0.0011	0.0002	0.0544	-0.0004	0.0002	0.0547
			σ_1	0.0549	0.1445	1.4755	0.0465	0.0312	0.6147	-0.0129	0.0238	0.5461	0.0677	0.0360	0.6423
			α_2	0.0021	0.0028	0.2065	-0.0005	0.0005	0.0887	-0.0018	0.0005	0.0880	-0.0001	0.0005	0.0887
			σ_2	0.0676	0.1726	1.6077	0.0594	0.0398	0.7138	-0.0090	0.0281	0.6174	0.0840	0.0472	0.7366
			θ	0.0050	0.0763	1.0832	-0.0217	0.0617	0.9242	-0.0536	0.0670	0.9668	-0.0110	0.0605	0.9099

Table 5: MLE and Bayesian of the Parameters of BFGME distribution under Accelerated Life Testing based on progressive Type-II censored sample: Case 2, simple ramp-stress.

$\alpha_1 = 2.5, \sigma_{01} = 1.6, \alpha_2 = 2.8, \sigma_{02} = 1.7, \theta = 0.5$															
$\Phi_0 = 30, \Phi_1 = 50, \Phi_2 = 100$				MLE			SELF			Linex (c=1.5)			Linex (c=-0.5)		
P	n_1,n_2	m_1,m_2		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
0.35	40,35	20,15	α_1	0.0387	0.1240	1.3726	0.0135	0.1027	0.5644	-0.0528	0.0391	0.5309	0.0353	0.1454	0.5769
			σ_{01}	0.0950	0.3887	2.4166	0.1002	0.1038	1.0153	-0.0393	0.0577	0.7755	0.1546	0.1401	1.0947
			α_2	0.0290	0.1810	1.6647	0.0027	0.0515	0.7076	-0.0736	0.0431	0.6616	0.0293	0.0549	0.7330
			σ_{02}	0.1603	0.5354	2.8001	0.1610	0.3656	1.1879	-0.0177	0.0800	0.9090	0.2373	0.8527	1.2979
			θ	0.2446	0.3132	1.9065	-0.0494	0.1599	1.5564	-0.1328	0.1386	1.6004	-0.0211	0.1548	1.5447
		25,30	α_1	0.0070	0.0736	1.0637	-0.0021	0.0140	0.4527	-0.0386	0.0135	0.4368	0.0105	0.0144	0.4592
			σ_{01}	0.1168	0.2719	1.9931	0.0895	0.0657	0.8706	-0.0181	0.0339	0.7259	0.1300	0.0841	0.9300
			α_2	0.0134	0.0914	1.1846	-0.0027	0.0165	0.4812	-0.0484	0.0163	0.4673	0.0132	0.0172	0.4868
			σ_{02}	0.1013	0.3025	2.1201	0.0857	0.0699	0.9122	-0.0343	0.0430	0.7494	0.1308	0.0899	0.9707
			θ	0.0260	0.2287	1.8729	-0.0503	0.1270	1.3734	-0.1145	0.1205	1.3970	-0.0288	0.1229	1.3713
	50,60	30,40	α_1	0.0120	0.0597	0.9575	-0.0010	0.0114	0.4236	-0.0305	0.0111	0.4179	0.0091	0.0117	0.4263
			σ_{01}	0.0877	0.2395	1.8882	0.0769	0.0586	0.8106	-0.0146	0.0368	0.6682	0.1109	0.0730	0.8636
			α_2	0.0042	0.0677	1.0202	-0.0048	0.0129	0.4315	-0.0418	0.0124	0.4221	0.0079	0.0132	0.4397
			σ_{02}	0.0962	0.2361	1.8678	0.0820	0.0575	0.8117	-0.0204	0.0363	0.6856	0.1201	0.0727	0.8713
			θ	0.0306	0.1452	1.4896	-0.0321	0.0938	1.1438	-0.0856	0.1062	1.1876	-0.0142	0.0912	1.1362
