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Extended Odd Weibull Pareto Distribution, Estimation and Applications

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Abstract: We introduce a new three-parameter continuous lifetime. It combines Pareto and Weibull distributions to formulate the extended odd Weibull Pareto distribution. This new distribution has many nice properties as it has a simple linear representation. We observe its hazard rate function, moments and moment generating function, in addition to mean residual and mean inactivity time. Different classical and Bayesian estimation methods are used to estimate the unknown parameters of extended odd Weibull Pareto distribution. Monte Carlo Markov chain method are used for numerical analysis, simulation is used to assess the use of estimation methods. Two real data examples are analyzed for illustrative purpose.

Keywords: Extended Odd Weibull, Pareto distribution, Monte Carlo Markov chain, Product Spacing, Weighted Least Square

1 Introduction

Many lifetime distributions were discussed in literature and many authors try to explore new distributions either by adding parameters to the original one or by combining two well known distribution together. The need of generating new life time distributions still essential as the amount of data available in nature has been growing increasingly, this urges statisticians to work more on distribution theory in order to better describe phenomenon or experiment under study and to predict more accurate future behaviors of the data based on an observed set of data. one of the main goals to establish new distributions is to provide more flexibility in modeling data under test. Although, several researchers worked on the field of statistical inference of the unknown parameters for several lifetime models, still, there is much space for new work on the generalizations of new models and using some classical and Bayesian inference for the new parameters under study.

Pareto distribution is a famous model, it was first studied by a professor of economics ”Vilfredo Pareto”. Many authors studied several forms of Pareto distribution and it was used extensively in many scientific applications such as actuarial sciences, finance, economic, life testing and climatology.

Recently, authors did several generalizations for Pareto distribution so that the new generalization is more flexible and posses good properties so it can be used to model more phenomenal data. For example, exponentiated Pareto by Stoppa [1], the beta-Pareto distribution by [2], the Kumaraswamy Pareto distribution by Bourguignon et al. [3], Weibull-Pareto distribution by Alzaatreh et al. [4], new Weibull-Pareto distribution by Nasiru and Luguterah [5] and Tahir et al. [6], the Marshall-Olkin Pareto distribution by Bdair and Haj Ahmad [7], Marshall-Olkin generalized Pareto by Haj Ahmad and Almetwally [8], Marshall-Olkin Alpha Power Pareto (MOAPP) by Almetwally and Haj Ahmad [9], and new four parameter distribution called new generalized Pareto distribution using transformation by Jayakumar et al. [10], and others.

Here we study a new model with three parameters, it is called extended odd Weibull Pareto (EOWP) distribution. In this paper we mainly work on two goals: (i) Study the main properties of the new model and compare it with other sub-models using real data examples. (ii) Find point and interval estimation for the EOWP parameters by using the three

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classical estimation methods and the Bayes estimation method. (ii) Test the efficiency of the estimation methods and obtain the best method of estimation that minimizes the mean square error (MSE) and Biases, for this purpose we used simulation analysis wih R-package.

The EOWP has several attractive properties that will be obtained throughout this paper and we summarize them as follows:

- The EOWP distribution is suitable of modeling constant, decreasing, increasing, upside down bathtub life times. Its density can be symmetric or right-skewed. Also its hazard rate can be decreasing, upside down or constant. Most distributions with these properties are much complicated.
- The EOWP is preferable to use for skewed data that may not be fitted properly by other distributions, also it can be used in many problems in applied areas, such as medicine, engineering, industrial reliability and survival analysis.
- Real data applications from medical and engineering fields prove that the EOWP model acts better than other related lifetime distributions which motivate its privilege in applied fields.
- The cumulative distribution function (CDF) and hazard rate function (HR) of EOWP have simple closed forms, hence it is useful to work with censored and complete samples as well.

The EOWP distribution is obtained based on the extended odd Weibull-G (EOW-G) family introduced by Alizadeh et al. [11]. We may also refer to Alzaatreh et al.[12] who introduced the basic T-X family and proved that it is well defined family of probability distributions. Let $\overline{G}(x;\theta) = 1 - G(x;\theta)$ and $g(x;\theta) = \frac{dG(x;\theta)}{dx}$ denote the survival function (S) and probability density function (PDF) of a baseline model with parameter vector θ respectively, so the CDF of the EOW-G family is given by:

$$F(x;\alpha,\beta,\theta) = 1 - \left\{1+\beta\left[\frac{G(x;\theta)}{\overline{G}(x;\theta)}\right]^{\alpha}\right\}^{\frac{-1}{\beta}}, x\in\mathbb{R}. \quad (1)$$

The corresponding PDF of (1) is defined by

$$f(x;\alpha,\beta,\theta) = \frac{\alpha\, g(x;\theta)\, G(x;\theta)^{\alpha-1}}{\overline{G}(x;\theta)^{\alpha+1}}\left\{1+\beta\left[\frac{G(x;\theta)}{\overline{G}(x;\theta)}\right]^{\alpha}\right\}^{\frac{-1}{\beta}-1}, x\in\mathbb{R}, \quad (2)$$

where α and β are positive shape parameters. The random variable with PDF (2) is denoted by X ∼EOW-G(α,β,θ).

Many authors have discussed different studies based on EOW-G family as: Afify and Mohamed [13] discussed EOW-exponential (EOWE) distribution, Almetwally [14] discussed EOW-inverse Rayleigh (EOWIR) distribution with application on carbon fibres, Alshenawy et al. [15] discussed progressive type-II censoring schemes of extended odd Weibull exponential (EOWE) distribution, and Almongy et al. [16] discussed EOW-Rayleigh (EOWR) distribution.

The rest of this paper is organized as follows. In Section 2, we define EOWP distribution. EOWP linear representation of its PDF is obtained in Section 3, along with some of its statistical properties. Five methods of point estimation are studied in Section 4. In Section 5, a simulation study is conducted in order to compare the performance of these estimation methods. Two real data sets from different life applications are used in section 6 to prove the efficiency of the EOWP distribution with respect to other distributions. Finally, conclusions are given in Section 7.

