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Statistical Analysis of Joint Progressive Censoring Data from Gompertz Distribution

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Abstract: In this paper, we consider the problems of estimating the unknown parameters as well as predicting the failure times of the removed units in multiple stages of the joint progressively censored sample coming from two Gompertz distributions. The likelihood, Bootstrap and Bayesian methods are applied for the estimation problem. In Bayesian contexts, the posterior densities are estimated by using Lindley's approximation, importance sampling and Metropolis-Hastings methods under different loss error functions. The confidence intervals based on the asymptotic normality and credible intervals based the Bayesian approach are discussed as well. A real life data is analyzed for illustrative purposes and Monte Carlo simulations are conducted to compare the performances of the all proposed methods.

Keywords: Bayesian estimation, bootstrap-t method, Gompertz distribution, joint progressive censoring scheme, Lindley's approximation, maximum likelihood estimation, Metropolis-Hastings algorithm, Monte Carlo simulation

1 Introduction

The Gompertz distribution is one of the most well-known distributions in analyzing skewed data. The Gompertz model was originally proposed by Benjamin Gompertz [1] and it has been used as a growth model, especially in epidemiological and biomedical studies, and also can be used for adequate tumor growth. In hydrology, the Gompertz distribution is applied to extreme events such as annual maximum rainfalls. More applications and survey of the Gompertz model can be found in Ahuja [2] and Chen [3]. It is assumed that the lifetimes of the items being tested have a Gompertz distribution with the probability density function (PDF)

$$f(t;\alpha,\theta)=\alpha\theta\exp(\theta t-\alpha(e^{\theta t}-1)),\ t\geq 0 \text{ and } \alpha>0,\ \ \theta>0, \tag{1}$$

and cumulative distribution function (CDF)

$$F(t;\alpha,\theta)=1-\exp(-\alpha(e^{\theta t}-1))\ \ ,t\geq 0\text{and } \alpha>0,\ \theta>0. \tag{2}$$

Here $\alpha>0,\theta>0$ are the shape and scale parameters, respectively. The Gompertz distribution with the shape and scale parameters as α and θ will be denoted by $\text{GO}(\alpha,\theta)$. The GO distribution has a decreasing and unimodal density but has an increasing failure rate function. Since the CDF of the GO distribution is in closed form, it has been used very effectively for analyzing censored lifetime data. Due to the time and cost constraints, the analysis of censored data arise naturally in various fields such as reliability, survival, and medical studies.
The progressive censoring scheme is popular mechanism of collecting data in lifetime analysis. For a detailed study on progressive censoring topic and related issues, one may refer to Balakrishnan and Aggarwala [4] and Balakrishnan and Cramer [5]. A brief description of the type-II progressive scheme can be simply put as follows: assuming n identical subjects are placed on a lifetime experiment. The integer k $(0<k<N)$ represents the number of failures to be observed in the test and $R_1,R_2,...,R_k$ are k prefixed non-negative integers such that $\sum_{i=1}^{k} R_i+k=N$. At the prefixed first time of failure,

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R_1 items are randomly selected from the remaining $N-1$ items, and eliminated from the test. Similarly, at the second time of failure, R_2 items are randomly selected from the remaining $N-R_1-2$ items, eliminated, and so on. At the final time of k^{th} failure, the remaining R_k items are eliminated, and the test is terminated. Inference on the progressive censoring data has been addressed frequently by many authors. Among them, Ng [6], Raqab et al. [7], Pradhan and Kundu [8], Valiollahi et al. [9], Maurya et al. [10].
In the practical experiments, there are situations where the experimenter would like to compare the life lengths of the units coming from different two populations. In this context, Rasouli and Balakrishnan [11] developed such comparison under the same environmental conditions. The resulting scheme coming from these distributions is called joint progressive censoring (JPC). Under the JPC scheme, the two samples taken from two populations (Group 1 and Group 2) of sizes m and n, respectively, are combined and put on a life testing experiment. The size of the combined sample is $N=m+n$. Let $k<N$ and $R_1,R_2,...,R_k$ are non-negative integers satisfying $\sum_{i=1}^{k} R_i+k=m+n$, where $R_i=S_i+W_i$ with S_i and W_i being the number of removals at the i-th stage from Group 1 and Group 2, respectively. Based on the combined sample, at the first failure time T_1, $R_1=S_1+W_1$ units are randomly withdrawn from the remaining $(N-1)$ surviving units where S_1 and W_1 are the number of removed units from Group 1 and Group 2, respectively. Proceeding similarly, at the second time of failure, T_2, $R_2=S_2+W_2$ items are chosen randomly and withdrawn from the remaining combined $N-2-R_1$ surviving units, and so on. In the final stage (at the time of the k^{th} failure), all the remaining $R_k=N-k-\sum_{i=1}^{k} R_i$ surviving units are withdrawn. The JPC scheme includes the complete sample and Type-II censoring schemes as special cases when $R_1=R_2=\ldots=R_k=0$, for all $i=1,2,...,k$ and $R_1=R_2=...=R_{k-1}=0$, so that $R_k=N-k$, respectively. Based on JPC data, Rasouli and Balakrishnan [11] considered the exact likelihood inference for two exponential populations. Parsi and Bairamov [12] investigated the expected number of failures in the experimental test. Parsi et al. [13] provided the conditional maximum likelihood and interval estimation of the parameters of two Weibull distributions. Mondal and Kundu [14] studied the estimation of the parameters of two Weibull distributions. See also, Doostparast et al. [15], Ashour and Abo-Kasem [16], and Mondal and Kundu [17].
The problem of estimation is the main problems considered in the real life situations. In the Bayesian inference, the performance of the estimator or predictor depends on the prior distribution and the loss function used as well. Our main aim is to develop the Bayes estimates of all parameters involved under different loss functions. The performances of the Bayes estimators based on the different loss functions are compared with the classical maximum likelihood estimators (MLEs) by extensive computer simulations. We further compute the symmetric credible intervals and compare them with the confidence intervals based on the asymptotic distributions of the MLEs, by extensive computer simulations.
The layout of the paper is organized as follows. In Section 1, we provide the models description, loss functions and priors. In Section 3, we derive the MLEs of the unknown parameters. The asymptotic and Bootstrap-t confidence intervals (CIs) are also developed in Section 3. The Bayes estimates using Lindley's approximation, importance and Metropolis-Hastings methods have been considered in Section 4. Analyses of Real data sets representing the survival times in months of Melanoma patients are analyzed and the performance of the different Bayes estimates and MLEs via simulation study are performed in Section 5. Finally, we conclude the paper in Section 6.

2 Models description, loss functions and priors

Let $X_1,X_2,...,X_m$ being independent and identically (iid) distributed m units from Group 1 (Gr-1) of GO lifetime distribution with CDF $F(x;\alpha_1,\theta)$, and PDF $f(x;\alpha_1,\theta)$. Furthermore, let $Y_1,Y_2,...,Y_n$ be iid n units from Group 2 (Gr-2) of GO lifetime distribution with CDF $F(y;\alpha_2,\theta)$ and PDF $f(y;\alpha_2,\theta)$. For a given $(R_1,\ldots,R_k)$, where k is a non-negative integer such that $(k\geq 0)$ satisfying $\sum_{i=1}^{k} R_i=N-k$, let $(\mathbf{t},\delta,\mathbf{S})=\{(\mathbf{t_1},\delta_1,\mathbf{S_1}),(\mathbf{t_2},\delta_2,\mathbf{S_2}),\ldots,(\mathbf{t_k},\delta_\mathbf{k},\mathbf{S_k})\}$, be the JPC data from the combined population. Here for $j=1,\ldots,k$, we have $\delta_j=1$ if the failure at t_j occurs from Gr-1, and $\delta_j=0$ if the failure at t_j occurs from Gr-2, S_j and then $W_j=R_j-S_j$, denotes the number of items removed at t_j from Gr-1 and Gr-2, respectively.

The likelihood function based on JPC sample is given by

$$L(data\mid\alpha_1,\alpha_2,\theta)=C\prod_{i=1}^{k}\Big[f(t_i,\alpha_1,\theta)\Big]^{\delta_i}\Big[f(t_i,\alpha_2,\theta)\Big]^{1-\delta_i}\Big[1-F(t_i,\alpha_1,\theta)\Big]^{S_i}\Big[1-F(t_i,\alpha_2,\theta)\Big]^{W_i},\tag{3}$$

where $k_1=\sum_{i=1}^{k}\delta_i$, $k_2=k-k_1$, $S_k=m-k_1-\sum_{i=1}^{k-1}S_i$, $C=D_1D_2$ with D_1 and D_2 being

$$D_1=\prod_{j=1}^{k}\left[\left(m-\sum_{i=1}^{j-1}\delta_i-\sum_{i=1}^{j-1}S_i\right)\delta_j+\left(n-\sum_{i=1}^{j-1}(1-\delta_i)-\sum_{i=1}^{j-1}W_i\right)(1-\delta_j)\right],$$

and

$$D_2=\prod_{j=1}^{k-1}\frac{\binom{m\sum_{i=1}^{j-1}\delta_i-\sum_{i=1}^{j-1}S_i}{S_j}\binom{n-\sum_{i=1}^{j-1}(1-\delta_i)-\sum_{i=1}^{j-1}W_i}{W_j}}{\binom{N-j-\sum_{i=1}^{j-1}R_i}{R_j}}.$$

It is well-known that the error loss functions play an important role in the Bayesian estimation. One of the most popular symmetric loss function is the squared error (SE) loss function which gives equal weight to overestimation as well as underestimation. In some cases, the use of symmetric loss function may be inappropriate, see for example, [18]. For this reason, asymmetric loss functions can be proposed. One of the well-known asymmetric loss function is the linear-exponential (LINEX) loss function which was introduced by Varian [19]. Another alternative function is the general entropy (GE) loss function (see, Ren et al. [20]). For estimating any parameter (say, ξ) by a decision rule d, using the Bayes approach, the following loss functions are considered.

SQUARE ERROR (SE) LOSS FUNCTION:

$$L_1=(\xi-d)^2.$$

The Bayes estimate under the SE loss function is

$$\hat{\xi}_S=E(\xi|\mathbf{t})$$

LINEAR-EXPONENTIAL (LINEX) ERROR LOSS FUNCTION:

$$L_2=e^{\omega(d-\xi)}-\omega(d-\xi)-1,\omega\neq 0.$$

With an appropriate LINEX parameter ν, we can reflect small (large) losses for underestimation and overestimation, which is not the case for SE loss function. For history and motivation, see for example, Zellner [21] and Christoffersen and Diebold [22]. The BEs of ξ under LINEX loss functions is

$$\hat{\xi}_L=E\left(e^{-\omega\xi}|\mathbf{t}\right).$$

ENTROPY LOSS FUNCTION:

Basu and Ebrahimi [18] defined a modified LINEX loss which does not change the characteristics of LINEX loss. A suitable alternative to the modified LINEX loss is the GE loss proposed by Calabria and Pulcini [23]. It is given by

$$L_3=(\frac{d}{\xi})^{\rho}-\rho\log(\frac{d}{\xi})-1,\quad \rho\neq 0,$$

which has a minimum at $d=\xi$. The Bayes estimate of ξ based on GO joint progressive data under the GE loss function may be defined as

$$\hat{\xi}_E=\left[E\left(\xi^{-\rho}|\mathbf{t}\right)\right]^{-1/\rho},$$

provided that $E\left(\xi^{-\rho}|\mathbf{t}\right)$ exists and is finite.
Now, we specify the prior distributions of α_1,α_2, and θ. It is desirable that the model parameters are independent such that all prior and posterior densities belong to similar families. These prior choices allow the posterior distribution to be analytically tractable and computationally efficient. A natural choice for the priors of α_1,α_2, and θ would be to assume that the three quantities are independent gamma $G(a_i,b_i), i=1,2$ and $G(a_0,b_0)$ distributions, respectively, with the following densities:

$$g_{a_i,b_i}(\alpha_i)=\frac{b_i^{a_i}}{\Gamma(a_i)}\,\alpha_i^{a_i-1}e^{-b_i\alpha_i}, \tag{4}$$

and

$$g_{a_0,b_0}(\theta)=\frac{b_0^{a_0}}{\Gamma(a_0)}\,\theta^{a_0-1}e^{-b_0\theta}, \tag{5}$$

where α_i, and $\theta>0$ and a_i, b_i, and a_0,b_0, are chosen to reflect prior knowledge about α_i and θ.