		45,50	α_1	0.0040	0.0436	0.8185	-0.0021	0.0078	0.3295	-0.0242	0.0081	0.3253	0.0054	0.0079	0.3338
			σ_{01}	0.0654	0.1512	1.5032	0.0486	0.0328	0.6374	-0.0133	0.0240	0.5663	0.0709	0.0384	0.6576
			α_2	0.0078	0.0519	0.8927	-0.0031	0.0095	0.3907	-0.0304	0.0101	0.3806	0.0062	0.0097	0.3944
			σ_{02}	0.0635	0.1592	1.5447	0.0567	0.0351	0.6409	-0.0128	0.0252	0.5626	0.0818	0.0417	0.6689
			θ	0.0051	0.1000	1.2403	-0.0277	0.0748	1.0360	-0.0705	0.0821	1.0547	-0.0134	0.0734	1.0212
0.85	40,35	20,15	α_1	0.0515	0.1503	1.5069	0.0068	0.0264	0.6331	-0.0535	0.0260	0.5973	0.0279	0.0285	0.6476
			σ_{01}	0.1140	0.4197	2.5012	0.1235	0.1161	1.0553	-0.0205	0.0607	0.8330	0.1799	0.1560	1.1765
			α_2	0.0443	0.1649	1.5832	0.0048	0.0273	0.6539	-0.0702	0.0288	0.6000	0.0313	0.0297	0.6645
			σ_{02}	0.1283	0.4508	2.5847	0.1221	0.1017	0.9941	-0.0359	0.0538	0.8037	0.1839	0.1398	1.1086
			θ	0.3345	0.6756	2.0480	-0.0531	0.1639	1.5989	-0.1370	0.1609	1.6107	-0.0247	0.1596	1.5766
		25,30	α_1	0.0320	0.0769	1.0803	0.0077	0.0151	0.4839	-0.0298	0.0150	0.4653	0.0207	0.0158	0.4885
			σ_{01}	0.0818	0.2619	1.9815	0.0862	0.0715	0.8397	-0.0232	0.0436	0.7054	0.1274	0.0908	0.9203
			α_2	0.0252	0.0975	1.2206	0.0011	0.0174	0.4989	-0.0447	0.0183	0.4816	0.0170	0.0181	0.5064
			σ_{02}	0.0925	0.2880	2.0734	0.0911	0.0722	0.8673	-0.0293	0.0406	0.7085	0.1364	0.0945	0.9497
			θ	0.0256	0.2336	1.8929	-0.0526	0.1175	1.3080	-0.1152	0.1104	1.3486	-0.0314	0.1136	1.2895
	50,60	30,40	α_1	0.0227	0.0654	0.9993	0.0044	0.0116	0.4120	-0.0250	0.0117	0.4050	0.0145	0.0120	0.4159
			σ_{01}	0.0771	0.2442	1.9142	0.0690	0.0534	0.7710	-0.0196	0.0352	0.6722	0.1018	0.0660	0.8054
			α_2	0.0142	0.0730	1.0583	-0.0016	0.0139	0.4514	-0.0387	0.0147	0.4398	0.0112	0.0143	0.4573
			σ_{02}	0.0802	0.2415	1.9015	0.0761	0.0606	0.8542	-0.0252	0.0390	0.7135	0.1139	0.0761	0.9046
			θ	0.0156	0.1559	1.5475	-0.0415	0.1029	1.2295	-0.0933	0.1135	1.2411	-0.0240	0.1007	1.2203
		45,50	α_1	0.0004	0.0411	0.7950	-0.0049	0.0072	0.3332	-0.0271	0.0077	0.3247	0.0026	0.0073	0.3340
			σ_{01}	0.0634	0.1339	1.4136	0.0524	0.0315	0.6356	-0.0111	0.0226	0.5665	0.0752	0.0372	0.6687
			α_2	-0.0094	0.0511	0.8859	-0.0061	0.0099	0.3760	-0.0333	0.0106	0.3665	0.0032	0.0100	0.3796
			σ_{02}	0.0964	0.1726	1.5851	0.0672	0.0410	0.6516	-0.0042	0.0279	0.5832	0.0931	0.0492	0.6873
			θ	0.0168	0.0921	1.1887	-0.0215	0.0733	1.0155	-0.0640	0.0812	1.0498	-0.0072	0.0717	1.0076