2 EOWP Distribution

The three-parameter EOWP distribution is a special model of EOW-G family with Pareto distribution as a baseline function. The Pareto distribution under consideration has PDF and CDF of the form $g(x;\delta) = \frac{\delta}{x^{\delta+1}}$ and $G(x;\delta) = 1 - \frac{1}{x^{\delta}}$, $x > 1$, $\delta > 0$. By substituting the CDF and PDF of the Pareto model in (1) and (2), we obtain the CDF and PDF of the EOWP distribution respectively as:

$$F(x;\alpha,\beta,\delta) = 1 - \left\{1+\beta\left[x^{\delta}-1\right]^{\alpha}\right\}^{\frac{-1}{\beta}}, x>1, \alpha,\beta,\delta>0. \quad (3)$$

$$f(x;\alpha,\beta,\delta) = \alpha\delta x^{\delta-1}\left(x^{\delta}-1\right)^{\alpha-1}\left[1+\beta\left(x^{\delta}-1\right)^{\alpha}\right]^{-\frac{1+\beta}{\beta}}, x>1, \alpha,\beta,\delta>0. \quad (4)$$

Therefore, a random variable with PDF (4) is denoted by X ∼EOWP(α,β,δ). The EOWP model reduces to the two parameter Weibull Pareto model when $\beta \to 0^{+}$. Another important characterization is (a) if $Y = \left[x^{\delta}-1\right]^{\alpha}$, then $F_Y(y) = 1-(1+\beta y)^{\frac{-1}{\beta}}$ and (b) if $Y = \left[x^{\delta}-1\right]$ then Y reduces to the Extended Weibull distribution.

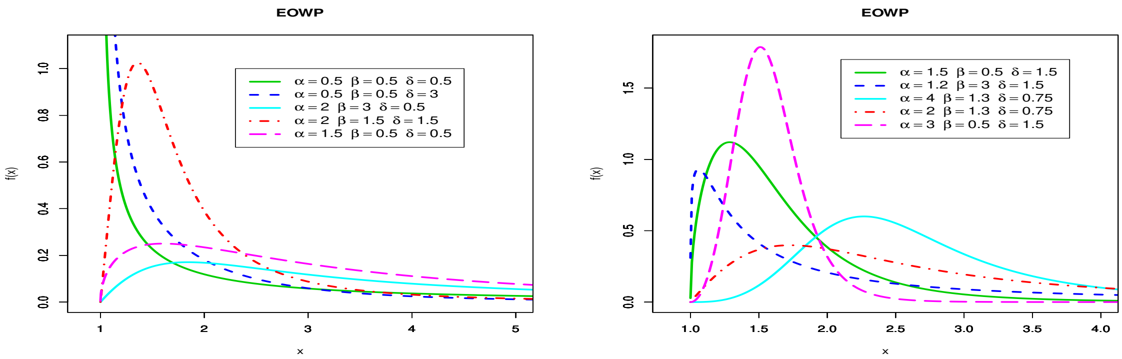

Fig. 1: Plots of the probability density function (PDF) of the EOWP distribution

The hazard rate function (HR) and the quantile function (Q) of the EOWP distribution are given by:

$$h(x;\alpha,\beta,\delta) = \frac{\alpha\delta x^{\delta-1}\left(x^{\delta}-1\right)^{\alpha-1}}{1+\beta\left(x^{\delta}-1\right)^{\alpha}}$$

and

$$Q(u) = \left(\left[\frac{1}{\beta}\left((1-u)^{-\beta}-1\right)\right]^{\frac{1}{\alpha}}+1\right)^{\frac{1}{\delta}},\ 0<u<1,$$

respectively. To check the monotonic behavior of the PDF of EOWP we find the derivative of the logarithmic PDF with respect to x then equate it to zero. Hence let

$$V = 1-\frac{1}{x^{\delta}}$$

and substitute it in (4) which will reduce to

$$f(x;\alpha,\beta,\delta) = \frac{\alpha V' V^{\alpha-1}}{(1-V)^{\alpha+1}}[1+\beta[\frac{V}{1-V}]^{\alpha}]^{-\frac{1}{\beta}-1}$$

. Now taking the derivative of the logarithmic function of the above formula we get:

$$\frac{\partial \log f(x;\alpha,\beta,\delta)}{\partial x} = \frac{V''}{V'}+(\alpha-1)\frac{V'}{V}+(\alpha+1)\frac{V'}{1-V}-\frac{\alpha\beta(\frac{1}{\beta}+1)V^{\alpha-1}V'}{(1-V)^{\alpha+1}+\beta V^{\alpha}(1-V)}$$

Since V is the CDF of the baseline function which is Pareto distribution, then it is clear that $V>0$ and $V'>0$ but $V''<0$. We will have two cases to discuss: Case(1) If $\alpha<1$, in this case the first term and last term in the above equation are both negative, while the second and third terms together will give a negative value. Hence the PDf of EOWP distribution is a decreasing function for this case. Case (2) If $\alpha>1$, the above derivative has a single root r_0, so the EOWP distribution is increasing when $x\in(0,r_0)$, but it is decreasing when $x\in(r_0,\infty)$. The above cases are illustrated in Figure (1). Figures 1 and 2 are different shapes of the PDF and HR of the EOWP distribution. These figures show that the PDF of the EOWP distribution can be right-skewed, symmetric or decreasing curves. The HR of the EOWP distribution has some important shapes, including, constant, decreasing, and upside down curve, which are attractive characteristics for any lifetime model. It can be noticed from the application section, that the EOWP distribution possesses great flexibility and can be used to model skewed data, hence widely applied in different areas such as biomedical studies, biology, reliability, physical engineering, and survival analysis.

3 Statistical Properties

In this section, we observe some statistical properties of the EOWP distribution namely, the linear representation of PDF, which is useful in finding the moments and moment generating function (MGF). Also we obtain the mean residual life and mean inactivity time.