3 Maximum likelihood estimation and its approximation

Here in this section, we consider the likelihood and bootstrap methods for estimating the unknown parameters and corresponding confidence intervals. By combining (1), (2), and (3), we write the likelihood function of JPC GO data which can be viewed as a function of α_1, α_2 and θ as

$$L(data|\alpha_1,\alpha_2,\theta) = C\ \alpha_1^{k_1}\,\alpha_2^{k_2}\ \theta^k\ e^{\theta\sum_{i=1}^{k} t_i - \alpha_1\sum_{i=1}^{k} u_i A_\theta(t_i) - \alpha_2 \sum_{i=1}^{k} v_i A_\theta(t_i)}, \tag{6}$$

where

$$A_\theta(t_i) = e^{\theta t_i} - 1,\ u_i = \delta_i + S_i \text{ and } v_i = 1 - \delta_i + W_i.$$

The corresponding log-likelihood function can be written as

$$l(data|\alpha_1,\alpha_2,\theta) \propto k_1\ log\ \alpha_1 + k\ log\ \theta + k_2\ log\ \alpha_2 + \theta\sum_{i=1}^{k} t_i - \alpha_1\sum_{i=1}^{k} u_i\, A_\theta(t_i) - \alpha_2\sum_{i=1}^{k} v_i A_\theta(t_i). \tag{7}$$

The MLEs can be obtained by taking the first partial derivatives of Eq.(7) concerning α_1, α_2 and θ and equating each to zero. That is, for $k_1 > 0$ and $k_2 > 0$, the likelihood equations can be obtained as follows:

$$\frac{\partial l(\alpha_1,\alpha_2,\theta)}{\partial\alpha_1} = \frac{k_1}{\alpha_1} - \sum_{i=1}^{k} u_i A_\theta(t_i) = 0, \tag{8}$$

$$\frac{\partial l(\alpha_1,\alpha_2,\theta)}{\partial\alpha_2} = \frac{k_2}{\alpha_2} - \sum_{i=1}^{k} v_i A_\theta(t_i) = 0, \tag{9}$$

$$\frac{\partial l(\alpha_1,\alpha_2,\theta)}{\partial\theta} = \frac{k}{\theta} + \sum_{i=1}^{k} t_i - \alpha_1\sum_{i=1}^{k} u_i t_i e^{\theta t_i} - \alpha_2\sum_{i=1}^{k} v_i t_i e^{\theta t_i} = 0. \tag{10}$$

It follows from (8) and (9) that the MLEs of α_1 and α_2 can be obtained, respectively,

$$\hat{\alpha}_1(\theta) = \frac{k_1}{\sum_{i=1}^{k} u_i A_\theta(t_i)},\ \hat{\alpha}_2(\theta) = \frac{k_2}{\sum_{i=1}^{k} v_i A_\theta(t_i)}. \tag{11}$$

Upon plugging $\hat{\alpha}_1$ and $\hat{\alpha}_2$ into Eq. (10), we immediately have

$$\frac{\partial l(\alpha_1,\alpha_2,\theta)}{\partial\theta} = \frac{k}{\theta} + \sum_{i=1}^{k} t_i - \frac{k_1\sum_{i=1}^{k} u_i t_i e^{\theta t_i}}{\sum_{i=1}^{k} u_i\, A_\theta(t_i)} - \frac{k_2\sum_{i=1}^{k} v_i t_i e^{\theta t_i}}{\sum_{i=1}^{k} v_i\, A_\theta(t_i)} = 0. \tag{12}$$

It can be checked that for given θ, the log-likelihood function in (7) is unimodal. This in turns out the estimates given in (11) are the MLEs of α_1 and α_2. Further, for given $\alpha_j (j = 1,2)$, $\frac{\partial l}{\partial\theta}$ is a monotone decreasing function starting from ∞ at 0 to a negative constant when $\theta \to \infty$. By using this fact and $\frac{\partial^2 l}{\partial\theta^2} < 0$, we conclude that the MLE of θ exists and unique. The non-linear equation (12) cannot be solved analytically and a numerical method such as Newton-Raphson method can be applied. Once the MLE of θ, $\hat{\theta}$ is computed, the MLEs $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are obtained directly using (11).

For special cases where $R_i = 0$, for all $i = 1,2,\ldots,k$, and then $S_i = W_i = 0$, the terms $\sum_{i=1}^{k} S_i A_\theta(t_i)$, $\sum_{i=1}^{k} W_i A_\theta(t_i)$, and $\sum_{i=1}^{k} S_i t_i e^{\theta t_i}$ in (8), (9), and (12) reduce to 0. This in turns out that the likelihood equations become

$$\frac{\partial l(\alpha_1,\alpha_2,\theta)}{\partial\alpha_1} = \frac{k_1}{\alpha_1} - \sum_{i=1}^{k} \delta_i A_\theta(t_i) = 0,\ \frac{\partial l(\alpha_1,\alpha_2,\theta)}{\partial\alpha_2} = \frac{k_2}{\alpha_2} - \sum_{i=1}^{k} (1-\delta_i) A_\theta(t_i) = 0,$$

and

$$\frac{\partial l(\alpha_1,\alpha_2,\theta)}{\partial\theta} = \frac{k}{\theta} + \sum_{i=1}^{k} t_i - \alpha_1\sum_{i=1}^{k} \delta_i t_i e^{\theta t_i} - \alpha_2\sum_{i=1}^{k} (1-\delta_i) t_i e^{\theta t_i} = 0.$$

It follows that

$$\hat{\alpha}_1 = \frac{k_1}{\sum_{i=1}^{k} \delta_i A_\theta(t_i)}, \; \hat{\alpha}_2 = \frac{k_2}{\sum_{i=1}^{k} (1-\delta_i) A_\theta(t_i)},$$

and

$$\hat{\theta} = \frac{k}{\sum_{i=1}^{k} t_i - \hat{\alpha}_1 \sum_{i=1}^{k} \delta_i t_i e^{\theta t_i} - \hat{\alpha}_2 \sum_{i=1}^{k} (1-\delta_i) t_i e^{\theta t_i}}.$$

Clearly, the MLEs are not expressed in closed forms and their respective variances cannot be obtained. Because of that, we propose to use the asymptotic variances of $\hat{\alpha}_1, \hat{\alpha}_2$, and $\hat{\theta}$. From the log-likelihood function in Eq.(7), we have

$$\begin{aligned}
&\frac{\partial^2 l}{\partial \alpha_1^2} = -\frac{k_1}{\alpha_1^2},\\
&\frac{\partial^2 l}{\partial \alpha_1\, \partial \alpha_2} = \frac{\partial^2 l}{\partial \alpha_2\, \partial \alpha_1} = 0,\\
&\frac{\partial^2 l}{\partial \alpha_2^2} = -\frac{k_2}{\alpha_2^2},\\
&\frac{\partial^2 l}{\partial \alpha_1\, \partial \theta} = \frac{\partial^2 l}{\partial \theta\, \partial \alpha_1} = -\sum_{i=1}^{k} u_i t_i e^{\theta t_i},\\
&\frac{\partial^2 l}{\partial \alpha_2\, \partial \theta} = \frac{\partial^2 l}{\partial \theta\, \partial \alpha_2} = -\sum_{i=1}^{k} v_i t_i e^{\theta t_i},\\
&\frac{\partial^2 l}{\partial \theta^2} = -\frac{k}{\theta^2} - \alpha_1 \sum_{i=1}^{k} u_i t_i^2 e^{\theta t_i} - \alpha_2 \sum_{i=1}^{k} v_i t_i^2 e^{\theta t_i}.
\end{aligned} \tag{13}$$

To this end, let us consider $\mathbf{Q} = (\alpha_1, \alpha_2, \theta)$. Under the usual regularity conditions (Lehmann and Casella [24]) and large k, the asymptotic normality of $\hat{\mathbf{Q}} = (\hat{\alpha}_1, \hat{\alpha}_2, \hat{\theta})$ that based on the convergence in distribution ($\xrightarrow{D}$) of $\hat{\mathbf{Q}}$ can be described as

$$\hat{\mathbf{Q}} \xrightarrow{D} N_3(\mathbf{Q}, \mathbf{I}^{-1}(\mathbf{Q})),$$

where $I^{-1}(\mathbf{Q})$ is the inverse of the Fisher information matrix of the unknown parameters $\mathbf{Q}$ [we find it by taking the negative expectation of the expressions in (13)]. In practical applications, one may use the approximation $\hat{\mathbf{Q}} \xrightarrow{D} N_3(\mathbf{Q}, \mathbf{J}^{-1}(\mathbf{Q}))$, where $J^{-1}(\mathbf{Q})$ denotes the inverse of the observed information matrix of $\mathbf{Q}$. Hence, we can define the observed information matrix $J(\mathbf{Q})$ by taking the negative of the expressions in (13) as follows:

$$J(\mathbf{Q}) = \begin{pmatrix} -\frac{\partial^2 l}{\partial \alpha_1^2} & -\frac{\partial^2 l}{\partial \alpha_1 \partial \alpha_2} & -\frac{\partial^2 l}{\partial \alpha_1 \partial \theta} \\ -\frac{\partial^2 l}{\partial \alpha_2 \partial \alpha_1} & -\frac{\partial^2 l}{\partial \alpha_2^2} & -\frac{\partial^2 l}{\partial \alpha_2 \partial \theta} \\ -\frac{\partial^2 l}{\partial \theta \partial \alpha_1} & -\frac{\partial^2 l}{\partial \theta \partial \alpha_2} & -\frac{\partial^2 l}{\partial \theta^2} \end{pmatrix}.$$

Therefore, the $100(1-\gamma)\%$ approximate CIs for α_1, α_2 and θ are

$$(\hat{\alpha}_1 - z_{1-\gamma/2}\sqrt{V_{11}}, \hat{\alpha}_1 + z_{1-\gamma/2}\sqrt{V_{11}}),\; (\hat{\alpha}_2 - z_{1-\gamma/2}\sqrt{V_{22}}, \hat{\alpha}_2 + z_{1-\gamma/2}\sqrt{V_{22}}),$$

and

$$(\hat{\theta} - z_{1-\gamma/2}\sqrt{V_{33}}, \hat{\theta} + z_{1-\gamma/2}\sqrt{V_{33}}),$$

respectively, where V_{11}, V_{22} and V_{33} are the elements of the main diagonal of $J^{-1}(\hat{\mathbf{Q}})$ and z_γ is $100\gamma^{th}$ percentile of the standard normal distribution. Usually, the CI based on the asymptotic results do not perform quite well for small sample size. For this reason, it is more appropriate to propose alternative CI, say, Bootstrap-t method (Boot-t method), see for example, Ahmed [25]. The following algorithm describes the steps for obtaining Boot-t CIs.

- **Step 1:** Estimate α_1, α_2 and θ using the maximum likelihood based on the observed informative sample (say $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\theta}$).

–**Step 2:** Using $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\theta}$ obtained in Step 1, generate a Bootstrap sample and then obtain the first k observed censored units, $B_1, B_2, \ldots, B_k$ under the GO model. Then compute the corresponding MLEs $\hat{\alpha}_1^*$, $\hat{\alpha}_2^*$ and $\hat{\theta}^*$ of α_1, α_2 and θ and the elements $(V_{11}^*, V_{22}^*, V_{33}^*)$ of the main diagonal of $J^{*-1}(\hat{\alpha}_1^*, \hat{\alpha}_2^*, \hat{\theta}^*)$.

–**Step 3:** Based on the Bootstrap sample in Step 2, define an estimated Bootstrap version

$$Q_1^* = \frac{\hat{\alpha}_1^* - \hat{\alpha}_1}{\sqrt{V_{11}^*}}, \qquad Q_2^* = \frac{\hat{\alpha}_2^* - \hat{\alpha}_2}{\sqrt{V_{22}^*}} \qquad and \qquad Q_3^* = \frac{\hat{\theta}^* - \hat{\theta}}{\sqrt{V_{33}^*}}.$$

–**Step 4:** Generate $M = 1000$ Bootstrap samples and versions of Q_1^*, Q_2^* and Q_3^* then we obtain the $100\gamma^{th}$ sample quantiles of Q_1^*, Q_2^* and Q_3^* (say $q_{1,\gamma}^*$, $q_{2,\gamma}^*$ and $q_{3,\gamma}^*$).