Table 6: MLE and Bayesian of the Parameters of BFGME distribution under Accelerated Life Testing based on progressive Type-II censored sample: Case 3, simple ramp-stress.

$\alpha_1 = 1.5, \sigma_{01} = 2.6, \alpha_2 = 1.8, \sigma_{02} = 2.7, \theta = -0.6$															
$\Phi_0 = 30, \Phi_1 = 50, \Phi_2 = 100$				MLE			SELF			Linex (c=1.5)			Linex (c=-0.5)		
P	n_1,n_2	m_1,m_2		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
0.35	40,35	20,15	α_1	0.0216	0.0485	0.8598	0.0026	0.0088	0.3631	-0.0192	0.0085	0.3537	0.0101	0.0091	0.3658
			σ_{01}	0.1923	1.0796	4.0046	0.1914	0.2931	1.6391	-0.1611	0.1452	1.1966	0.3414	0.4673	1.9033
			α_2	0.0223	0.0677	1.0168	0.0083	0.0700	0.4295	-0.0270	0.0254	0.4126	0.0194	0.0781	0.4385
			σ_{02}	0.2244	1.1325	4.0798	0.1966	0.3477	1.7321	-0.1652	0.1678	1.2655	0.3535	0.5836	2.0578
			θ	-0.2038	0.3998	2.0074	0.0685	0.1832	1.6720	-0.0323	0.1732	1.6353	0.1038	0.1921	1.6634
		25,30	α_1	0.0027	0.0269	0.6433	-0.0021	0.0051	0.2827	-0.0155	0.0051	0.2751	0.0025	0.0052	0.2842
			σ_{01}	0.1720	0.7361	3.2966	0.1445	0.1870	1.4464	-0.1279	0.1083	1.1176	0.2562	0.2824	1.6308
			α_2	0.0128	0.0401	0.7840	-0.0002	0.0069	0.3330	-0.0190	0.0069	0.3264	0.0062	0.0070	0.3365
			σ_{02}	0.1516	0.7681	3.3854	0.1354	0.1995	1.4717	-0.1447	0.1142	1.0782	0.2499	0.3048	1.6063
			θ	-0.0739	0.2423	1.9085	0.0483	0.1302	1.4553	-0.0253	0.1212	1.4084	0.0737	0.1363	1.4669
	50,60	30,40	α_1	0.0000	0.0231	0.5966	-0.0024	0.0041	0.2558	-0.0131	0.0042	0.2512	0.0012	0.0042	0.2561
			σ_{01}	0.1775	0.6916	3.1863	0.1274	0.1479	1.2947	-0.1012	0.0889	1.0123	0.2172	0.2093	1.4242
			α_2	0.0029	0.0294	0.6726	-0.0015	0.0058	0.2898	-0.0167	0.0058	0.2838	0.0036	0.0059	0.2911
			σ_{02}	0.1623	0.6314	3.0508	0.1335	0.1459	1.2954	-0.1078	0.0905	1.0075	0.2294	0.2106	1.4198
			θ	-0.0238	0.1436	1.4834	0.0597	0.1070	1.2386	-0.0029	0.0958	1.2004	0.0812	0.1131	1.2478
		45,50	α_1	-0.0001	0.0143	0.4695	-0.0018	0.0027	0.2005	-0.0097	0.0027	0.1990	0.0009	0.0027	0.2016
			σ_{01}	0.1075	0.3546	2.2971	0.0866	0.0865	1.0288	-0.0713	0.0607	0.8820	0.1461	0.1133	1.1093
			α_2	0.0061	0.0244	0.6122	0.0002	0.0046	0.2629	-0.0112	0.0046	0.2579	0.0040	0.0047	0.2644
			σ_{02}	0.1128	0.4388	2.5600	0.0975	0.0924	1.0788	-0.0727	0.0619	0.8921	0.1617	0.1231	1.1538
			θ	-0.0091	0.0928	1.1939	0.0502	0.0768	1.0792	-0.0020	0.0694	1.0487	0.0681	0.0811	1.0960
0.85	40,35	20,15	α_1	0.0259	0.0475	0.8492	0.0059	0.0086	0.3523	-0.0161	0.0082	0.3393	0.0134	0.0090	0.3592
			σ_{01}	0.2127	1.3314	4.4479	0.1980	0.3149	1.6124	-0.1558	0.1445	1.1468	0.3516	0.5108	1.8934
			α_2	0.0299	0.0736	1.0573	0.0052	0.0121	0.4253	-0.0249	0.0117	0.4067	0.0156	0.0128	0.4308
			σ_{02}	0.2329	1.4529	4.6384	0.1801	0.2787	1.6286	-0.1784	0.1467	1.2043	0.3321	0.4510	1.9149
			θ	-0.2426	0.3720	1.9991	0.0742	0.1995	1.6765	-0.0299	0.1847	1.6156	0.1105	0.2106	1.6928
		25,30	α_1	0.0000	0.0292	0.6696	-0.0041	0.0051	0.2779	-0.0175	0.0052	0.2713	0.0004	0.0052	0.2807
			σ_{01}	0.2117	0.8010	3.4105	0.1637	0.1975	1.4334	-0.1126	0.1073	1.0841	0.2762	0.2964	1.6087
			α_2	0.0040	0.0381	0.7651	-0.0036	0.0075	0.3301	-0.0227	0.0077	0.3265	0.0029	0.0077	0.3315
			σ_{02}	0.1721	0.7368	3.2981	0.1576	0.1915	1.4415	-0.1325	0.1070	1.0910	0.2769	0.2954	1.6685
			θ	-0.0438	0.2610	1.9963	0.0569	0.1395	1.4441	-0.0211	0.1295	1.4049	0.0838	0.1464	1.4611
	50,60	30,40	α_1	-0.0009	0.0220	0.5814	-0.0025	0.0042	0.2501	-0.0131	0.0042	0.2465	0.0011	0.0042	0.2510
			σ_{01}	0.1806	0.6027	2.9612	0.1422	0.1485	1.2956	-0.0880	0.0836	0.9814	0.2342	0.2177	1.4351
			α_2	0.0113	0.0311	0.6906	0.0017	0.0056	0.2837	-0.0134	0.0056	0.2792	0.0069	0.0057	0.2847
			σ_{02}	0.1076	0.6222	3.0648	0.0957	0.1390	1.2809	-0.1378	0.0936	1.0281	0.1888	0.2024	1.4150
			θ	-0.0255	0.1382	1.4546	0.0683	0.1137	1.2882	0.0022	0.1011	1.2303	0.0911	0.1205	1.2928
		45,50	α_1	0.0061	0.0146	0.4734	0.0002	0.0026	0.1979	-0.0077	0.0026	0.1967	0.0029	0.0026	0.1992
			σ_{01}	0.0914	0.3760	2.3781	0.0852	0.0786	0.9678	-0.0717	0.0548	0.7909	0.1442	0.1035	1.0485
			α_2	0.0045	0.0213	0.5719	-0.0019	0.0040	0.2395	-0.0133	0.0041	0.2366	0.0019	0.0041	0.2411
			σ_{02}	0.0915	0.3737	2.3704	0.0878	0.0879	1.0306	-0.0813	0.0625	0.8827	0.1513	0.1156	1.0820
			θ	-0.0235	0.0947	1.2033	0.0453	0.0849	1.1068	-0.0072	0.0770	1.0662	0.0633	0.0892	1.1270

Table 7: MLE and Bayesian of the Parameters of BFGME distribution under Accelerated Life Testing based on progressive Type-II censored sample: Case 2, multiple ramp-stress.