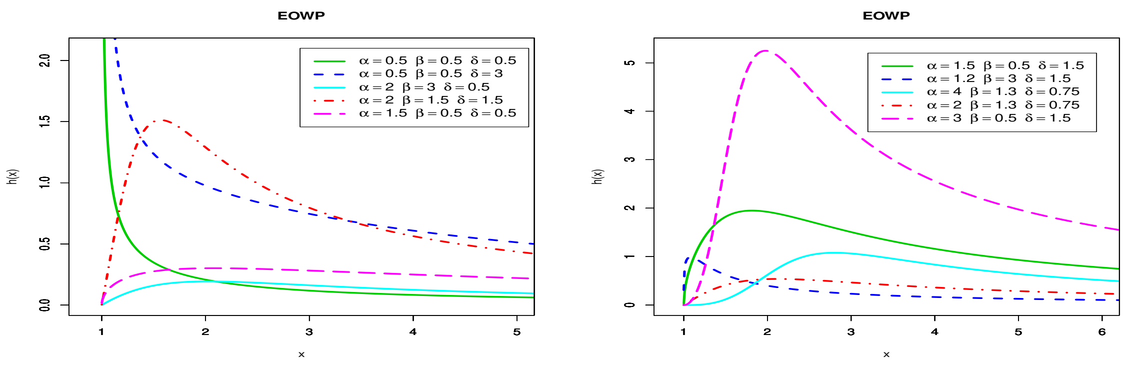

Fig. 2: Plots of the hazard rate function (HR) of the EOWP distribution

3.1 Linear Representation

Linear representation for the EOWP density using series techniques is useful for finding many statistical values and properties of the needed distribution. Alizadeh et al. [11] showed that EOW-G family has the following mixture representation of its density:

$$f(x) = \sum_{j,k=0}^{\infty} a_{j,k}\, h_{\alpha j+k}(x),$$

where $a_{j,k} = \frac{-\beta^j \Gamma(\alpha j+k)(-1/\beta)_j}{k!j!\Gamma(\alpha j)}$ and $h_{\alpha j+k}(x) = (\alpha j+k)\,G(x)^{\alpha j+k-1} g(x)$ is the Exponential-G density with positive power parameter $\alpha j+k$. Now substituting the PDF and CDF of the Pareto distribution, the above equation can be written as:

$$f(x) = \sum_{j,k=0}^{\infty} a_{j,k} \frac{\delta\,(\alpha j+k)}{x^{\delta+1}} \left(1-\frac{1}{x^{\delta}}\right)^{\alpha j+k-1}.$$

Applying the binomial expansion, the last equation reduces to:

$$f(x) = \sum_{j,k=0}^{\infty}\sum_{m=0}^{\infty} \binom{\alpha j+k-1}{m} (-1)^m\, a_{j,k} \frac{(\alpha j+k)}{1+m} \frac{\delta+m\delta}{x^{\delta+m\delta+1}}. \tag{5}$$

Equation (5) can be written as:

$$f(x) = \sum_{m=0}^{\infty} \upsilon_m\, g_{m+1}(x;\delta), \tag{6}$$

where

$$\upsilon_m = \sum_{j,k=0}^{\infty} \frac{(-1)^m a_{j,k}}{m+1} (\alpha j+k) \binom{\alpha j+k-1}{m}$$

and $g_{m+1}(x;\delta) = \frac{(m+1)\delta}{x^{(m+1)\delta+1}}$ denotes Pareto density with $(m+1)\delta$ as a scale parameter. Hence the PDF of EOWP can be expressed as a linear combination of Pareto distribution. Let X be a random variable having Pareto distribution with PDF $g(x;\delta) = \frac{\delta}{x^{\delta+1}}$, $x>1$, $\delta>0$. Then, the ith ordinary moment, incomplete moments, and MGF of X are

$$\mu'_{i,X} = \frac{\delta}{\delta-i},\ \delta > i, \tag{7}$$

$$\varphi_{i,X}(t) = \frac{\delta}{i-\delta} t^{i-\delta}, \delta > i$$

$$M_X(t) = \delta(-t^{\delta})\Gamma(-\delta,-t),\ t<0,$$

respectively, where $\Gamma(-\delta,-t) = \int_{-t}^{\infty} t^{-\delta-1} e^{-t} dt$ denotes the upper incomplete Gamma function.

Table 1: The numerical values of μ, σ^2, γ_1 and γ_2 for the EOWP distribution

	$\alpha=0.5,\beta=0.5$	$\alpha=2,\beta=0.5$	$\alpha=2,\beta=2$	$\alpha=0.5,\beta=2$	$\alpha=5,\beta=0.5$\$
μ	2.0943	1.4175	1.7836	30.1339	1.4028
σ^2	7.4962	0.0665	1.3889	212.9285	0.0105
γ_1	0.5766	0.1211	0.3407	0.7752	0.0149
γ_2	1.7284	0.3442	1.0935	4.3189	0.0482