–**Step 5:** Compute the approximate $100(1-\gamma)\%$ CIs for α_1, α_2 and θ as

$$(\hat{\alpha}_1 - q_{1,1-\gamma/2}^*\sqrt{V_{11}}, \hat{\alpha}_1 - q_{1,\gamma/2}^*\sqrt{V_{11}}), \quad (\hat{\alpha}_2 - q_{2,1-\gamma/2}^*\sqrt{V_{22}}, \hat{\alpha}_2 - q_{2,\gamma/2}^*\sqrt{V_{22}}),$$

and

$$(\hat{\theta} - q_{3,1-\gamma/2}^*\sqrt{V_{33}}, \hat{\theta} - q_{3,\gamma/2}^*\sqrt{V_{33}}).$$

4 Bayesian estimation of the parameters

In this section, we formulate the posterior densities of the parameters α_1, α_2, and θ based on joint progressive censored sample coming from from two-parameter *GO* distribution and then obtain the corresponding Bayes estimators (BEs) of these unknown parameters as well as the credible intervals (CrIs) under different error loss functions, L_1, L_2, and L_3, with respect to the priors as described in Section 2.

By combining the prior distributions given in (4) and (5) and likelihood function in (6), we can express the joint posterior density of α_1, α_2 and θ as

$$\pi(\alpha_1, \alpha_2, \theta|data) \propto \alpha_1^{k_1+a_1-1}\, e^{-\alpha_1\left(b_1+\sum_{i=1}^{k} u_i A_\theta(t_i)\right)} \times \alpha_2^{k_2+a_2-1}\, e^{-\alpha_2\left(b_2+\sum_{i=1}^{k} v_i A_\theta(t_i)\right)} \times \theta^{k+a_0-1} e^{-\theta\left(b_0-\sum_{i=1}^{k} t_i\right)}.$$

By setting

$$\Delta_\theta(\mathbf{t}) = \sum_{i=1}^{k} u_i A_\theta\,(t_i) \;\text{ and }\; \overline{\Delta}_\theta(\tilde{t}) = \sum_{i=1}^{k} v_i\, A_\theta(t_i),$$

we may rewrite the joint posterior distribution as

$$\pi(\alpha_1, \alpha_2, \theta|\mathbf{t}) \propto \mathbf{p_1}(\alpha_1|\theta, \mathbf{t}) \times \mathbf{p_2}(\alpha_2|\theta, \mathbf{t}) \times \mathbf{p_3}(\theta|\mathbf{t}), \tag{14}$$

where $p_1(\alpha_1|\theta, \mathbf{t})$, and $p_2(\alpha_2|\theta, \mathbf{t})$, are PDFs of $G(k_1+a_1, b_1+\Delta_\theta(\mathbf{t}))$, and $G\left(k_2+a_2, b_2+\overline{\Delta}_\theta(\mathbf{t})\right)$, respectively, while $p_3(\theta|\mathbf{t})$ is defined by

$$p_3(\theta|\mathbf{t}) = \mathbf{g_1}(\theta|\mathbf{t}) \times \mathbf{g_2}(\theta, \mathbf{t}), \tag{15}$$

with $g_1(\theta|\mathbf{t})$ being the PDF of $G(k+a_0, b_0)$ and $g_2(\theta, \mathbf{t}) = \frac{e^{\theta \sum_{i=1}^{k} t_i}}{(b_1+\Delta_\theta(\mathbf{t}))^{k_1+a_1}\left(b_2+\overline{\Delta}_\theta(\mathbf{t})\right)^{k_2+a_2}}$.

Under the error loss functions, L_1, L_2 and L_3, the BE of any function of α_1, α_2 and θ (say, $\lambda(\alpha_1, \alpha_2, \theta)$), respectively, takes the following form:

$$\hat{\lambda}_S(\alpha_1, \alpha_2, \theta) = \frac{\int_0^\infty \int_0^\infty \int_0^\infty \lambda(\alpha_1, \alpha_2, \theta) p_1(\alpha_1|\theta, \mathbf{t}) \mathbf{p_2}(\alpha_2|\theta, \mathbf{t}) \mathbf{p_3}(\theta|\mathbf{t}) \mathbf{d\alpha_1 d\alpha_2 d\theta}}{\int_0^\infty \int_0^\infty \int_0^\infty p_1(\alpha_1|\theta, \mathbf{t}) \mathbf{p_2}(\alpha_2|\theta, \mathbf{t}) \mathbf{p_3}(\theta|\mathbf{t}) \mathbf{d\alpha_1 d\alpha_2 d\theta}}, \tag{16}$$

$$\hat{\lambda}_L(\alpha_1, \alpha_2, \theta) = -\frac{1}{v} \log\left(\frac{\int_0^\infty \int_0^\infty \int_0^\infty e^{-v\lambda(\alpha_1, \alpha_2, \theta)} p_1(\alpha_1|\theta, \mathbf{t}) \mathbf{p_2}(\alpha_2|\theta, \mathbf{t}) \mathbf{p_3}(\theta|\mathbf{t}) \mathbf{d\alpha_1 d\alpha_2 d\theta}}{\int_0^\infty \int_0^\infty \int_0^\infty p_1(\alpha_1|\theta, \mathbf{t}) \mathbf{p_2}(\alpha_2|\theta, \mathbf{t}) \mathbf{p_3}(\theta|\mathbf{t}) \mathbf{d\alpha_1 d\alpha_2 d\theta}}\right), \tag{17}$$

and

$$\hat{\lambda}_E(\alpha_1, \alpha_2, \theta) = \left(\frac{\int_0^\infty \int_0^\infty \int_0^\infty (\lambda(\alpha_1, \alpha_2, \theta))^{-v} p_1(\alpha_1|\theta, \mathbf{t}) \mathbf{p_2}(\alpha_2|\theta, \mathbf{t}) \mathbf{p_3}(\theta|\mathbf{t}) \mathbf{d\alpha_1 d\alpha_2 d\theta}}{\int_0^\infty \int_0^\infty \int_0^\infty p_1(\alpha_1|\theta, \mathbf{t}) \mathbf{p_2}(\alpha_2|\theta, \mathbf{t}) \mathbf{p_3}(\theta|\mathbf{t}) \mathbf{d\alpha_1 d\alpha_2 d\theta}}\right)^{-\frac{1}{v}}. \tag{18}$$

It is clear to note that Eq.'s (16), (17) and (18) cannot be simplified into closed-form expressions and produce the BEs of α_1, α_2 and θ and must be calculated numerically. Among the various methods suggested to approximate the joint posterior density, Lindley approximation that may be used to approximate the ratio of two integrals. Although, this method can produce BEs of the parameters using numerical integration or approximation, but it is not possible to construct CrIs for the parameters involved. For this reason, alternative method called importance sampling will be proposed to provide sample based estimates of the parameters.

4.1 Lindley approximation method

Lindley [26] proposed an approximation procedure to evaluate the expressions given in Eq.'s (16), (17) and (18). Several authors have used this approximation for obtaining the BEs based on some lifetime distributions under the considered prior distribution (see, Howlader and Hossain in [27] and Jaheen [28]). If k is sufficiently large, according to Lindley [26], any ratio of the integrals of the form

$$I(data) = \frac{\int_0^\infty \int_0^\infty \int_0^\infty \lambda(\alpha_1, \alpha_2, \theta)\, e^{l(data|\alpha_1,\alpha_2,\theta)+\rho(\alpha_1,\alpha_2,\theta)} d\alpha_1 d\alpha_2 d\theta}{\int_0^\infty \int_0^\infty \int_0^\infty e^{l(data|\alpha_1,\alpha_2,\theta)+\rho(\alpha_1,\alpha_2,\theta)} d\alpha_1 d\alpha_2 d\theta}, \tag{19}$$

can be approximated as

$$\begin{aligned} I(data) = {} & \lambda(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\theta}) + (\lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3 + c_4 + c_5) + \frac{1}{2}[A(\lambda_1 \sigma_{11} + \lambda_2 \sigma_{12} + \lambda_3 \sigma_{13}) \\ & + B(\lambda_1 \sigma_{21} + \lambda_2 \sigma_{22} + \lambda_3 \sigma_{23}) + C(\lambda_1 \sigma_{31} + \lambda_2 \sigma_{32} + \lambda_3 \sigma_{33})], \end{aligned} \tag{20}$$

where $\hat{\alpha}_1$, $\hat{\alpha}_2$ and $\hat{\theta}$ are the MLEs of α_1, α_2 and θ, respectively, and other terms in the (19) and (20) are defined to be

$l(data|\alpha_1, \alpha_2, \theta)$ is the log-likelihood function,

$\rho(\alpha_1, \alpha_2, \theta)$ is the logarithm of the joint prior density $\pi^*(\alpha_1, \alpha_2, \theta) = \pi_1(\alpha_1)\pi_2(\alpha_2)\pi_3(\theta)$,

$c_i = \rho_1 \sigma_{i1} + \rho_2 \sigma_{i2} + \rho_3 \sigma_{i3}$, $i = 1, 2, 3$,

$c_4 = \lambda_{12} \sigma_{12} + \lambda_{13} \sigma_{13} + \lambda_{23} \sigma_{23}$,

$c_5 = \frac{1}{2}(\lambda_{11} \sigma_{11} + \lambda_{22} \sigma_{22} + \lambda_{33} \sigma_{33})$,

$A = \sigma_{11} l_{111} + 2\sigma_{12} l_{121} + 2\sigma_{13} l_{131} + 2\sigma_{23} l_{231} + \sigma_{22} l_{221} + \sigma_{33} l_{331}$,

$B = \sigma_{11} l_{112} + 2\sigma_{12} l_{122} + 2\sigma_{13} l_{132} + 2\sigma_{23} l_{232} + \sigma_{22} l_{222} + \sigma_{33} l_{332}$,

$C = \sigma_{11} l_{113} + 2\sigma_{12} l_{123} + 2\sigma_{13} l_{133} + 2\sigma_{23} l_{233} + \sigma_{22} l_{223} + \sigma_{33} l_{333}$,

$\rho_i = \frac{\partial \log \pi^*}{\partial \theta_i}$, $\quad \lambda_i = \frac{\partial \lambda}{\partial \theta_i}$, $\quad \lambda_{ij} = \frac{\partial^2 \lambda}{\partial \theta_i \partial \theta_j}$, $\quad i, j = 1, 2, 3$,

$l_{ij} = \frac{\partial^2 l}{\partial \theta_i \partial \theta_j}$, $\quad l_{ijk} = \frac{\partial^3 l}{\partial \theta_i \partial \theta_j \partial \theta_k}$, $\quad i, j, k = 1, 2, 3$,

σ_{ij} is the $(i, j)^{th}$ elements of the inverse of the matrix having elements $[-l_{ij}]$ calculated at $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\theta})$. Further, all the expressions in Eq.(20) are also evaluated at $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\theta})$. This implies that,

$$\rho(\alpha_1, \alpha_2, \theta) \propto (a_1 - 1)\log \alpha_1 + (a_2 - 1)\log \alpha_2 + (a_0 - 1)\log \theta - b_1 \alpha_1 - b_2 \alpha_2 - b_0 \theta,$$

and also

$$\rho_1 = \frac{a_1 - 1}{\alpha_1} - b_1, \;\; \rho_2 = \frac{a_2 - 1}{\alpha_2} - b_2, \;\; \rho_3 = \frac{a_0 - 1}{\theta} - b_0, \;\; l_{11} = -\frac{k_1}{\alpha_1^2}, \;\; l_{12} = l_{21} = 0,$$

$$l_{13} = l_{31} = -\sum_{i=1}^{k} u_i t_i e^{\theta t_i}, \;\; l_{22} = -\frac{k_2}{\alpha_2^2}, \; l_{23} = l_{32} = -\sum_{i=1}^{k} v_i t_i e^{\theta t_i},$$

$$l_{33} = -\frac{k}{\theta^2} - \alpha_1 \sum_{i=1}^{k} u_i t_i^2 e^{\theta t_i} - \alpha_2 \sum_{i=1}^{k} v_i t_i^2 e^{\theta t_i},$$

and the values of l_{ijk} for all $i,j,k=1,2,3$, are computed to be

$$l_{111}=2\frac{k_1}{\alpha_1^3},\qquad l_{222}=2\frac{k_2}{\alpha_2^3},\qquad l_{133}=l_{313}=l_{331}=-\sum_{i=1}^{k}u_i t_i^2 e^{\theta t_i},$$

$$l_{233}=l_{323}=l_{332}=-\sum_{i=1}^{k}v_i t_i^2 e^{\theta t_i},\qquad l_{333}=2\frac{k}{\theta^3}-\alpha_1\sum_{i=1}^{k}u_i t_i^3 e^{\theta t_i}-\alpha_2\sum_{i=1}^{k}v_i t_i^3 e^{\theta t_i},$$

$$l_{112}=l_{113}=l_{121}=l_{122}=l_{123}=l_{131}=l_{132}=l_{211}=l_{212}=l_{213}=l_{221}=l_{223}=l_{231}=l_{232}=l_{311}=l_{312}=l_{321}=l_{322}=0.$$