$\alpha_1 = 2.5, \sigma_1 = 1.6, \alpha_2 = 2.8, \sigma_{02} = 1.7, \theta = 0.5$															
$\Phi_0 = 30, \Phi_1 = 50, \Phi_2 = 80, \Phi_3 = 100, \Phi_4 = 130$															
				MLE			SELF			Linex (c=1.5)			Linex (c=-0.5)		
P	n_1,n_2,n_3,n_4	m_1,m_2,m_3,m_4		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
0.35	20,20,15,20	10,12,10,13	α_1	-0.0008	0.0678	1.0214	-0.0124	0.0141	0.4605	-0.0427	0.0141	0.4499	-0.0019	0.0142	0.4649
			σ_{01}	0.1768	0.5089	2.7106	0.1396	0.1463	1.1983	-0.0107	0.0758	0.9602	0.1966	0.1893	1.2802
			α_2	0.0058	0.0919	1.1886	-0.0107	0.0168	0.4894	-0.0499	0.0161	0.4740	0.0029	0.0170	0.4960
			σ_{02}	0.2048	0.7409	3.2788	0.1559	0.1521	1.2325	-0.0170	0.0776	0.9856	0.2233	0.2059	1.3499
			θ	0.0842	0.5774	2.9618	-0.0276	0.1224	1.3698	-0.0880	0.1236	1.3948	-0.0073	0.1200	1.3531
		15,18,12,16	α_1	0.0131	0.0515	0.8888	-0.0058	0.0097	0.3871	-0.0292	0.0091	0.3776	0.0022	0.0098	0.3881
			σ_{01}	0.0992	0.2891	2.0727	0.0940	0.0737	0.9047	-0.0114	0.0461	0.7621	0.1331	0.0917	0.9732
			α_2	0.0307	0.0683	1.0183	0.0012	0.0128	0.4319	-0.0285	0.0132	0.4278	0.0113	0.0131	0.4392
			σ_{02}	0.0738	0.3349	2.2512	0.0814	0.0825	0.9823	-0.0349	0.0514	0.8099	0.1249	0.1049	1.0458
			θ	0.0324	0.1780	1.6500	-0.0324	0.1028	1.2042	-0.0829	0.1013	1.2142	-0.0153	0.1007	1.2111
	30,35,25,40	20,22,18,30	α_1	0.0072	0.0320	0.7014	-0.0025	0.0062	0.3091	-0.0174	0.0064	0.3085	0.0025	0.0063	0.3102
			σ_{01}	0.0691	0.1944	1.7080	0.0603	0.0425	0.7295	-0.0138	0.0303	0.6392	0.0870	0.0506	0.7681
			α_2	0.0085	0.0395	0.7786	-0.0009	0.0075	0.3278	-0.0199	0.0078	0.3269	0.0056	0.0076	0.3279
			σ_{02}	0.0709	0.2031	1.7454	0.0627	0.0506	0.7754	-0.0229	0.0351	0.6750	0.0940	0.0614	0.8222
			θ	0.0291	0.1067	1.2759	-0.0148	0.0672	0.9779	-0.0496	0.0607	0.9971	-0.0031	0.0661	0.9662
		25,30,22,35	α_1	0.0056	0.0269	0.6429	-0.0028	0.0053	0.2791	-0.0151	0.0054	0.2774	0.0014	0.0053	0.2795
			σ_{01}	0.0615	0.1511	1.5054	0.0564	0.0370	0.6296	-0.0038	0.0269	0.5752	0.0778	0.0429	0.6578
			α_2	0.0023	0.0339	0.7219	-0.0041	0.0066	0.3206	-0.0195	0.0068	0.3146	0.0011	0.0066	0.3203
			σ_{02}	0.0705	0.1694	1.5903	0.0562	0.0397	0.6969	-0.0111	0.0292	0.6376	0.0802	0.0463	0.7171
			θ	0.0017	0.0845	1.1399	-0.0281	0.0596	0.9295	-0.0588	0.0517	0.9467	-0.0178	0.0581	0.9181
0.85	20,20,15,20	10,12,10,13	α_1	0.0148	0.0667	1.0115	-0.0088	0.0128	0.4465	-0.0393	0.0124	0.4391	0.0018	0.0130	0.4510
			σ_{01}	0.1391	0.4633	2.6132	0.1339	0.1224	1.1449	-0.0134	0.0640	0.9066	0.1903	0.1629	1.2634
			α_2	0.0193	0.0894	1.1700	-0.0041	0.0171	0.4998	-0.0424	0.0168	0.4939	0.0091	0.0175	0.5019
			σ_{02}	0.1404	0.4973	2.7103	0.1254	0.1273	1.1562	-0.0358	0.0690	0.9121	0.1876	0.1709	1.2589
			θ	0.0895	0.7823	3.4510	-0.0383	0.1230	1.3601	-0.1006	0.1204	1.3949	-0.0172	0.1195	1.3451
		15,18,12,16	α_1	0.0245	0.0524	0.8922	-0.0006	0.0098	0.3849	-0.0245	0.0101	0.3763	0.0076	0.0100	0.3875
			σ_{01}	0.0761	0.2638	1.9921	0.0859	0.0719	0.8994	-0.0200	0.0434	0.7410	0.1250	0.0904	0.9633
			α_2	0.0001	0.0602	0.9619	-0.0136	0.0120	0.4302	-0.0430	0.0133	0.4241	-0.0035	0.0120	0.4351
			σ_{02}	0.1179	0.3417	2.2456	0.1012	0.0843	0.9431	-0.0205	0.0505	0.8099	0.1465	0.1075	1.0046
			θ	0.0556	0.2034	1.7554	-0.0183	0.1007	1.2287	-0.0699	0.1009	1.2439	-0.0010	0.0992	1.2228
	30,35,25,40	20,22,18,30	α_1	0.0034	0.0283	0.6601	-0.0056	0.0056	0.2891	-0.0205	0.0052	0.2864	-0.0006	0.0056	0.2886
			σ_{01}	0.0693	0.1828	1.6545	0.0640	0.0456	0.7287	-0.0115	0.0312	0.6516	0.0913	0.0549	0.7671
			α_2	0.0134	0.0373	0.7557	-0.0020	0.0072	0.3370	-0.0208	0.0075	0.3348	0.0044	0.0072	0.3385
			σ_{02}	0.0487	0.1863	1.6821	0.0561	0.0461	0.7640	-0.0280	0.0320	0.6511	0.0866	0.0559	0.8093
			θ	0.0092	0.1104	1.3023	-0.0267	0.0713	1.0023	-0.0622	0.0708	1.0256	-0.0148	0.0698	0.9859
		25,30,22,35	α_1	0.0136	0.0261	0.6310	-0.0016	0.0050	0.2733	-0.0140	0.0051	0.2684	0.0026	0.0050	0.2733
			σ_{01}	0.0424	0.1398	1.4571	0.0528	0.0332	0.6466	-0.0078	0.0242	0.5857	0.0744	0.0388	0.6688
			α_2	0.0158	0.0320	0.6986	0.0011	0.0061	0.2995	-0.0143	0.0063	0.2981	0.0062	0.0062	0.2983
			σ_{02}	0.0431	0.1579	1.5492	0.0497	0.0376	0.6679	-0.0162	0.0280	0.6122	0.0734	0.0441	0.7055
			θ	0.0178	0.0796	1.1047	-0.0202	0.0645	0.9549	-0.0506	0.0609	0.9618	-0.0101	0.0635	0.9474