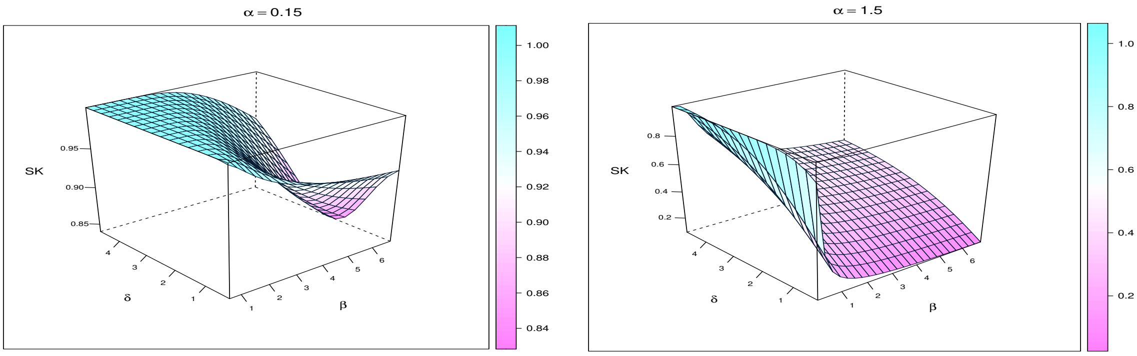

Fig. 3: Plots of skewness of EOWP distribution

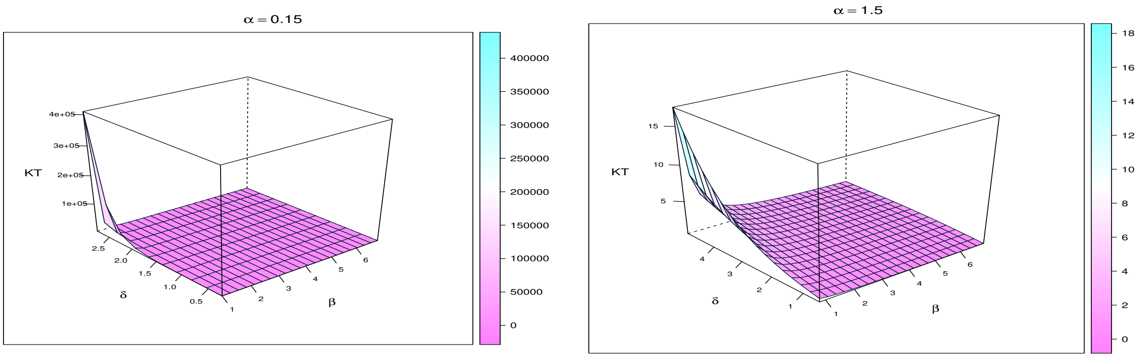

Fig. 4: Plots of kurtosis of EOWP distribution

3.2 Moments and Moment Generating Functions

The ith moment of X follows directly from Equations (6) and (7)

$$\mu_i = E(X^i) = \sum_{m=0}^{\infty} \upsilon_m \frac{(m+1)\delta}{(m+1)\delta - i}. \tag{8}$$

Table 1 lists some numerical values such as the first moment or the mean (μ), variance (σ^2), skewness (γ_1), and kurtosis (γ_2) of the EOWP density, with $\delta = 2$ and some suggested values of α and β. The values appear in Table 1 specifies that the skewness of the EOWP which is ranging between 0.0149 and 0.7752 indicate that the EOWP is either skewed to the right or nearly symmetry, while the dispersion of its kurtosis has much bigger ranging in the interval $(0.0482, 4.3189)$, so it is said that it is leptokurtic ($\gamma_2 > 0$). Therefore, the EOWP distribution can be used to model right skewed and symmetric data. Figures 3 and 4 show different shapes of skewness and kurtosis of EOWP distribution.

The ith incomplete moment of X can be obtained from Equations (6) and (7) as

$$\varphi_i(t) = \sum_{m=0}^{\infty} \upsilon_m \frac{(m+1)\delta}{i-(m+1)\delta} t^{i-(m+1)\delta}, (m+1)\delta > i.$$

For $i=1$ the first incomplete moment of X is

$$\varphi_1(t)=\sum_{m=0}^{\infty}\upsilon_m\frac{(m+1)\delta}{1-(m+1)\delta}t^{1-(m+1)\delta},(m+1)\delta>1. \tag{9}$$

Referring to Equation (6), the MGF of the EOWP distribution is given by:

$$M(t)=\sum_{m=0}^{\infty}\upsilon_m(m+1)\delta(-t^{(m+1)\delta})\Gamma(-(m+1)\delta,-t),\ t\neq 0.$$

3.3 Mean Residual Life and Mean Inactivity Time

The expected life of a unit at age t is called the mean residual life (MRL) which also represents the expected extra life length for a unit, which is still alive at age t and is defined as $m_X(t)=E(X-t\,|\,X>t)$, for $t>0$.

The MRL of X is

$$m_X(t)=[1-\varphi_1(t)]/S(t)-t, \tag{10}$$

where $\varphi_1(t)$ is given by (9) and $S(t)$ is the survival function of the EOWP distribution. Substituting Equation (9) in (10), we have

$$m_X(t)=\frac{1}{S(t)}[1-\sum_{m=0}^{\infty}\upsilon_m\frac{(m+1)\delta}{1-(m+1)\delta}t^{1-(m+1)\delta}]-t.$$

The mean inactivity time (MIT) function is important measure in reliability analysis and actuarial studies, it is usually known as the mean past lifetime or the mean waiting time functions. It is defined as

$$n_X(t)=\int_0^t\frac{F(x)}{F(t)}dx,t>0$$

$$=\int_0^t\frac{1-\left\{1+\beta\left[x^{\delta}-1\right]^{\alpha}\right\}^{\frac{-1}{\beta}}}{1-\left\{1+\beta\left[t^{\delta}-1\right]^{\alpha}\right\}^{\frac{-1}{\beta}}}dx$$

4 Parameter Estimation

In this section, we use different point estimation methods to estimate the unknown parameters of the EOWP. We use classical and non-classical (Bayes) methods. The classical methods are: maximum likelihood estimator (MLE), Least Square (LS), Weighted Least Square (WLS) and maximum product of spacing estimator (MPS). In addition to the non-classical method which is Bayesian estimation method. In the last few years, parameter estimation using different estimation methods got great attention by many authors such as Bdair and Haj Ahmad [7], Almetwally and Almongy [17], Haj Ahmad and Almetwally [8], Basheer et al. [18] and Afify and Mohamed [13].