As a consequence of that, the BEs under L_1, L_2 and L_3 can be expressed as follows:

1. **Under SE loss function:**
   - If $\lambda(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\theta})=\hat{\alpha}_1$ then $\hat{\alpha}_{1S}=\hat{\alpha}_1+c_1+\frac{1}{2}[A\sigma_{11}+B\sigma_{21}+C\sigma_{31}]$.
   - If $\lambda(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\theta})=\hat{\alpha}_2$ then $\hat{\alpha}_{2S}=\hat{\alpha}_2+c_2+\frac{1}{2}[A\sigma_{12}+B\sigma_{22}+C\sigma_{32}]$.
   - If $\lambda(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\theta})=\hat{\theta}$ then $\hat{\theta}_S=\hat{\theta}+c_3+\frac{1}{2}[A\sigma_{13}+B\sigma_{23}+C\sigma_{33}]$.
2. **Under LINEX loss function:**
   - If $\lambda(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\theta})=e^{-\nu\hat{\alpha}_1}$ then $\hat{\alpha}_{1L}=\hat{\alpha}_1-\frac{1}{\nu}\log\left(1-\nu(c_1-\frac{\nu}{2}\sigma_{11}+\frac{1}{2}[A\sigma_{11}+B\sigma_{21}+C\sigma_{31}])\right)$.
   - If $\lambda(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\theta})=e^{-\nu\hat{\alpha}_2}$ then $\hat{\alpha}_{2L}=\hat{\alpha}_2-\frac{1}{\nu}\log\left(1-\nu(c_2-\frac{\nu}{2}\sigma_{22}+\frac{1}{2}[A\sigma_{12}+B\sigma_{22}+C\sigma_{32}])\right)$.
   - If $\lambda(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\theta})=e^{-\nu\hat{\theta}}$ then $\hat{\theta}_L=\hat{\theta}-\frac{1}{\nu}\log\left(1-\nu(c_3-\frac{\nu}{2}\sigma_{33}+\frac{1}{2}[A\sigma_{13}+B\sigma_{23}+C\sigma_{33}])\right)$.
3. **Under Entropy loss function:**
   - If $\lambda(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\theta})=\alpha_1^{-\nu}$ then $\hat{\alpha}_{1E}=\hat{\alpha}_1\left[1-\frac{\nu}{\hat{\alpha}_1}(c_1-\frac{\nu+1}{2\hat{\alpha}_1}\sigma_{11}+\frac{1}{2}[A\sigma_{11}+B\sigma_{21}+C\sigma_{31}])\right]^{-\frac{1}{\nu}}$.
   - If $\lambda(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\theta})=\alpha_2^{-\nu}$ then $\hat{\alpha}_{2E}=\hat{\alpha}_2\left[1-\frac{\nu}{\hat{\alpha}_2}(c_2-\frac{\nu+1}{2\hat{\alpha}_2}\sigma_{22}+\frac{1}{2}[A\sigma_{12}+B\sigma_{22}+C\sigma_{32}])\right]^{-\frac{1}{\nu}}$.
   - If $\lambda(\hat{\alpha}_1,\hat{\alpha}_2,\hat{\theta})=\hat{\theta}^{-\nu}$ then $\hat{\theta}_E=\hat{\theta}\left[1-\frac{\nu}{\hat{\theta}}(c_3-\frac{\nu+1}{2\hat{\theta}}\sigma_{33}+\frac{1}{2}[A\sigma_{13}+B\sigma_{23}+C\sigma_{33}])\right]^{-\frac{1}{\nu}}$.

4.2 Bayesian sample based methods

In light of the JPC sample, one can update the prior information about the shape and scale parameters via the posterior model, it is possible to provide approximate BEs of α_1,α_2 and θ based on importance sampling technique. In the context of the approach introduced by Geman and Geman [29], we generate the posterior distribution of α_1,α_2 and θ based on the conditional arguments in (14) as follows:

Algorithm of Bayesian sample-based method:

- **Step 1:** Generate θ from $g_1(\theta|\mathbf{t})$;
- **Step 2:** Generate α_1 from $\pi_1(\alpha_1|\theta,\mathbf{t})$ and α_2 from $\pi_2(\alpha_2|\theta,\mathbf{t})$;
- **Step 3:** Repeat steps 1-3, M times and then obtain $(\theta_1,\alpha_{11},\alpha_{21}),\ldots,(\theta_M,\alpha_{1M},\alpha_{2M})$;
- **Step 4:** Compute

$$w_i(\theta_i,\alpha_{1i},\alpha_{2i})=\frac{g_2(\theta_i,\mathbf{t})}{\sum_{i=1}^{M}g_2(\theta_i,\mathbf{t})},$$

and then compute an approximate BE of a function of α_1,α_2 and θ (say, $\lambda(\alpha_1,\alpha_2,\theta)$) under a SE loss function as

$$\hat{\lambda}_S=\sum_{i=1}^{M}w_i(\alpha_{1i},\alpha_{2i},\theta_i)\lambda(\alpha_{1i},\alpha_{2i},\theta_i).$$

The other Bayesian estimates under L_2 and L_3 can be computed readily based on the following expressions:

$$\hat{\lambda}_L=-\frac{1}{\nu}log\left[\sum_{i=1}^{M}w_i(\alpha_{1i},\alpha_{2i},\theta_i)e^{-\nu\lambda(\alpha_{1i},\alpha_{2i},\theta_i)}\right],$$

and

$$\hat{\lambda}_E=\left[\sum_{i=1}^{M}w_i(\alpha_{1i},\alpha_{2i},\theta_i)(\lambda(\alpha_{1i},\alpha_{2i},\theta_i))^{-\nu}\right]^{-1/\nu}.$$

Using the percentile-based argument used in Chen and Shao [30], we can provide CrI of any function of the parameters, $\lambda(\alpha_1,\alpha_2,\theta)$. The 100γth $(0<\gamma<1)$ quantile of λ is λ_γ such that $P(\lambda\leq\lambda_\gamma)=\gamma$. Let us assume that $\lambda_{(1)},\ldots,\lambda_{(M)}$ be order statistics of $\lambda_1,\ldots,\lambda_M$ and $w^{(1)},\ldots,w^{(M)}$ be the values associated with $\lambda_{(1)},\ldots,\lambda_{(M)}$. This implies that the consistent sample based estimate of λ_γ is $\hat{\lambda}_\gamma=\lambda_{(\kappa)}$, where κ is an integer satisfying

$$\sum_{i=1}^{\kappa-1} w^{(i)} \leq \gamma \leq \sum_{i=1}^{\kappa} w^{(i)}.$$

This in turns that $(1-\gamma)100\%$ CrI of λ can be computed as $(\hat{\lambda}_{\frac{\gamma}{2}},\hat{\lambda}_{1-\frac{\gamma}{2}})$, say, C-S CrI. It is important to point out that the CrIs obtained by the previous approach does not specify whether the values of λ within these intervals have highest probability than that of the values outside the intervals. It is more desirable to have CrI of $\lambda(\alpha_1,\alpha_2,\lambda)$ with the highest posterior density (HPD). For any probability content, $1-\gamma$, the HPD interval is of the shortest width and the posterior density for every point outside the interval is less than that for every point inside the interval. For M sufficiently large, the $100(1-\gamma)\%$ HPD interval for λ may be chosen as the shortest of the intervals $C_\kappa(M)$, $\kappa=1,2,\ldots,M-[(1-\gamma)M]$, where $[x]$ is the largest integer that is less than or equal to x, with

$$C_\kappa(M)=\left(\lambda_{(\kappa)},\lambda_{(\kappa+[(1-\gamma)M])}\right).$$

Therefore, the HPD intervals of the three parameters are computed in this way.

Another Bayesian sampling algorithm is to estimate the posterior distribution based on Gibbs sampler approach. This approach requires being able to sample from the full conditional distributions from each parametric quantity involved. This can be applied for α_1 and α_2 but not for θ. Consequently, Metropolis-Hastings (M-H) steps are proposed into the Gibbs sampler so that α_1 and α_2 are sampled directly from their full conditional distributions, whereas θ can be updated via a M-H steps as explained in Tierney [31], using $G(k+a_0,b_1)$ as a proposal distribution. The M-H steps proceed as follows:

Algorithm of M-H method:

–**Step 1:** Select an initial guess θ_0;
–**Step 2:** For $t=1,2,\ldots,M$, repeat:
(a) Draw candidate θ^* from $G(k+a_0,b_0)$ with its pdf $g_{k+a_0,b_1}(\theta^*|\theta_{t-1})$ and u from $U(0,1)$;
(b) Compute the acceptance probability

$$\varepsilon=\frac{p_3(\theta^*)/g_{k+a_0,b_1}(\theta^*|\theta_{t-1})}{p_3(\theta_{t-1})/g_{k+a_0,b_0}(\theta_{t-1}|\theta^*)}=\frac{p_3(\theta^*)}{p_3(\theta_{t-1})}\frac{g_{k+a_0,b_0}(\theta_{t-1})}{g_{k+a_0,b_0}(\theta^*)}=h(\theta^*,\theta_{t-1})\times e^{(\theta^*-\theta_{t-1})\sum_{i=1}^{k}t_i},$$

where

$$h(\theta^*,\theta_{t-1})=\frac{\left(b_1+\Delta_{\theta_{t-1}}(\mathbf{t})\right)^{k_1+a_1}\left(b_2+\overline{\Delta}_{\theta_{t-1}}(\mathbf{t})\right)^{k_2+a_2}}{\left(b_1+\Delta_{\theta^*}(\mathbf{t})\right)^{k_1+a_1}\left(b_2+\overline{\Delta}_{\theta^*}(\mathbf{t})\right)^{k_2+a_2}}.$$

(c) If $u<\min(1,\varepsilon)$, then set $\theta_t=\theta^*$, else go to (a).

Once the posterior samples have been obtained, the simulation consistent Bayes estimates under different loss functions can be computed. The associated CrIs of the parameters can also be constructed.

5 Simulation results and data analysis

Here in this section, we perform a comprehensive simulation study to assess the performance of the estimates developed in the previous sections. Also, we analyze a real data set from GO distribution for illustrative purposes. All computations are performed using R software.

5.1 Data analysis

In this section, we present the analysis of real data sets to illustrate the performance of the obtained methods. The data represent the survival time in months of stage 4 Melanoma patients based on their gender who received treatment at the University of Oklahoma Health Sciences Center from 1974-1978. The data were originally reported by Lee et al.[32] and for more details, see for example Lee and Wang [33]. The sets of data are:
Data set 1 (female patients):

1.3 2.7 3.8 4.2 7.4 9.3 10.5 11.4 13.3 13.8 13.8 20 22.2

Data set 2 (male patients):

0.4 0.9 1.2 1.5 1.6 1.7 2.5 2.5 3.9 3.9 4 4.2 4.5 5.8 5.9 6.3 7.3 7.4 8.3 9.8 11 11.1 16.1 20.5

Table 1 presents the ML estimate of the unknown parameters, the goodness-of-fit tests based on Kolmogrov-Smirnov (K-S) and Cramer-von Mises (CvM) statistics. It is easily seen that the GO model fits both data sets very well. This conclusion is also supported by diagnostic plots of the empirical and fitted distribution functions in Figures 1 and 2. In addition, it is of interest to study the null hypothesis $H_0 : \theta_1 = \theta_2 = \theta$ (i.e., the scale parameters are equal) versus the alternative hypothesis $H_1 : \theta_1 \neq \theta_2$ using the likelihood ratio test. For the given data, the test statistic is computed as $\Delta = L1/L2 = 0.465558$, $-2\log(\Delta) = 1.529034$ and the p-value of the test is $P(\chi^2_{(1)} > 0.465558) = 0.495065$. Hence, the assumption of equality of the scale parameters cannot be rejected.