Table 8: MLE and Bayesian of the Parameters of BFGME distribution under Accelerated Life Testing based on progressive Type-II censored sample: Case 3, multiple ramp-stress.

$\alpha_1=1.5,\sigma_{01}=2.6,\alpha_2=1.8,\sigma_{02}=2.7,\theta=-0.6$															
$\Phi_0=30,\Phi_1=50,\Phi_2=80,\Phi_3=100,\Phi_4=130$															
				MLE			SELF			Linex (c=1.5)			Linex (c=-0.5)		
P	n_1,n_2,n_3,n_4	m_1,m_2,m_3,m_4		Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI	Bias	MSE	L.CI
0.35	20,20,15,20	10,12,10,13	α_1	0.0053	0.0239	0.6060	-0.0068	0.0045	0.2578	-0.0178	0.0046	0.2546	-0.0030	0.0045	0.2586
			σ_{01}	0.2215	1.2075	4.2212	0.2166	0.3387	1.8543	-0.1478	0.1597	1.3461	0.3679	0.5333	2.0848
			α_2	0.0274	0.0371	0.7478	0.0027	0.0124	0.3294	-0.0136	0.0084	0.3236	0.0081	0.0135	0.3297
			σ_{02}	0.1588	1.1620	4.1816	0.1779	0.3398	1.8472	-0.1812	0.1708	1.3411	0.3304	0.5503	2.0981
			θ	-0.0965	0.4036	2.4626	0.0623	0.1387	1.4668	-0.0132	0.1268	1.4148	0.0885	0.1459	1.4699
		15,18,12,16	α_1	0.0125	0.0199	0.5504	-0.0007	0.0038	0.2355	-0.0092	0.0038	0.2322	0.0022	0.0038	0.2367
			σ_{01}	0.1299	0.7402	3.3355	0.1416	0.1986	1.4507	-0.1203	0.1116	1.1415	0.2462	0.2918	1.6341
			α_2	0.0087	0.0244	0.6119	-0.0034	0.0048	0.2695	-0.0155	0.0049	0.2678	0.0007	0.0049	0.2702
			σ_{02}	0.1303	0.7346	3.3224	0.1289	0.1786	1.4349	-0.1410	0.1136	1.1045	0.2360	0.2605	1.6365
			θ	-0.0584	0.2163	1.8096	0.0371	0.1230	1.3200	-0.0257	0.1162	1.3026	0.0587	0.1275	1.3218
	30,35,25,40	20,22,18,30	α_1	0.0052	0.0108	0.4074	-0.0013	0.0022	0.1765	-0.0067	0.0022	0.1745	0.0005	0.0022	0.1767
			σ_{01}	0.1061	0.4774	2.6776	0.1103	0.1252	1.2135	-0.0782	0.0804	1.0029	0.1817	0.1672	1.2926
			α_2	0.0048	0.0155	0.4880	-0.0018	0.0030	0.2175	-0.0094	0.0030	0.2164	0.0007	0.0030	0.2177
			σ_{02}	0.1105	0.5323	2.8284	0.1019	0.1353	1.2333	-0.0961	0.0929	1.0507	0.1768	0.1798	1.3262
			θ	-0.0203	0.0960	1.2126	0.0411	0.0746	1.0397	-0.0025	0.0675	0.9963	0.0559	0.0782	1.0602
		25,30,22,35	α_1	0.0081	0.0101	0.3937	-0.0001	0.0019	0.1685	-0.0049	0.0019	0.1683	0.0015	0.0019	0.1675
			σ_{01}	0.0700	0.3842	2.4153	0.0827	0.0892	1.0323	-0.0741	0.0620	0.8731	0.1413	0.1163	1.0957
			α_2	0.0041	0.0138	0.4602	-0.0018	0.0027	0.2042	-0.0087	0.0028	0.2028	0.0005	0.0027	0.2047
			σ_{02}	0.0934	0.3932	2.4319	0.0913	0.1055	1.1876	-0.0745	0.0749	0.9704	0.1535	0.1360	1.2580
			θ	-0.0236	0.0896	1.1704	0.0334	0.0675	1.0033	-0.0040	0.0634	0.9752	0.0460	0.0697	1.0029
0.85	20,20,15,20	10,12,10,13	α_1	0.0031	0.0249	0.6183	-0.0049	0.0046	0.2682	-0.0160	0.0047	0.2649	-0.0012	0.0046	0.2686
			σ_{01}	0.2335	1.2223	4.2382	0.1855	0.3026	1.7920	-0.1645	0.1569	1.2944	0.3337	0.4849	2.1332
			α_2	0.0124	0.0354	0.7363	-0.0023	0.0068	0.3253	-0.0175	0.0069	0.3217	0.0030	0.0069	0.3261
			σ_{02}	0.2422	1.3176	4.4006	0.2048	0.3382	1.8699	-0.1582	0.1726	1.3837	0.3549	0.5317	2.1320
			θ	-0.0942	0.3521	2.2977	0.0664	0.1350	1.3976	-0.0059	0.1216	1.3273	0.0912	0.1424	1.4279
		15,18,12,16	α_1	0.0062	0.0174	0.5163	-0.0031	0.0033	0.2269	-0.0116	0.0034	0.2257	-0.0003	0.0033	0.2281
			σ_{01}	0.1496	0.7307	3.3008	0.1456	0.1976	1.5386	-0.1115	0.1206	1.2042	0.2468	0.2762	1.6746
			α_2	0.0188	0.0268	0.6376	-0.0012	0.0052	0.2785	-0.0133	0.0052	0.2747	0.0029	0.0052	0.2814
			σ_{02}	0.0979	0.7249	3.3170	0.1351	0.2014	1.4164	-0.1375	0.1197	1.1040	0.2445	0.2954	1.5883
			θ	-0.0164	0.1744	1.6368	0.0518	0.1159	1.2883	-0.0107	0.1050	1.2343	0.0734	0.1220	1.2948
	30,35,25,40	20,22,18,30	α_1	0.0057	0.0108	0.4076	-0.0010	0.0020	0.1732	-0.0063	0.0021	0.1731	0.0008	0.0021	0.1736
			σ_{01}	0.0886	0.4865	2.7135	0.0881	0.1215	1.1748	-0.0941	0.0854	0.9874	0.1567	0.1581	1.2569
			α_2	-0.0015	0.0165	0.5038	-0.0044	0.0029	0.2057	-0.0119	0.0030	0.2041	-0.0018	0.0029	0.2067
			σ_{02}	0.1472	0.5692	2.9021	0.1092	0.1254	1.1602	-0.0890	0.0834	0.9823	0.1857	0.1703	1.2564
			θ	-0.0104	0.1105	1.3031	0.0450	0.0792	1.0920	0.0004	0.0732	1.0600	0.0602	0.0824	1.0977
		25,30,22,35	α_1	0.0047	0.0101	0.3937	-0.0001	0.0019	0.1663	-0.0049	0.0019	0.1662	0.0015	0.0019	0.1664
			σ_{01}	0.0731	0.3662	2.3560	0.0694	0.0858	1.0174	-0.0870	0.0632	0.8405	0.1281	0.1113	1.0923
			α_2	0.0097	0.0141	0.4644	-0.0007	0.0026	0.1985	-0.0075	0.0027	0.1989	0.0016	0.0026	0.1985
			σ_{02}	0.0683	0.4051	2.4817	0.0888	0.0978	1.0892	-0.0765	0.0692	0.9216	0.1508	0.1273	1.1442
			θ	-0.0179	0.0745	1.0680	0.0364	0.0658	0.9731	-0.0025	0.0605	0.9608	0.0495	0.0684	0.9777