4.1 Maximum Likelihood Method

Let $x_1,\cdots,x_n$ be a random sample from the EOWP distribution with parameters α,β, and δ. The likelihood function can be written as:

$$L(\Theta)=\alpha^n\delta^n\prod_{i=1}^{n}x_i^{\delta-1}\left(x_i^{\delta}-1\right)^{\alpha-1}\left[1+\beta\left(x_i^{\delta}-1\right)^{\alpha}\right]^{-\frac{1+\beta}{\beta}}, \tag{11}$$

The log-likelihood function is

$$\ell(\Theta)=n\log(\alpha)+n\log(\delta)+(\delta-1)\sum_{i=1}^{n}\log(x_i)+(\alpha-1)\sum_{i=1}^{m}\log(H_i)-\frac{\beta+1}{\beta}\sum_{i=1}^{n}\log(1+\beta H_i^{\alpha}), \tag{12}$$

where $H_i=(x_i^{\delta}-1)$ and $\Theta=(\alpha,\ \beta,\ \delta)$ is a vector of the EOWP parameters. The MLE are obtained by solving the following normal equations,

$$\frac{\partial \ell(\Theta)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(H_i) - (\beta+1)\sum_{i=1}^{n} \frac{H_i^{\alpha}\log(H_i)}{1+\beta H_i^{\alpha}} = 0,$$

$$\frac{\partial \ell(\Theta)}{\partial \delta} = \frac{n}{\delta} + \sum_{i=1}^{n} \log(x_i) + (\alpha-1)\sum_{i=1}^{n} \frac{x_i^{\alpha}\log(x_i)}{H_i} - \alpha(\beta+1)\sum_{i=1}^{n} \frac{H_i^{\alpha-1}x_i^{\delta}\log(x_i)}{1+\beta H_i^{\alpha}} = 0,$$

and

$$\frac{\partial \ell(\Theta)}{\partial \beta} = \frac{1}{\beta^2}\sum_{i=1}^{n} \log\left(1+\beta H_i^{\alpha}\right) - \frac{\beta+1}{\beta}\sum_{i=1}^{n} \frac{H_i^{\alpha}}{1+\beta H_i^{\alpha}} = 0.$$

These equations cannot be solved explicitly, hence a nonlinear optimization algorithm as Newton Raphson method is used.

4.2 Maximum Product Spacing

According to Cheng and Amin [19], the maximum product spacing method (MPS) is an efficient estimation method that proved to have some advantages with respect to other point estimation methods. So we use MPS in this section to have point estimation of the unknown parameters of EOWP distribution. This can be obtained by solving the normal equations resulted from taking partial derivatives of logarithm of product spacing function $G(\Theta)$ which is written as:

$$G(\Theta) = \left\{ \left(1 - \left\{1+\beta\,[H_1]^{\alpha}\right\}^{\frac{-1}{\beta}}\right)\left(1 - \left\{1+\beta\,[H_n]^{\alpha}\right\}^{\frac{-1}{\beta}}\right)\prod_{i=2}^{n}\left[\left\{1+\beta\,[H_{i-1}]^{\alpha}\right\}^{\frac{-1}{\beta}} + \left\{1+\beta\,[H_i]^{\alpha}\right\}^{\frac{-1}{\beta}}\right]\right\}^{\frac{1}{n+1}}.$$

and the logarithmic function is $G(\Theta)$

$$\log G(\Theta) \propto \log\left[1 - \left\{1+\beta\,[H_1]^{\alpha}\right\}^{\frac{-1}{\beta}}\right] + \log\left[1 - \left\{1+\beta\,[H_n]^{\alpha}\right\}^{\frac{-1}{\beta}}\right] + \sum_{i=2}^{n}\log\left[\left\{1+\beta\,[H_{i-1}]^{\alpha}\right\}^{\frac{-1}{\beta}} + \left\{1+\beta\,[H_i]^{\alpha}\right\}^{\frac{-1}{\beta}}\right] \tag{13}$$

The MPS estimators of Θ are obtained by differentiating the log-product equation (13) with respect to each parameter separately, then we solve the nonlinear system of equations found by using any iterative procedure techniques such as conjugate-gradient algorithms. This developed in last few year to estimation parameter of model under censoring scheme as Ng et al. [20], Basu et al. [21], Alotaibi et al. [22], Almetwally et al. [17,23], and El-Sherpieny et al. [24].

4.3 Least Square and Weighted Least Square Estimators

The least square (LS) and weighted least square (WLS) estimation methods were first introduced by Swain in 1988. This method is based on the ordered sample $y_1 < \cdots < y_n$ from the random sample $x_1,...,x_n$ with EOWP distribution, the least squares method are obtained by minimizing

$$LS(\Theta) = \sum_{i=1}^{n}\left(1 - \left\{1+\beta\left[y_i^{\delta}-1\right]^{\alpha}\right\}^{\frac{-1}{\beta}} - \frac{i}{n+1}\right)^2. \tag{14}$$

After differentiating equation (14) with respect to the parameters α, β and δ and equating them to zero, we get a system of normal equations that can be solved Numerically. Hence we obtain the LS of α, β and δ denoted by $\widehat{\alpha}_{LS}, \widehat{\beta}_{LS}$ and $\widehat{\delta}_{LS}$ respectively.

Similarly we can use the WLS procedure to estimate the parameters α, β and δ of the EOWP distribution.

$$WLS(\gamma) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left(1 - \left\{1+\beta\left[y_i^{\delta}-1\right]^{\alpha}\right\}^{\frac{-1}{\beta}} - \frac{i}{n+1}\right)^2, \tag{15}$$

Here we need to minimize (15) with respect to the parameters α, β and δ.

After differentiating Equation (15) with respect to parameters α, β and δ and equating them to zero we get the normal equations that are solved numerically, hence the WLS estimators of α, β and δ are $\widehat{\alpha}_{WLS}, \widehat{\beta}_{WLS}$ and $\widehat{\delta}_{WLS}$ respectively.