Table 1: MLEs, K-S and CvM goodness-of-fit tests.

Data Set	Scale Parameter	Shape Parameter	K-S(p-value)	CvM(p-value)
1	0.9900692	1.0227568	0.13925933(0.9931)	0.03012802(0.899)
2	0.9900692	0.4378163	0.09860765(0.9368)	0.0285854(0.9954)

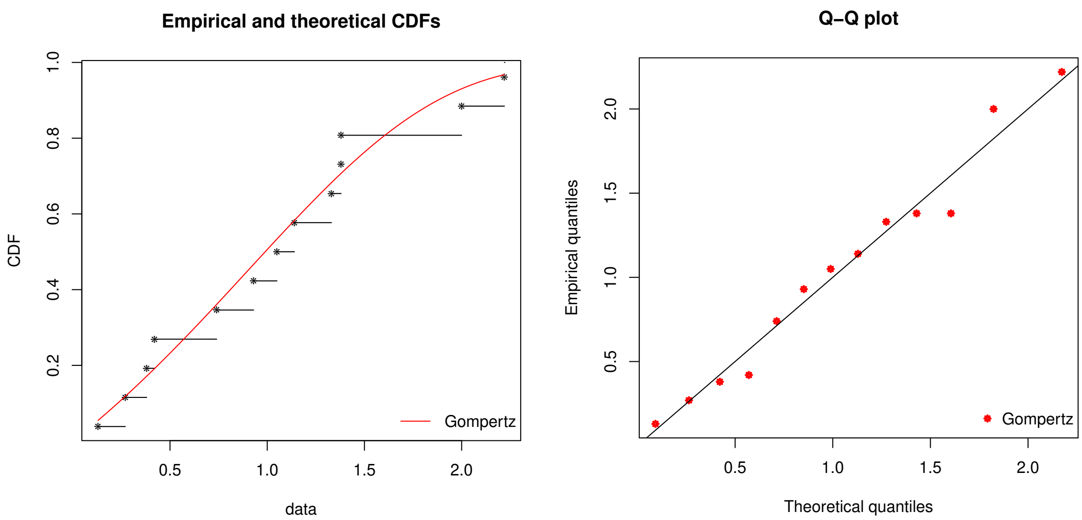

Fig. 1: Empirical and fitted distribution functions and Q-Q Plots for Data Set 1.

For explanation purposes, we suggest the following joint progressive type-II censored sample with $m = 13$ and $n = 24$, $k = 10$, $R_i = 2$, $i = 1,...,3$ and $R_i = 3$, $i = 4,\ldots,10$. The resulting data set is recorded as follows:

(0.4,0,0), (0.9,0,2), (1.2, 0,1), (1.3,1,0), (1.5,0,1), (1.6,0,2), (1.7,0,1), (2.5,0,1), (2.7,1,1), (3.8,1,2).

Based on the above observed joint progressive type II censored data, we obtain the MLEs and BEs of α_1, α_2 and θ under SE, LI and GE loss functions. We have generated 5000 observations to compute the BEs of α_1, α_2 and θ based on the importance sampler after discarding the initial 500 burn-in samples. Note that, for computing the BEs and HPD CrIs, we assume that the priors of α_1, α_2 and θ are improper, i.e. $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$, since we do not have any prior information. The M-H algorithm is also used to compute the BEs of θ where we use the gamma distribution as a proposal distribution. Here, the best fitted model for the full conditional distribution can be concluded by managing the choice of the parameters for the proposal distribution. Therefore, to generate numbers from the target probability distribution, we use the M-H algorithm with gamma proposal distribution. We assumed the initial value of θ to be its MLE, $\hat{\theta}$ which is computed using EM-algorithm. Here, we generated $50,000$ random variates and we checked the acceptance rate for this

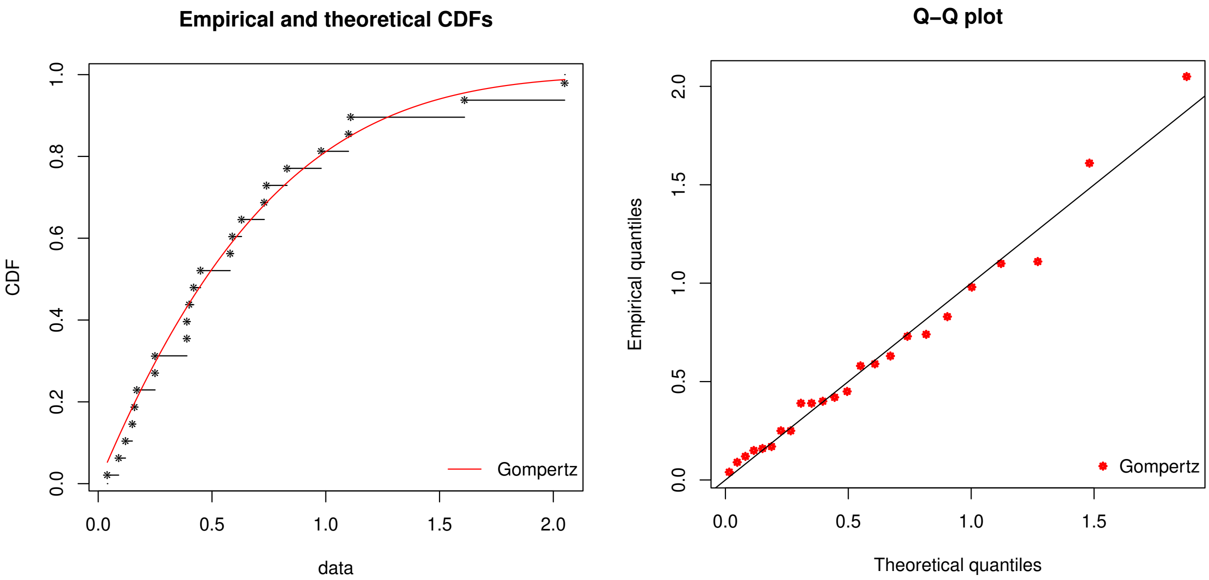

Fig. 2: Empirical and fitted distribution functions and Q-Q Plots for Data Set 2.

choice of variance to be 68.27% which is quite satisfactory. We discarded the initial 5000 burn-in samples and computed the BEs based on the remaining observations.

Graphical diagnostics tools involving trace and ACF plots are used to check the convergence of M-H algorithm. Figure 3 shows the trace and ACF plots for θ. From the trace plot, we can easily observe a random scatter about some mean value represented by a solid line with a fine mixing of the chains for the simulated values of θ. The ACF plot shows that chains have low autocorrelations. As a result, these plots indicate the rapid convergence of the M-H algorithm based on the proposed gamma distribution.

The results for MLEs and BEs using importance and M-H samplers along with the 95% Boot-t CI, asymptotic CI and HPD CrIs for α_1, α_2 and θ are presented in Tables 2 and 3.

Table 2: MLEs and BEs of α_1, α_2 and θ based on joint progressive censored data.

	MLE	Method	SE	LINEX		GE	
				ω= -0.5	ω=0.5	ρ= -0.5	ρ=0.5
α_1	0.1366	Lindley	0.1904	0.1918	0.1890	0.1894	0.1876
		Importance	0.1213	0.1216	0.1210	0.1180	0.1095
		M-H	0.1685	0.1707	0.1665	0.1572	0.1347
α_2	0.2940	Lindley	0.3136	0.3222	0.3049	0.2961	0.2633
		Importance	0.2640	0.2648	0.2632	0.2604	0.2515
		M-H	0.3274	0.3311	0.3238	0.3161	0.2927
θ	0.4498	Lindley	0.4990	0.4992	0.4991	0.4985	0.4982
		Importance	0.4515	0.4518	0.4511	0.4508	0.4497
		M-H	0.3992	0.3993	0.3991	0.3990	0.3985

5.2 Numerical comparisons

Now, we compare the performances of the different methods of estimation based on Monte Carlo simulations. We compare the performance of the MLEs, and BEs in terms of biases and MSEs. In this simulation, the values of GO parameters are

Table 3: The corresponding 95% *CIs* for α_1, α_2 and θ

	Approx.	Boot-t	Loss		HPD (Importance)	HPD (M-H)
α_1	(0.0528,0.7261)	(0.0245,0.6258)	SE		(0.0268,0.2650)	(0.1126,0.3113)
			LINEX	ω= - 0.5	(0.0268,0.2050)	(0.0929,0.2332)
				ω= 0.5	(0.0268,0.1925)	(0.0932,0.2213)
			GE	ρ= - 0.5	(0.0227,0.2126)	(0.0919,0.2409)
				ρ= 0.5	(0.0227,0.2052)	(0.0956,0.2324)
α_2	(0.0901,1.2781)	(0.0413,0.9146)	SE		(0.0827,0.3226)	(0.1773,0.3781)
			LINEX	ω= - 0.5	(0.1086,0.3187)	(0.1816,0.3535)
				ω= 0.5	(0.1089,0.2912)	(0.1784,0.3354)
			Entropy	ρ= - 0.5	(0.0986,0.3223)	(0.1863,0.3667)
				ρ= 0.5	(0.1093,0.3032)	(0.1772,0.3462)
θ	(0.1295,1.1700)	(0.2788,0.7127)	SE		(0.2128,0.5202)	(0.3308,0.5694)
			LINEX	ω= - 0.5	(0.2352,0.4735)	(0.3411,0.5412)
				ω= 0.5	(0.2928,0.4702)	(0.3584,0.5186)
			GE	ρ= - 0.5	(0.2223,0.4915)	(0.3345,0.5504)
				ρ= 0.5	(0.2628,0.4616)	(0.3419,0.5231)

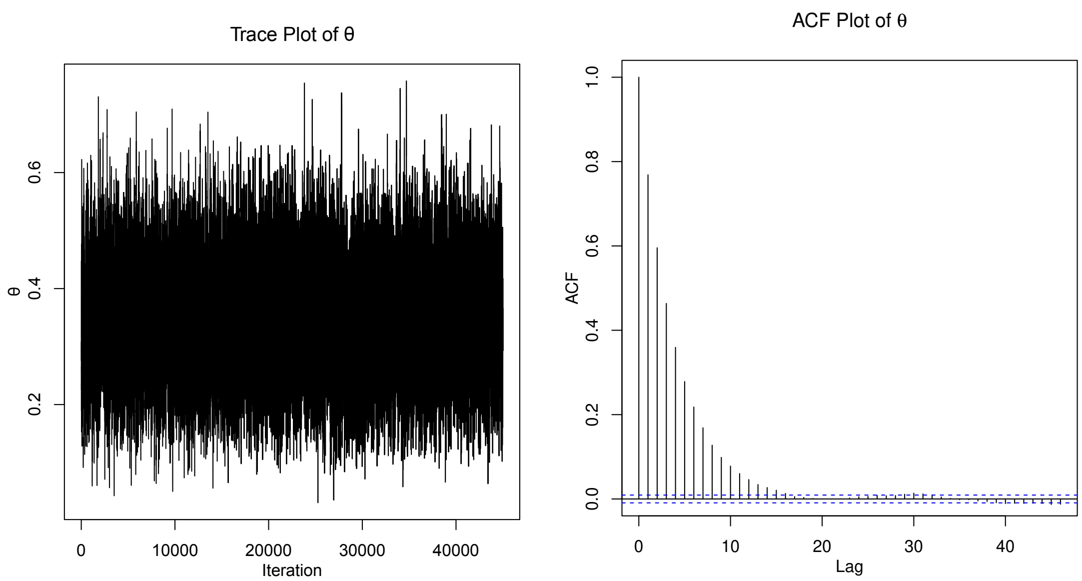

Fig. 3: Plots of Metropolis-Hastings Markov chains for θ.

considered as $\alpha_1 = 2$, $\alpha_2 = 1.5$ and $\theta = 2$. Here, we consider different effective sample sizes, $k = 20, 25$ and different censoring schemes. For conducting the Bayesian analysis, we assume two priors. For the first prior (Prior 0), we assume that $a_1 = b_1 = a_2 = b_2 = a_3 = b_3 = 0$. Then we assume an additional prior; Prior 1 with $a_1 = a_2 = a_3 = 1$, $b_1 = b_2 = b_3 = 2$. We use the following notation for a particular progressive censoring scheme. For example $k = 6$ and $R = (4, 0_{(5)})$ means $R_1 = 4$, $R_2 = R_3 = R_4 = R_5 = R_6 = 0$.