Table 9: Bivariate acceleration data with 4 level of stress.

Φ	50			80			100			130		
i	$x_{i:m:n}$	$y_{[i:m:n]}$	R_i	$x_{i:m:n}$	$y_{[i:m:n]}$	R_i	$x_{i:m:n}$	$y_{[i:m:n]}$	R_i	$x_{i:m:n}$	$y_{[i:m:n]}$	R_i
1	0.0127	0.3816	3	0.0035	0.1158	1	0.0040	0.0257	0	0.0004	0.0345	0
2	0.0201	0.0782	6	0.0056	0.0627	10	0.0065	0.0055	7	0.0075	0.0343	4
3	0.0291	0.1399	1	0.0084	0.0141	2	0.0068	0.0623	0	0.0080	0.0218	1
4	0.0347	0.6711	0	0.0233	0.0106	0	0.0124	0.1271	0	0.0137	0.0003	0
5	0.0648	0.0921	0	0.0261	0.0707	0	0.0193	0.1075	0	0.0146	0.0048	0
6	0.0706	0.2637	0	0.0399	0.0105	0	0.0199	0.0406	0	0.0296	0.0094	0
7	0.0736	0.0575	0	0.0539	0.0014	0	0.0260	0.1083	0	0.0318	0.0267	0
8	0.1162	0.0075	0	0.0804	0.0023	0	0.0475	0.0466	0	0.0405	0.0168	0
9	0.1199	0.3196	0	0.0835	0.0420	0	0.0491	0.0582	0	0.0407	0.0089	0
10	0.1232	0.2699	0	0.0951	0.1330	0	0.0525	0.0162	0	0.0490	0.0066	0
11	0.1263	0.0529	0	0.1069	0.0247	0	0.0532	0.0079	0	0.0498	0.0156	0
12	0.1602	0.4838	0	0.1069	0.0735	0	0.0562	0.0110	0	0.0640	0.0044	0
13	0.1691	0.4784	0	0.1143	0.2026	0	0.0822	0.0035	0	0.0736	0.0036	0
14	0.1696	0.2139	0	0.1190	0.0022	0	0.0997	0.0089	0	0.0819	0.0105	0
15	0.1853	0.1602	0	0.1290	0.0086	0	0.1665	0.0343	0	0.1860	0.0165	0
16	0.2493	0.0245	0	0.1634	0.0881	0	0.2322	0.0043	0			
17	0.2544	0.1043	0	0.1716	0.0001	0	0.3489	0.0369	0			
18	0.2662	0.3628	0	0.1849	0.0456	0	0.4743	0.0538	0			
19	0.3884	0.1848	0	0.1921	0.1601	0						
20	0.4144	0.1818	0	0.2288	0.1305	0						
21				0.2376	0.0666	0						
22				0.2872	0.0606	0						

Table 10: MLE and Bayesian estimation with point and interval estimation.