4.4 Bayesian estimation

Bayesian methods is a statistical inference that depends on the choice of the prior distribution and the loss function. In this method all parameters are considered as random variables with certain distribution called prior distribution. If prior information is not available which is usually the case, we need to select one. Since the selection of prior distribution plays an important role in estimation of the parameters, our choice for the priors are the independent gamma distributions. On the other hand, the loss function is important in Bayesian methods. Most of the Bayesian inference procedures are developed under the symmetric and asymmetric loss functions. One of the most common symmetric loss function is the squared error loss function. The independent joint prior density function of Θ can be written as follows:

$$\pi(\Theta) = \frac{b_1^{a_1}}{\Gamma(a_1)}\frac{b_2^{a_2}}{\Gamma(a_2)}\frac{b_3^{a_3}}{\Gamma(a_3)}\alpha^{a_1-1}\beta^{a_2-1}\delta^{a_3-1}e^{-(b_1\alpha+b_2\beta+b_3\delta)}. \tag{16}$$

The joint posterior density function of Θ is obtained from (11) and (16) as follows:

$$\pi(\Theta|\underline{x}) = \frac{\ell(\underline{x}|\Theta).\pi(\Theta)}{\int_{\Theta}\ell(\underline{x}|\Theta).\pi(\Theta)d\Theta}. \tag{17}$$

The Bayes estimators of Θ, say $(\widehat{\alpha}_B, \widehat{\beta}_B, \widehat{\delta}_B)$ based on squared error loss function is given by

$$\begin{aligned}\hat{p}_{B-SEL}(\alpha,\beta,\delta) &= E_{(\alpha,\beta,\delta|\underline{x})}[p(\alpha,\beta,\delta)] \\ &= \int_0^\infty\int_0^\infty\int_0^\infty p(\alpha,\beta,\delta)\times\pi(\Theta|\underline{x})d\alpha d\beta d\delta.\end{aligned} \tag{18}$$

For more details about the Bayesian estimation, see for example, El-Sherpieny et al. [25], Nassr et al. [26], El-Morshedy et al. [27], and Almetwally [28]. It is noticed that the integrals given by (18) can't be obtained explicitly. Because of that we use the Markov Chain Monte Carlo technique (MCMC) to find an approximate value of integrals in (18). Many of studies used MCMC technique such as, Almetwally et al. [23,29].

5 Simulation Analysis

In this section Monte-Carlo simulation procedure is performed for comparison between the classical estimation methods: MLE, LS,WLS, MPS and Bayesian estimation method under square error loss function based on MCMC, for estimating parameters of EOWP distribution in life time by R language. Monte-Carlo experiments are carried out based on data-generated 10000 random samples from EOWP distribution, where x has EOWP life time for different actual values of parameters and different sample sizes n:(25, 50, 100 and 200). We could define the best estimators methods as which minimizes the bias and mean squared error (MSE) of estimators.

Tables 2 and 3 summarizes the simulation results of point estimation methods proposed in this paper. We consider the bias and the MSE values in order to perform the needed comparison between different point estimation methods. The following remarks can be noted from these tables:

1. As sample size increases the biases and MSEs decrease .
2. In Table 2, within the classical methods of estimation we find that the WLS estimation performs better for estimating α, β and δ with respect to MSE, while MPS is preferable for estimating α and β with respect to bias. MLE is considered better for estimating δ with respect to bias.
3. To assess the performance between classical estimation methods and Bayesian method under SEL, we look for least value of bias and MSE through which we realize that Bayesian estimation performs better than all other classical ones for estimating α and β. For estimating δ, Bayesian method performs better than classical methods with respect to MSE, but MLE method proves to behave better with respect to bias.
4. In table 3, the best classical method for estimating α, β and δ is MPS with respect to bias and MSE, while when comparing with Bayesian method, Bayes estimation is superior to the proposed classical method.

Table 2: Bias and MSE of EOWP distribution for MLE, LS, WLS, MPS and Bayesian when $\alpha = 0.5$