To conduct the comparison process, we have randomly generated six different joint progressive censored schemes from the GO distribution and for $k = 20$ and $k = 25$ summarized in Table 4. We then compute the average biases, mean square errors (MSEs) for the estimates and CIs for GO parameters. Under the non-informative and informative priors, the biases and MSEs of estimates over 1000 replications are computed for various censoring schemes of $k = 20$ and $k = 25$.

Table 4: Censoring schemes.

	k	C.S.
R_1	20	$R=(7,0_{(18)},15)$
R_2	20	$R=(0_{(9)},7,0_{(9)},15)$
R_3	20	$R=(0_{(18)},7,15)$
R_4	25	$R=(7,0_{(23)},10)$
R_5	25	$R=(0_{(11)},7,0_{(12)},10)$
R_6	25	$R=(0_{(23)},7,10)$

The average biases and MSEs of the MLEs and BEs under SE, LI and EE loss functions of α_1, α_2 and θ with Prior 0 and Prior 1 are computed over 5000 replications and displayed in Tables 5 and 6. The MLEs are computed by maximizing the likelihood function and so by solving the likelihood equations (11) and (12). For the LI and GE loss functions, the BEs are computed using different values of ω and ρ (say, $\omega=-0.5,0.5$, $\rho=-0.5,0.5$), respectively. As seen in Tables 5 and 6, the BEs perform well for $k=20$ and $k=25$ in the sense of bias and MSE. As expected, the BEs under Prior 1 are better than that under Prior 0. Further, we can easily notice that the M-H method beats the Lindley's approximation and importance sampling methods in the sense of MSEs and biases values for all parameters and under all error loss functions applied. Under LI error loss function, the estimates based on M-H algorithm show the least MSEs and biases values over SE and GE error loss functions in all parameters and all methods of estimation.

Tables 7, 8 and 9 present the average lengths (ALs) and coverage probabilities (CPs) of 95% CIs for α_1, α_2 and θ based on Boot-t, asymptotic maximum likelihood and Bayesian methods with Prior 0 and Prior 1 under SE, LI and GE loss functions. From Tables 7, 8 and 9, it is observed that the HPD CrIs are shorter than the asymptotic and Boot-t CIs under all priors and all loss functions for $k=20$ and $k=25$. It can also be noticed that M-H CrIs are the shortest over all other intervals. Furthermore, the CrIs under LI loss function is better than SE and GE in all types of estimation used. The performances of HPD CrI tend to be high under the informative prior when compared to HPD CrI under noninformative prior, asymptotic and Boot-t CIs. While the Boot-t method performs well when compared to asymptotic method for estimating of all parameters. It can be also observed that all CIs are shorter for $k=25$ when compared to $k=20$.

In summary, it is clear that the BEs based on Lindley's approximation, importance sampling and M-H methods under different error loss functions and priors work better than the MLEs in all the cases considered for estimating the parameters. The MSEs and biases of the BEs obtained by M-H algorithm are smaller than that of the BEs computed from the Lindley's approximation and importance sampling methods under the different loss functions. When comparing the BEs under LI and GE loss functions, we can notice the LI loss function provides better results than GE. It is realized that the HPD intervals based on M-H method compete the ones based on importance sampling method in terms of ALs and CPs criteria under SE, LI and GE loss functions. It is also checked that the ALs and CPs of HPD CrIs based on LI and GE loss functions tend to be close.

6 Conclusions

In this work, the estimation problem of the parameters based on joint Type-II progressive censoring scheme when their lifetimes follow Gompertz distributions with different shape parameters. It is shown that the maximum likelihood estimators of the model parameters and their asymptotic confidence intervals can be obtained. We have also proposed different Bayesian procedures to estimating the parameters involved, namely, Lindley's approximation, importance sampling procedure and Metropolis-Hastings algorithm. The corresponding credible intervals are also discussed. The performance of all methods presented in this paper are evaluated and compared via Monte Carlo simulations. It is observed that the Bayes estimates under Metropolis-Hastings method outperform the frequentist methods as well as the Bayes ones under the Lindley's approximation and importance sampling methods in the sense of bias and mean square error for all parameters under the error loss functions applied here. Under LINEX error loss function, the MSEs and biases of Metropolis-Hastings based estimates tend to be smaller over square and general entropy error losses. By considering the average length and coverage probability as optimality criteria for the credible intervals of the parameters, it is also evident that the highest posterior density credible intervals using Metropolis-Hastings compete the approximate, Bootstrap-t confidence intervals, and the posterior density credible intervals based on importance sampling.

Although, we have mainly restricted our attention to the joint Type-II progressive censoring scheme produced from the two populations, but the so developed procedures can be extended to more than two populations as well. More investigation is needed along this line.

Table 5: Biases and MSEs of the MLEs and BEs of α_1, α_2 and θ under Prior 0 (the entries in parentheses are MSEs).

C. S.		MLE		BEs (SE)	BEs (LINEX)		BEs (GE)	
					$\omega=-0.5$	$\omega=0.5$	$\rho=-0.5$	$\rho=0.5$
R_1	α_1	-2.2299(2.21170)	Lindley	-1.5607(1.1748)	-0.9853(0.8337)	-1.1243(0.9041)	-1.1462(0.9778)	1.5313(1.0404)
			Importance	-1.1463(1.0433)	-0.7778(0.6839)	-0.8332(0.7366)	-0.8804(0.7836)	-0.9081(0.8251)
			M-H	-0.9865(0.9551)	-0.7496(0.6056)	-0.7945(0.6948)	-0.8329(0.7755)	-0.8467(0.8145)
	α_2	-0.8938(0.3016)	Lindley	-0.1883(0.1608)	-0.1240(0.1040)	-0.1369(0.1254)	-0.1595(0.1385)	-0.1697(0.1443)
			Importance	0.1475(0.0237)	0.1138(0.0199)	0.1211(0.0220)	0.1392(0.0223)	0.1433(0.0230)
			M-H	0.1275(0.0223)	0.1129(0.0159)	0.1184(0.0192)	0.1226(0.0199)	0.1251(0.0210)
	θ	2.3451(1.0226)	Lindley	1.8790(0.8841)	1.8239(0.8589)	1.8301(0.8664)	1.8414(0.8702)	1.8493(0.8758)
			Importance	-0.5623(0.3181)	-0.5614(0.3171)	-0.5615(0.3172)	-0.5616(0.3173)	-0.5618(0.3175)
			M-H	-0.5493(0.3043)	-0.5468(0.3017)	-0.5472(0.3021)	-0.5476(0.3025)	-0.5479(0.3029)
R_2	α_1	-2.8720(2.6084)	Lindley	-1.8800(1.5875)	-1.5352(1.3135)	-1.6297(1.3641)	-1.7125(1.4552)	-1.7896(1.4667)
			Importance	-1.8505(1.3601)	-1.0555(1.0417)	-1.2110(1.0738)	-1.2793(1.1116)	-1.3536(1.1631)
			M-H	-1.5459(1.2136)	-1.0265(0.8845)	-1.1542(1.0646)	-1.2367(1.0858)	-1.3352(1.1101)
	α_2	1.3110(0.3244)	Lindley	-0.2140(0.1859)	-0.1431(0.1150)	-0.2074(0.1368)	-0.2092(0.1553)	-0.2107(0.1677)
			Importance	0.1576(0.0262)	0.1338(0.0225)	0.1468(0.0245)	0.1491(0.0249)	0.1533(0.0256)
			M-H	0.1337(0.0252)	0.1202(0.0183)	0.1277(0.0219)	0.1291(0.0226)	0.1314(0.0239)
	θ	2.5163(1.0646)	Lindley	2.1510(0.9523)	2.0042(0.8684)	2.0240(0.8775)	2.0657(0.8851)	2.0973(0.8899)
			Importance	-0.5680(0.3246)	-0.5672(0.3238)	-0.5673(0.3239)	-0.5674(0.3240)	-0.5676(0.3241)
			M-H	-0.5494(0.3045)	-0.5471(0.3021)	-0.5475(0.3024)	-0.5478(0.3028)	-0.5481(0.3032)
R_3	α_1	-2.8802(2.6612)	Lindley	2.0111(1.7321)	1.6318(1.5084)	1.7402(1.5978)	1.7587(1.6882)	1.8175(1.6991)
			Importance	-1.9933(1.4117)	-1.4519(1.2045)	-1.5022(1.2253)	-1.5449(1.2426)	-1.6496(1.2841)
			M-H	-1.6882(1.2854)	-1.4295(1.1707)	-1.4511(1.1818)	-1.4713(1.1921)	-1.5193(1.2121)
	α_2	2.0606(0.3579)	Lindley	-0.3941(0.2883)	-0.3245(0.2244)	-0.3258(0.2304)	-0.2566(0.3259)	-0.2697(0.3365)
			Importance	0.2365(0.0677)	0.2138(0.0613)	0.2199(0.0647)	0.2314(0.0651)	0.2340(0.0664)
			M-H	0.1831(0.0339)	0.1581(0.0253)	0.1717(0.0298)	0.1739(0.0360)	0.1785(0.0322)
	θ	2.7829(1.2112)	Lindley	2.7413(1.1987)	2.5482(1.1321)	2.6898(1.1784)	2.6917(1.1840)	2.7260(1.1918)
			Importance	-0.5688(0.3256)	-0.5681(0.3248)	-0.5682(0.3249)	-0.5683(0.3250)	-0.5684(0.3251)
			M-H	-0.5509(0.3062)	-0.5487(0.3038)	-0.5491(0.3042)	-0.5494(0.3045)	-0.5497(0.3049)

Table 5: Continued (the entries in parentheses are MSEs).