	MLE				SELF			
	estimate	SE	Lower	Upper	estimate	SE	Lower	Upper
α_1	1.3318	0.1147	1.1070	1.5566	1.4757	0.0826	1.3252	1.6358
σ_1	4.3052	1.1665	2.0189	6.5915	3.1742	0.4907	2.3986	4.0788
α_2	1.9656	0.1566	1.6587	2.2725	1.8846	0.1239	1.6431	2.1146
σ_2	2.4440	0.6252	1.2186	3.6695	2.7576	0.5562	1.8814	3.8237
θ	-0.2793	0.3749	-0.9991	0.4556	-0.5289	0.3598	-0.9920	0.1166

Table 11: Bayesian etimation with different loss function.

	SELF		Linex (c=1.5)		Linex (c=-0.5)	
	estimate	loss	estimate	loss	estimate	loss
α_1	1.4757	0.0283	1.4706	0.0010	1.4774	0.0001
σ_1	3.1742	2.9078	3.0086	0.2328	3.2353	0.0455
α_2	1.8846	0.0274	1.8731	0.0062	1.8884	0.0010
σ_2	2.7576	0.0655	2.5621	0.0200	2.8385	0.0023
θ	-0.5289	0.1028	-0.6483	0.0026	-0.4900	0.0015

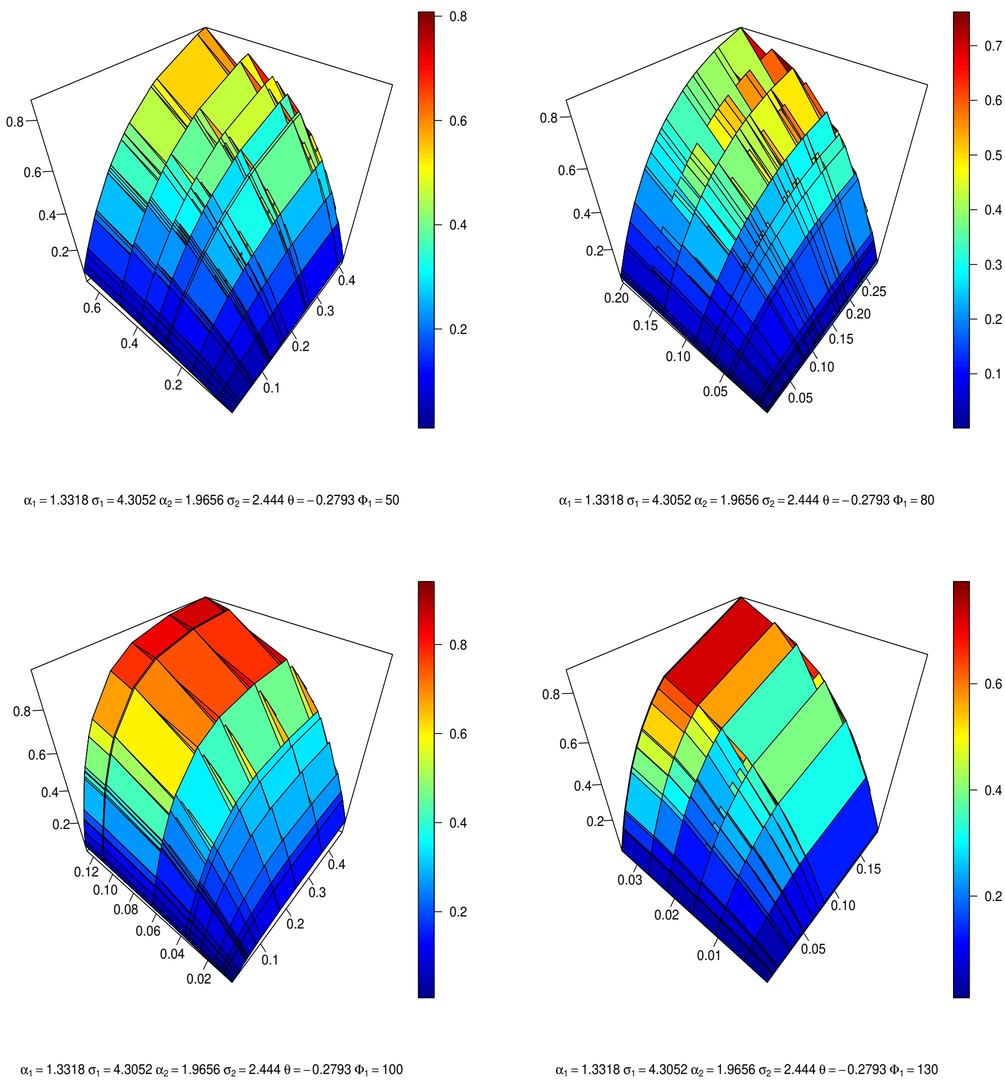

Fig. 1: Reliability function under different stress level.