$\alpha = 0.5$				MLE		LS		MPS		WLS		Bayesian	
β	δ	n		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.5	0.5	25	α	0.0328	0.0197	-0.0032	0.0193	-0.0161	0.0104	0.0084	0.0208	0.0280	0.0101
			β	0.1037	0.9125	-0.0325	0.0763	0.1847	0.3753	0.0520	0.2168	0.1210	0.1990
			δ	0.1239	0.2101	0.0597	0.0703	0.1220	0.1127	0.0848	0.0796	0.0099	0.0012
		50	α	0.0283	0.0136	0.0093	0.0058	0.0083	0.0096	0.0184	0.0075	0.0232	0.0060
			β	0.0971	0.5344	0.0170	0.0915	0.2968	0.4828	0.1143	0.1404	0.0652	0.0515
			δ	0.0917	0.1173	0.0549	0.0367	0.1692	0.1281	0.0851	0.0452	0.0188	0.0019
		100	α	0.0007	0.0046	0.0022	0.0038	-0.0079	0.0040	0.0065	0.0040	0.0093	0.0036
			β	-0.0517	0.3644	0.0177	0.0648	0.0402	0.1984	0.0466	0.1259	0.0467	0.0464
			δ	0.0038	0.0770	0.0191	0.0233	0.0272	0.0377	0.0301	0.0310	0.0088	0.0017
		200	α	0.0062	0.0024	0.0060	0.0022	-0.0011	0.0022	0.0058	0.0019	0.0069	0.0019
			β	0.0406	0.1544	0.0433	0.0592	0.0685	0.1178	0.0353	0.0746	0.0542	0.0391
			δ	0.0236	0.0281	0.0191	0.0155	0.0304	0.0209	0.0186	0.0178	0.0141	0.0024
	2	25	α	0.0088	0.0135	0.0192	0.0155	-0.0225	0.0107	0.0121	0.0153	0.0306	0.0095
			β	-0.3057	0.6527	-0.0213	0.1793	-0.1245	0.3658	0.0270	0.3274	0.1378	0.1432
			δ	-0.5326	0.7515	-0.1869	0.2843	-0.3373	0.4908	-0.1096	0.3488	-0.0022	0.0003
		50	α	0.0123	0.0117	0.0019	0.0097	-0.0169	0.0105	-0.0015	0.0095	0.0242	0.0087
			β	-0.0214	0.6066	-0.0082	0.1184	-0.0298	0.2330	-0.0763	0.1873	0.1537	0.1355
			δ	-0.0315	0.7086	-0.0453	0.2246	-0.0824	0.3717	-0.1121	0.3357	0.0057	0.0002
		100	α	-0.0030	0.0038	-0.0032	0.0031	-0.0110	0.0041	0.0012	0.0029	0.0026	0.0031
			β	-0.0816	0.1540	0.0420	0.0444	-0.0171	0.1149	0.0314	0.0426	0.0157	0.0289
			δ	-0.0647	0.3922	0.0612	0.1552	-0.0394	0.2101	0.0688	0.1852	-0.0004	0.0003
		200	α	0.0002	0.0016	0.0043	0.0018	-0.0021	0.0016	0.0057	0.0017	0.0073	0.0018
			β	-0.0571	0.1004	0.0003	0.0299	0.0127	0.0456	0.0133	0.0610	0.0268	0.0167
			δ	-0.0712	0.2781	-0.0189	0.1049	0.0011	0.1102	0.0127	0.1937	0.0007	0.0003
2	0.5	25	α	0.0295	0.0206	0.0063	0.0262	-0.0399	0.0155	0.0058	0.0219	0.0326	0.0120
			β	0.1981	1.6083	-0.1150	0.1846	-0.0192	0.3655	0.0048	0.7132	0.1612	0.1205
			δ	0.0829	0.1319	0.0190	0.0549	0.0395	0.0693	0.0406	0.0673	0.0117	0.0009
		50	α	0.0210	0.0079	0.0076	0.0097	-0.0196	0.0065	0.0094	0.0084	0.0284	0.0067
			β	0.1271	0.4311	0.0051	0.1791	0.0076	0.1824	-0.0092	0.2403	0.1805	0.1317
			δ	0.0410	0.0402	0.0147	0.0242	0.0188	0.0291	0.0170	0.0441	0.0140	0.0010
		100	α	0.0103	0.0042	-0.0028	0.0050	-0.0122	0.0035	0.0008	0.0042	0.0157	0.0037
			β	0.1082	0.4131	-0.0388	0.1621	0.0065	0.0850	0.0288	0.2566	0.1161	0.0907
			δ	0.0370	0.0380	0.0135	0.0150	0.0155	0.0128	0.0295	0.0255	0.0160	0.0011
		200	α	0.0044	0.0019	-0.0027	0.0027	-0.0087	0.0017	-0.0010	0.0022	0.0083	0.0019
			β	0.0956	0.3260	-0.0529	0.0411	-0.0069	0.0371	-0.0207	0.1752	0.1054	0.0690
			δ	0.0268	0.0189	-0.0025	0.0051	0.0033	0.0051	0.0033	0.0101	0.0157	0.0011
	2	25	α	0.0487	0.0386	0.0182	0.0391	-0.0246	0.0173	0.0177	0.0340	0.0473	0.0141
			β	0.1944	2.5057	-0.0897	0.3576	-0.0925	0.3584	0.0015	1.1841	0.1737	0.1799
			δ	0.1970	1.8015	-0.0178	0.2679	-0.0275	0.3381	0.0728	0.6695	0.0029	0.0001
		50	α	0.0232	0.0093	0.0038	0.0126	-0.0162	0.0068	0.0094	0.0113	0.0292	0.0065
			β	0.1548	1.3779	-0.0553	0.2253	-0.0394	0.1625	0.1637	1.4828	0.1552	0.1469
			δ	0.1083	0.8110	-0.0319	0.1449	-0.0505	0.1567	0.1399	0.7692	0.0031	0.0001
		100	α	0.0066	0.0039	0.0024	0.0062	-0.0145	0.0035	0.0065	0.0055	0.0166	0.0035
			β	0.0626	0.4919	-0.0232	0.2072	-0.0373	0.0949	0.1490	0.8744	0.1132	0.1175
			δ	0.0841	0.5531	-0.0242	0.1371	-0.0228	0.0899	0.1260	0.5653	0.0036	0.0002
		200	α	0.0013	0.0022	-0.0033	0.0032	-0.0103	0.0019	-0.0009	0.0024	0.0066	0.0019
			β	0.0389	0.3805	0.0144	0.2395	-0.0187	0.0561	0.0412	0.2968	0.0652	0.0632
			δ	0.0424	0.2818	0.0234	0.1318	-0.0111	0.0515	0.0490	0.2347	0.0046	0.0001

Table 3: Bias and MSE of EOWP distribution for MLE, LS, WLS, MPS and Bayesian when $\alpha = 1.5$