C. S.		MLE		BEs (SE)	BEs (LINEX)		BEs (GE)	
					$\omega=-0.5$	$\omega=0.5$	$\rho=-0.5$	$\rho=0.5$
R_4	α_1	-2.00034(1.9505)	Lindley	-1.1695(1.0797)	-0.9154(0.7257)	-1.0048(0.7772)	-1.0561(0.7984)	-1.0657(0.8728)
			Importance	-1.0891(0.9215)	-0.6279(0.4572)	-0.6510(0.5651)	-0.6854(0.6650)	-0.8091(0.7050)
			M-H	-0.7669(0.6280)	-0.2102(0.1626)	-0.3279(0.3204)	-0.4763(0.4432)	-0.4982(0.4909)
	α_2	-0.5653(0.2172)	Lindley	-0.1045(0.0647)	-0.0750(0.0455)	-0.0908(0.0510)	-0.0939(0.0581)	-0.0986(0.0600)
			Importance	0.1034(0.0147)	0.0648(0.0124)	0.0750(0.0136)	0.0760(0.0139)	0.0787 (0.0143)
			M-H	0.0759(0.0109)	0.0608(0.0058)	0.0693(0.0085)	0.0711(0.0090)	0.0735(0.0100)
	θ	2.2216(0.9134)	Lindley	1.5885(0.8498)	1.4893(0.7356)	1.5059(0.8173)	1.5411(0.8314)	1.5576(0.8365)
			Importance	-0.4122(0.1725)	-0.4116(0.1720)	-0.4117(0.1721)	-0.4118(0.1722)	-0.4119(0.1723)
			M-H	-0.4006(0.1644)	-0.3968(0.1615)	-0.3975(0.1621)	-0.3982(0.1626)	-0.3986(0.1629)
R_5	α_1	-2.4630(1.9543)	Lindley	-1.2940(1.1556)	-1.0583(0.9041)	-1.0987(1.0021)	-1.1219(1.0964)	-1.1485(1.1270)
			Importance	-1.2816(1.1374)	-0.9002(0.8845)	-0.9420(0.9514)	-1.0171(1.0078)	-1.1292(1.0334)
			M-H	-1.1971(1.0740)	-0.8814(0.8097)	-0.9345(0.9070)	-0.9652(0.9415)	-0.9989(0.9839)
	α_2	0.5822(0.2396)	Lindley	-0.1177(0.0920)	-0.1026(0.0595)	0.1084(0.0596)	-0.1130(0.0642)	-0.1154(0.0845)
			Importance	0.1076(0.0187)	0.0830(0.0156)	0.0949(0.0173)	0.0981(0.0176)	0.1028(0.0181)
			M-H	0.0987(0.0118)	0.0790(0.0065)	0.0919(0.0092)	0.0937(0.0098)	0.0962(0.0108)
	θ	2.3007(0.9792)	Lindley	1.8172(0.8950)	1.7102(0.8144)	1.7121(0.8514)	1.7509(0.8716)	1.7699(0.8808)
			Importance	-0.4174(0.1768)	-0.4168(0.1764)	-0.4169(0.1765)	-0.4170(0.1765)	-0.4171(0.1766)
			M-H	-0.4040(0.1670)	-0.4007(0.1644)	-0.4013(0.1649)	-0.4019(0.1653)	-0.4022(0.1656)
R_6	α_1	-2.6101(2.0283)	Lindley	-1.8125(1.4257)	-1.5121(1.2412)	-1.5627(1.2997)	-1.6510(1.3088)	-1.6587(1.3163)
			Importance	-1.6278(1.2757)	-1.2467(1.0694)	-1.2822(1.0930)	-1.3161(1.1283)	-1.3908(1.1789)
			M-H	-1.4259(1.1816)	-1.0489(1.0075)	-1.1962(1.0911)	-1.2621(1.1072)	-1.2974(1.1229)
	α_2	-0.6615(0.2427)	Lindley	-0.1790(0.1367)	-0.1175(0.1013)	-0.1294(0.1254)	-0.1386(0.1291)	-0.1512 (0.1334)
			Importance	0.1411(0.0244)	0.1105(0.0195)	0.1263(0.0218)	0.1315(0.0222)	0.1357(0.0232)
			M-H	0.1390(0.0201)	0.0920(0.0124)	0.1149(0.0164)	0.1292(0.0172)	0.1342(0.0186)
	θ	2.6249(1.3120)	Lindley	1.8656(0.9650)	1.7712(0.9288)	1.7782(0.9337)	1.8164(0.9426)	1.8363(0.9495)
			Importance	-0.4175(0.1771)	-0.4170(0.1767)	-0.4171(0.1768)	-0.4172(0.1769)	-0.4172(0.1769)
			M-H	-0.4055(0.1680)	-0.4029(0.1660)	-0.4034(0.1664)	-0.4039(0.1667)	-0.4041(0.1669)

Table 6: Biases and MSEs of the MLEs and BEs of α_1, α_2 and θ under Prior 1(the entries in parentheses are MSEs).

C. S.			BEs (SE)	BEs (LINEX)		BEs (GE)	
				$\omega = -0.5$	$\omega = 0.5$	$\rho = -0.5$	$\rho = 0.5$
R_1	α_1	Lindley	-1.2589(1.1205)	-0.9593(0.8247)	-1.0111(0.8440)	-1.0604(0.9300)	-1.1041(0.9674)
		Importance	-1.1289(1.0428)	-0.7405(0.6344)	-0.8273(0.7128)	-0.8439(0.7587)	-0.9850(0.9585)
		M-H	-0.9552(0.7871)	-0.7311(0.5352)	-0.7921(0.6871)	-0.8285(0.7436)	-0.8708(0.7747)
	α_2	Lindley	-0.1347(0.0794)	-0.1110(0.0512)	-0.1240(0.0556)	-0.1264(0.0632)	-0.1305(0.0730)
		Importance	0.1145(0.0188)	0.1041(0.0137)	0.1126(0.0160)	0.1139(0.0166)	0.1143(0.0177)
		M-H	0.0869(0.0160)	0.0753(0.0129)	0.0819(0.0149)	0.0832(0.0151)	0.0850(0.0156)
	θ	Lindley	1.0418(0.5610)	0.9765(0.5605)	0.9814(0.5606)	0.9977(0.5607)	1.0094(0.5607)
		Importance	-0.5586(0.3169)	-0.5293(0.3164)	-0.5348(0.3165)	-0.5378(0.3166)	-0.5473(0.3167)
		M-H	-0.5363(0.2910)	-0.5172(0.2836)	-0.5305(0.2847)	-0.5315(0.2859)	-0.5326(0.2870)
R_2	α_1	Lindley	-1.7315(1.3326)	-1.4479(1.2406)	-1.4754(1.2415)	1.5764(1.2443)	1.7276(1.2986)
		Importance	-1.7125(1.3155)	-1.0395(0.9172)	-1.0703(0.9850)	-1.0965(1.0178)	-1.1318(1.0620)
		M-H	-1.2687(1.1013)	-0.8925(0.7973)	-0.9423(0.8738)	-0.9814(0.9701)	-1.0189(0.9992)
	α_2	Lindley	0.1591(0.1335)	0.1199(0.0954)	0.1331(0.0963)	0.1387(0.0965)	0.1413(0.1261)
		Importance	0.1440(0.0214)	0.1088(0.0156)	0.1269(0.0183)	0.1355(0.0190)	0.1397(0.0201)
		M-H	0.0985(0.0183)	0.0850(0.0149)	0.0926(0.0171)	0.0941(0.0173)	0.0963(0.0178)
	θ	Lindley	1.2626(0.6562)	1.1476(0.6199)	1.2157(0.6294)	1.2196(0.6350)	1.2335(0.6425)
		Importance	-0.5615(0.3188)	-0.5611(0.3182)	-0.5612(0.3183)	-0.5612(0.3184)	-0.5613(0.3187)
		M-H	-0.5449(0.2996)	-0.5399(0.2942)	-0.5407(0.2951)	-0.5414(0.2959)	-0.5422(0.2967)
R_3	α_1	Lindley	-1.9507(1.6870)	-1.5933(1.3221)	-1.7276(1.4775)	-1.7459(1.5342)	-1.7616(1.6082)
		Importance	-1.8672(1.3662)	-1.2824(1.1321)	-1.3197(1.1485)	-1.3534(1.1631)	-1.4149(1.1893)
		M-H	-1.6128(1.2472)	-1.1025(1.0841)	-1.2163(1.0960)	-1.2742(1.1114)	-1.3767(1.1377)
	α_2	Lindley	-0.2473(0.1882)	-0.2230(0.1231)	-0.2249(0.1451)	-0.2252(0.1615)	-0.2283(0.1708)
		Importance	0.2273(0.0608)	0.2109(0.0531)	0.2184(0.0573)	0.2211(0.0578)	0.2242(0.0593)
		M-H	0.1558(0.0249)	0.1311(0.0177)	0.1446(0.0215)	0.1470(0.0221)	0.1514(0.0235)
	θ	Lindley	1.4046(0.8491)	1.3724(0.8003)	1.4049(0.8150)	1.4064(0.8235)	1.4275(0.8339)
		Importance	-0.5638(0.3201)	-0.5633(0.3196)	-0.5634(0.3197)	-0.5635(0.3198)	-0.5635(0.3199)
		M-H	-0.5498(0.3049)	-0.5458(0.3006)	-0.5464(0.3013)	-0.5470(0.3020)	-0.5476(0.3026)

Table 6: Continued (the entries in parentheses are MSEs).

C. S.			BEs (SE)	BEs (LINEX)		BEs (GE)	
				$\omega=-0.5$	$\omega=0.5$	$\rho=-0.5$	$\rho=0.5$
R_4	α_1	Lindley	-1.1537(1.0704)	-0.6858(0.6377)	-0.7257(0.6994)	-0.7595(0.7412)	-0.8125(0.7687)
		Importance	-0.9891(0.9186)	-0.6173(0.3998)	-0.6489(0.5071)	-0.6367(0.6092)	-0.6901(0.6460)
		M-H	-0.6449(0.5509)	-0.2120(0.1608)	-0.2170(0.2581)	-0.4191(0.3721)	-0.4227(0.4170)
	α_2	Lindley	-0.0767(0.0594)	-0.0466(0.0414)	-0.0606(0.0494)	-0.0659(0.0497)	-0.0762(0.0499)
		Importance	0.0631(0.0087)	0.0464(0.0065)	0.0558(0.0077)	0.0579(0.0079)	0.0605(0.0083)
		M-H	0.0449(0.0023)	0.0208(0.0007)	0.0344(0.0015)	0.0375(0.0017)	0.0412(0.0020)
	θ	Lindley	0.9225(0.5355)	0.6394(0.4661)	0.8977(0.5049)	0.9055(0.5145)	0.9101(0.5222)
		Importance	-0.3964(0.1602)	-0.3960(0.1598)	-0.3961(0.1599)	-0.3962(0.1600)	-0.3963(0.1601)
		M-H	0.3833(0.1507)	0.3810(0.1490)	0.3814(0.1493)	0.3819(0.1495)	0.3821(0.1496)
R_5	α_1	Lindley	-1.2763(1.1925)	-0.9969(0.9411)	-1.0241(0.9741)	-1.1079(0.9571)	-1.1410(1.0919)
		Importance	-1.2632(1.1237)	-0.8684(0.7944)	-0.9130(0.8753)	-0.9509(0.9281)	-0.9870(0.9800)
		M-H	-1.2099(1.0621)	-0.7549(0.7474)	-0.8342(0.8061)	-0.9047(0.8606)	-0.9608(0.8831)
	α_2	Lindley	-0.1121(0.0769)	0.0708(0.0485)	-0.0767(0.0548)	-0.0811(0.0599)	-0.0897(0.0678)
		Importance	0.0869(0.0091)	0.0609(0.0068)	0.0754(0.0080)	0.0784(0.0083)	0.0826(0.0087)
		M-H	0.0636(0.0078)	0.0461(0.0039)	0.0561(0.0059)	0.0582(0.0064)	0.0609(0.0070)
	θ	Lindley	1.0855(0.6118)	0.7880(0.4980)	1.0264(0.5479)	1.0633(0.5723)	1.0740(0.5848)
		Importance	-0.3969(0.1608)	-0.3966(0.1605)	-0.3967(0.1606)	-0.3967(0.1606)	-0.3968(0.1607)
		M-H	-0.3834(0.1513)	-0.3814(0.1480)	-0.3818(0.1494)	-0.3821(0.1497)	-0.3823(0.1498)
R_6	α_1	Lindley	-1.6394(1.2466)	-1.4162(1.0146)	-1.5608(1.0653)	-1.5981(1.1126)	-1.6201(1.1493)
		Importance	-1.3863(1.1772)	-0.9894(0.9648)	-1.0075(0.9944)	-1.0480(1.0235)	-1.1088(1.0528)
		M-H	-1.2819(1.1216)	-0.9875(0.9638)	-0.9994(0.9854)	-1.0472(1.0089)	-1.0896(1.0295)
	α_2	Lindley	-0.1542(0.1080)	-0.1105(0.0777)	-0.1273(0.0866)	-0.1304(0.0935)	-0.1431(0.0980)
		Importance	0.1392(0.0238)	0.1065(0.0123)	0.1158(0.0216)	0.1199(0.0221)	0.1353(0.0230)
		M-H	0.1231(0.0195)	0.0917(0.0105)	0.1080(0.0161)	0.1175(0.0168)	0.1203(0.0181)
	θ	Lindley	1.3641(0.7472)	1.2404(0.6148)	1.3111(0.7007)	1.3237(0.7032)	1.3463(0.7065)
		Importance	-0.4010(0.1639)	-0.4006(0.1635)	-0.4007(0.1636)	-0.4008(0.1637)	-0.4009(0.1638)
		M-H	-0.3885(0.1543)	-0.3869(0.1532)	-0.3872(0.1534)	-0.3875(0.1536)	-0.3877(0.1537)

Table 7: ALs and CPs of 95% approximate and Boot-t CIs of α_1, α_2 and λ when $m = 20$ and $n = 25$.