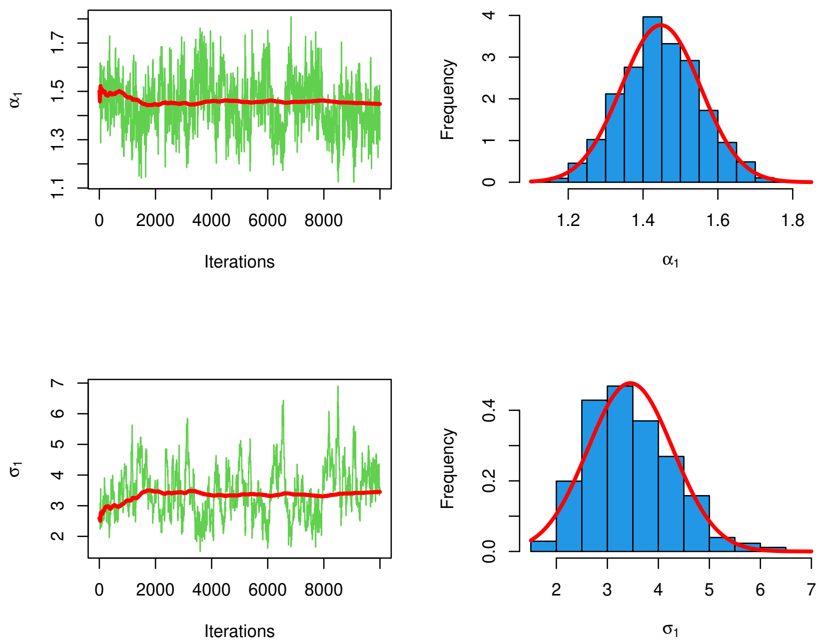

Fig. 2: MCMC results for α_1, σ_1.

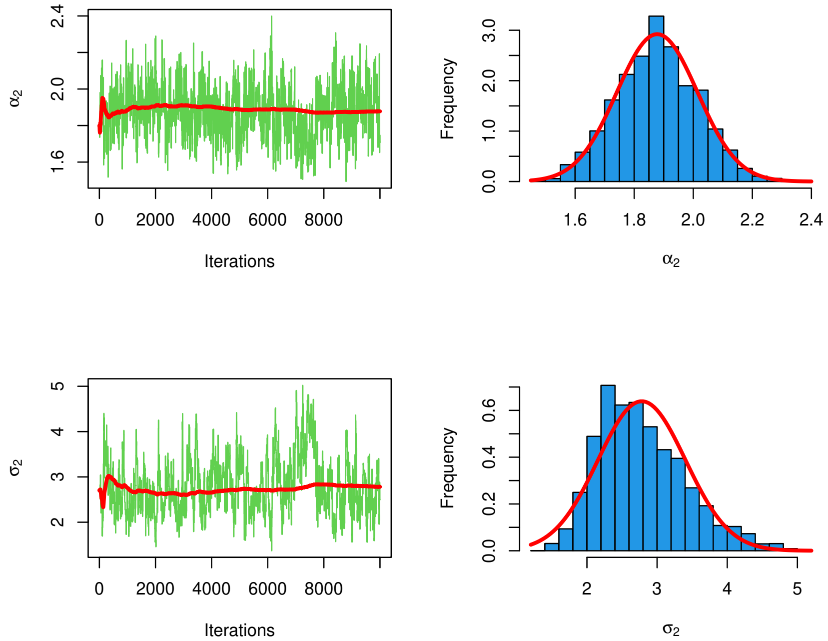

Fig. 3: MCMC results for α_2, σ_2.

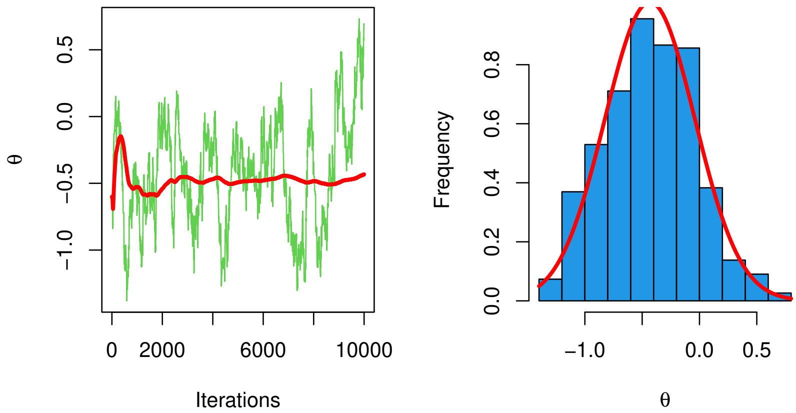

Fig. 4: MCMC results for θ.

6 Conclusion

In this paper, we have discussed a constant-stress accelerated bivariate model under progressively Type-II censoring based on the copula function. The simulation findings show that statistical inference modes are getting closer to genuine values, with bivariate model dependence becoming greater. As a result, the copula approach is a useful and practical tool for analyzing high-reliability products. A simulated stress data-set was presented to illustrate the application of the proposed model in practice. Numerical examples are given and Monte Carlo simulation studies are done for illustration reasons. The following observations are made based on the findings:

–When the censored sample size (m) is increased the MSE of all estimates are decreases.
–The bias, MSE, and length of CI are decreased when the sample size (n) increased.
–The bias, MSE, and length of CI are increased when the parameter of binomial removal increased in most parameters.
–Bayesian estimation is the best estimation method of bivariate based on CSALT under PT-II censored samples.
–the Linex loss function is better than the SELF of bivariate based on CSALT under PT-II censored samples.
–When the index of LINEX loss function (c) is increased the MSE of all estimates are decreases.
–The reliability of the bivariate exponential model based on CSALT under PT-II censored samples is increased with increasing stress levels.

Conflict of Interest

The authors declare that they have no conflicts of interests regarding the publication of this paper.

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