C. S.			Approx	Boot-t
R_1	α_1	AL	1.8292	1.6115
		CP	0.6285	0.6652
	α_2	AL	1.3666	0.7526
		CP	0.6126	0.6865
	θ	AL	1.5442	1.3756
		CP	0.6153	0.7111
R_2	α_1	AL	1.9264	1.7950
		CP	0.6616	0.7135
	α_2	AL	1.4377	0.9382
		CP	0.6567	0.7023
	θ	AL	1.7112	1.5641
		CP	0.6591	0.7112
R_3	α_1	AL	2.0057	1.8869
		CP	0.7180	0.7560
	α_2	AL	1.5039	1.0467
		CP	0.7126	0.7622
	θ	AL	1.8848	1.7137
		CP	0.7112	0.7153
R_4	α_1	AL	1.6714	1.4578
		CP	0.5395	0.6308
	α_2	AL	0.9944	0.7411
		CP	0.5472	0.5619
	θ	AL	0.9153	0.8792
		CP	0.5429	0.5486
R_5	α_1	AL	1.6867	1.5641
		CP	0.5468	0.6432
	α_2	AL	1.0007	0.7412
		CP	0.5407	0.5808
	θ	AL	1.2853	0.9249
		CP	0.5479	0.6301
R_6	α_1	AL	1.9288	1.8593
		CP	0.6180	0.6633
	α_2	AL	1.1787	0.8443
		CP	0.5623	0.5866
	θ	AL	1.2980	1.0277
		CP	0.6068	0.7057

Table 8: ALs and CPs of 95% CIs of α_1, α_2 and θ when $m = 20$ and $n = 25$ under Prior 0.

C. S.			HPD (Importance)					HPD M-H				
			SE	LINEX		GE		SE	LINEX		GE	
				$\omega=-0.5$	$\omega=0.5$	$\rho=-0.5$	$\rho=0.5$		$\omega=-0.5$	$\omega=0.5$	$\rho=-0.5$	$\rho=0.5$
R_1	α_1	AL	0.9935	0.7431	0.9156	0.9214	0.9808	0.1251	0.1097	0.1126	0.1191	0.1213
		CP	0.8424	0.8957	0.8941	0.8925	0.8908	0.9502	0.9721	0.9683	0.9603	0.9600
	α_2	AL	0.3610	0.3559	0.3566	0.3569	0.3588	0.0660	0.0620	0.0615	0.0642	0.0648
		CP	0.8955	0.9124	0.9107	0.9090	0.9074	0.9760	0.9808	0.9806	0.9804	0.9761
	θ	AL	0.1935	0.1909	0.1923	0.1926	0.1930	0.1722	0.1716	0.1718	0.1719	0.1720
		CP	0.8571	0.8633	0.8617	0.8602	0.8586	0.9603	0.9754	0.9640	0.9607	0.9605
R_2	α_1	AL	1.6631	1.1173	1.3498	1.3976	1.4986	0.1621	0.1150	0.1165	0.1215	0.1357
		CP	0.8775	0.9107	0.8957	0.8941	0.8909	0.9603	0.9807	0.9801	0.9800	0.9645
	α_2	AL	0.3692	0.3573	0.3672	0.3676	0.3684	0.0918	0.0880	0.0903	0.0904	0.0912
		CP	0.9000	0.9137	0.9113	0.9099	0.9085	0.9778	0.9813	0.9811	0.9807	0.9794
	θ	AL	0.2275	0.2250	0.2263	0.2264	0.2269	0.2077	0.2062	0.2069	0.2070	0.2074
		CP	0.8909	0.8992	0.8976	0.8960	0.8944	0.9720	0.9767	0.9725	0.9724	0.9723
R_3	α_1	AL	1.8401	1.1961	1.4714	1.4909	1.6857	0.1689	0.1215	0.1326	0.1364	0.1484
		CP	0.9090	0.9157	0.9140	0.9124	0.9107	0.9725	0.9847	0.9805	0.9801	0.9760
	α_2	AL	0.4195	0.4102	0.4166	0.4169	0.4171	0.1005	0.0967	0.0983	0.0986	0.0994
		CP	0.9074	0.9141	0.9124	0.9107	0.9090	0.9877	0.9887	0.9885	0.9883	0.9878
	θ	AL	0.2357	0.2330	0.2344	0.2345	0.2351	0.2298	0.2294	0.2295	0.2296	0.2297
		CP	0.8940	0.9160	0.9040	0.9040	0.8980	0.9800	0.9807	0.9805	0.9803	0.9802
R_4	α_1	AL	0.7672	0.5516	0.7318	0.7387	0.7540	0.1219	0.0556	0.0849	0.0950	0.1075
		CP	0.8243	0.8743	0.8727	0.8273	0.8258	0.8608	0.9107	0.8957	0.8941	0.8909
	α_2	AL	0.3579	0.3556	0.3563	0.3566	0.3576	0.0631	0.0607	0.0608	0.0619	0.0620
		CP	0.8752	0.8955	0.8921	0.8905	0.8888	0.9563	0.9606	0.9605	0.9604	0.9567
	θ	AL	0.1907	0.1887	0.1898	0.1899	0.1903	0.1692	0.1686	0.1687	0.1688	0.1689
		CP	0.8433	0.8617	0.8602	0.8586	0.8571	0.9600	0.9643	0.9607	0.9605	0.9603
R_5	α_1	AL	1.0998	0.8349	0.9685	0.9913	1.0835	0.1333	0.0972	0.1078	0.1131	0.1263
		CP	0.8608	0.8974	0.8941	0.8909	0.8792	0.9527	0.9730	0.9720	0.9680	0.9605
	α_2	AL	0.3672	0.3566	0.3599	0.3600	0.3641	0.0751	0.0699	0.0703	0.0721	0.0732
		CP	0.8888	0.8955	0.8938	0.8921	0.8905	0.9600	0.9685	0.9683	0.9608	0.9601
	θ	AL	0.2184	0.2156	0.2171	0.2172	0.2178	0.1775	0.1771	0.1772	0.1773	0.1774
		CP	0.8453	0.8639	0.8612	0.8592	0.8581	0.9680	0.9688	0.9687	0.9685	0.9683
R_6	α_1	AL	1.6783	0.8471	1.3516	1.3636	1.3867	0.1666	0.0867	0.1177	0.1218	0.1424
		CP	0.8909	0.9107	0.8957	0.8941	0.8925	0.9601	0.9807	0.9803	0.9685	0.9683
	α_2	AL	0.3732	0.3625	0.3669	0.3675	0.3700	0.0970	0.0946	0.0950	0.0952	0.0958
		CP	0.8895	0.9141	0.8938	0.8921	0.8905	0.9647	0.9752	0.9700	0.9681	0.9680
	θ	AL	0.2329	0.2300	0.2315	0.2316	0.2323	0.2101	0.2096	0.2098	0.2099	0.2100
		CP	0.8750	0.8812	0.8797	0.8781	0.8765	0.9760	0.9767	0.9765	0.9763	0.9763

Table 9: ALs and CPs of 95% CIs of α_1, α_2 and θ when $m = 20$ and $n = 25$ under Prior 1.

C. S.			HPD (Importance)					HPD M-H				
			SE	LINEX		GE		SE	LINEX		GE	
				$\omega = -0.5$	$\omega = 0.5$	$\rho = -0.5$	$\rho = 0.5$		$\omega = -0.5$	$\omega = 0.5$	$\rho = -0.5$	$\rho = 0.5$
R_1	α_1	AL	0.7496	0.6535	0.6581	0.6726	0.6972	0.0940	0.0710	0.0756	0.0791	0.0802
		CP	0.8544	0.8727	0.8547	0.8546	0.8545	0.9325	0.9447	0.9403	0.9401	0.9363
	α_2	AL	0.2748	0.2637	0.2709	0.2710	0.2727	0.0609	0.0574	0.0587	0.0590	0.0601
		CP	0.8703	0.8812	0.8752	0.8736	0.8719	0.9400	0.9408	0.9405	0.9403	0.9402
	θ	AL	0.1917	0.1896	0.1906	0.1909	0.1913	0.1692	0.1686	0.1687	0.1688	0.1689
		CP	0.8243	0.8453	0.8438	0.8407	0.8392	0.9400	0.9407	0.9405	0.9404	0.9403
R_2	α_1	AL	0.9228	0.7541	0.8461	0.8472	0.8792	0.0960	0.0762	0.0786	0.0798	0.0865
		CP	0.8545	0.8791	0.8775	0.8743	0.8727	0.9400	0.9607	0.9520	0.9487	0.9401
	α_2	AL	0.3319	0.3240	0.3300	0.3301	0.3307	0.0727	0.0704	0.0712	0.0715	0.0718
		CP	0.9378	0.9440	0.9438	0.9430	0.9320	0.9525	0.9601	0.9563	0.9560	0.9527
	θ	AL	0.2203	0.2182	0.2193	0.2194	0.2198	0.1904	0.1900	0.1901	0.1902	0.1903
		CP	0.8392	0.8453	0.8438	0.8422	0.8407	0.9400	0.9607	0.9407	0.9405	0.9404
R_3	α_1	AL	1.2695	0.9335	1.0856	1.0977	1.1733	0.0983	0.0800	0.0840	0.0846	0.0908
		CP	0.8576	0.8974	0.8775	0.8759	0.8743	0.9405	0.9752	0.9607	0.9502	0.9500
	α_2	AL	0.3843	0.3678	0.3809	0.3811	0.3821	0.0908	0.0867	0.0876	0.0880	0.0892
		CP	0.9498	0.9520	0.9510	0.9490	0.9480	0.9600	0.9752	0.9607	0.9605	0.9603
	θ	AL	0.2295	0.2266	0.2282	0.2283	0.2289	0.2080	0.2069	0.20757	0.2076	0.2078
		CP	0.8392	0.8633	0.8438	0.8422	0.8407	0.9600	0.9754	0.9605	0.9604	0.9603
R_4	α_1	AL	0.7136	0.6014	0.6575	0.6587	0.6971	0.0937	0.0636	0.0744	0.0770	0.0794
		CP	0.8000	0.8424	0.8226	0.8211	0.8014	0.9163	0.9440	0.9241	0.9240	0.9205
	α_2	AL	0.2670	0.2630	0.2646	0.2660	0.2661	0.0478	0.0458	0.0467	0.0468	0.0471
		CP	0.8703	0.8768	0.8752	0.8736	0.8719	0.9252	0.9407	0.9405	0.9403	0.9400
	θ	AL	0.1829	0.1802	0.1816	0.1820	0.1825	0.1675	0.1669	0.1670	0.1672	0.1674
		CP	0.7857	0.7913	0.7899	0.7885	0.7871	0.9004	0.9207	0.9205	0.9203	0.9200
R_5	α_1	AL	0.8861	0.7315	0.8155	0.8211	0.8653	0.0945	0.0723	0.0760	0.0775	0.0859
		CP	0.8196	0.8608	0.8409	0.8394	0.8363	0.9365	0.9561	0.9405	0.9403	0.9367
	α_2	AL	0.2927	0.2616	0.2742	0.2749	0.2845	0.0697	0.0674	0.0675	0.0676	0.0687
		CP	0.8518	0.8955	0.8752	0.8736	0.8534	0.9200	0.9407	0.9252	0.9206	0.9204
	θ	AL	0.1951	0.1922	0.1938	0.1941	0.1946	0.1721	0.1716	0.1718	0.1719	0.1720
		CP	0.8035	0.8038	0.8037	0.8036	0.8036	0.9200	0.9407	0.9205	0.9204	0.9203
R_6	α_1	AL	1.0759	0.8189	0.9089	0.9171	0.9836	0.0976	0.0744	0.0769	0.0794	0.0862
		CP	0.8545	0.8974	0.8775	0.8759	0.8743	0.9400	0.9607	0.9603	0.9405	0.9403
	α_2	AL	0.3414	0.3378	0.3378	0.3387	0.3411	0.0752	0.0708	0.0724	0.0727	0.0732
		CP	0.8703	0.8955	0.8782	0.8781	0.8750	0.9360	0.9443	0.9441	0.9407	0.9405
	θ	AL	0.2241	0.2221	0.2231	0.2232	0.2236	0.1958	0.1950	0.1955	0.1956	0.1957
		CP	0.8214	0.8273	0.8258	0.8243	0.8228	0.9400	0.9607	0.9405	0.94030	0.9403

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Conflicts of Interests

The authors declare that they have no conflicts of interests

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