$\alpha = 1.5$				MLE		LS		MPS		WLS		Bayesian	
β	δ	n		Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
0.5	0.5	25	α	0.2334	0.7820	-0.0113	0.0789	0.0428	0.2166	0.0586	0.2475	0.0340	0.0380
			β	0.2315	1.9521	0.0797	0.1047	0.2334	0.5406	0.1670	0.5606	0.0965	0.1120
			δ	0.0179	0.0243	0.0167	0.0093	0.0282	0.0153	0.0219	0.0118	0.0065	0.0015
		50	α	0.0476	0.0957	0.0084	0.0408	0.0046	0.0853	0.0305	0.0683	0.0532	0.0357
			β	-0.0207	0.2495	0.0252	0.0630	0.0993	0.2555	0.0452	0.1610	0.1087	0.1156
			δ	-0.0024	0.0081	0.0076	0.0043	0.0139	0.0080	0.0083	0.0052	0.0119	0.0019
		100	α	0.0196	0.0422	0.0048	0.0264	-0.0062	0.0377	0.0072	0.0278	0.0243	0.0239
			β	-0.0063	0.1015	0.0200	0.0503	0.0607	0.1011	0.0130	0.0546	0.0658	0.0598
			δ	-0.0028	0.0033	0.0015	0.0023	0.0067	0.0033	0.0006	0.0023	0.0072	0.0016
		200	α	0.0025	0.0173	-0.0029	0.0100	0.0005	0.0192	-0.0046	0.0129	0.0148	0.0171
			β	-0.0136	0.0416	0.0144	0.0134	0.0447	0.0479	-0.0060	0.0199	0.0420	0.0436
			δ	-0.0033	0.0014	0.0011	0.0009	0.0048	0.0014	-0.0017	0.0009	0.0041	0.0010
	2	25	α	0.1527	0.2911	0.1456	0.1745	-0.0583	0.1257	0.2323	0.2644	0.1839	0.0791
			β	0.0471	1.3203	0.0033	1.1758	-0.0510	0.8935	0.2014	1.4017	0.4021	0.7364
			δ	-0.1396	0.1568	-0.1422	0.1396	-0.1487	0.1150	-0.0706	0.1349	0.0036	0.0035
		50	α	-0.0963	0.0934	0.0188	0.1030	-0.1343	0.1071	-0.0643	0.0707	-0.0740	0.0475
			β	-0.1906	0.1349	0.1785	0.3095	-0.0685	0.1121	-0.0115	0.0982	-0.0126	0.0495
			δ	-0.1787	0.0996	0.0386	0.1240	-0.0820	0.0740	-0.0475	0.0790	-0.0096	0.0025
		100	α	-0.0152	0.0407	0.0098	0.0481	-0.0354	0.0374	0.0016	0.0418	0.0145	0.0214
			β	-0.0439	0.1138	0.0611	0.1791	0.0298	0.1069	0.0233	0.1238	0.0536	0.0463
			δ	-0.0329	0.0463	0.0207	0.0598	0.0091	0.0427	0.0038	0.0451	0.0043	0.0020
		200	α	0.0091	0.0207	0.0079	0.0327	0.0005	0.0193	0.0114	0.0250	0.0146	0.0143
			β	-0.0064	0.0453	0.0003	0.0870	0.0420	0.0448	0.0079	0.0625	0.0263	0.0235
			δ	-0.0079	0.0216	-0.0077	0.0312	0.0192	0.0212	-0.0017	0.0261	0.0019	0.0012
2	0.5	25	α	0.2156	1.1127	-0.0233	0.0921	-0.0825	0.1335	-0.0025	0.0979	0.0537	0.0313
			β	0.3688	7.9755	0.0173	0.1811	-0.0200	0.9846	0.0248	0.4031	0.1559	0.1116
			δ	0.0238	0.0450	0.0153	0.0162	-0.0034	0.0231	0.0139	0.0189	0.0129	0.0014
		50	α	0.1564	0.4076	0.0072	0.0606	-0.0534	0.0724	0.0228	0.0545	0.0533	0.0270
			β	0.4448	6.5051	0.0534	0.1988	0.0143	0.5612	0.0884	0.3485	0.1825	0.1108
			δ	0.0288	0.0245	0.0096	0.0083	-0.0048	0.0114	0.0125	0.0103	0.0130	0.0013
		100	α	0.0243	0.0500	0.0064	0.0257	-0.0465	0.0437	0.0231	0.0284	0.0382	0.0203
			β	-0.0047	0.4007	0.0266	0.0869	-0.0540	0.2953	0.0565	0.1384	0.1344	0.0951
			δ	0.0020	0.0073	0.0087	0.0042	-0.0054	0.0056	0.0107	0.0046	0.0126	0.0012
		200	α	0.0392	0.0322	0.0126	0.0180	-0.0182	0.0212	0.0179	0.0157	0.0429	0.0105
			β	0.0847	0.3409	0.0343	0.0966	-0.0119	0.1690	0.0359	0.1108	0.1407	0.0865
			δ	0.0098	0.0055	0.0066	0.0025	-0.0012	0.0032	0.0064	0.0028	0.0116	0.0011
	2	25	α	0.5949	2.1576	-0.0168	0.1433	-0.0468	0.2002	0.0973	0.2960	0.0617	0.0339
			β	1.7423	21.0769	-0.0272	0.6616	0.0907	1.5038	0.3816	2.9921	0.1654	0.1418
			δ	0.3133	1.4757	-0.0104	0.2856	-0.0379	0.3651	0.1039	0.6139	0.0092	0.0021
		50	α	0.1329	0.2930	-0.0037	0.0861	-0.0710	0.0975	0.0547	0.1602	0.0548	0.0259
			β	0.3364	3.0759	0.0362	0.6843	-0.0553	0.7690	0.2128	1.5510	0.1366	0.1200
			δ	0.0726	0.4384	0.0251	0.2596	-0.0526	0.2164	0.0616	0.3511	0.0046	0.0008
		100	α	0.1341	0.1930	0.0396	0.0636	-0.0161	0.0533	0.0619	0.0859	0.0358	0.0213
			β	0.4231	2.4133	0.1289	0.5425	0.0553	0.4511	0.2054	0.8652	0.0924	0.0838
			δ	0.1243	0.2950	0.0500	0.1345	0.0012	0.1036	0.0696	0.1731	0.0045	0.0004
		200	α	0.0289	0.0457	0.0207	0.0294	-0.0357	0.0265	0.0029	0.0347	0.0171	0.0128
			β	0.1092	0.5297	0.1133	0.2883	-0.0190	0.2407	0.0424	0.3591	0.0689	0.0592
			δ	0.0426	0.1102	0.0477	0.0695	-0.0117	0.0619	0.0195	0.0825	0.0076	0.0005

6 Applications to Engineering and Medical Data

In this section, two real data examples from physical engineering and medical sciences are given to test the goodness of the EOWP distribution. The EOWP model is compared with other related models such as, Pareto, Marshall-Olkin alpha power Pareto (MOAPP) [9] and Marshall-Olkin Pareto (MOP) distributions [7]. Tables 4 and 5 provide values of log likelihood (ℓ), Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion

(BIC), Hannan-Quinn information criterion (HQIC) and Kolmogorov-Smirnov (KS) statistic along with its P-value for all models fitted based on two real data sets.

The medical data set consists of the relief times of 20 patients receiving an analgesic which was studied by Nadarajah et al. [30]. The data are as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, and 2.0.

Table 4: MLE estimates, LL, AIC, CAIC, BIC HQIC, K-S and P-values for medical data

model	α	β	δ	LL	AIC	CAIC	BIC	HQIC	KS	p-value
EOWP	2.0071	0.6977	1.2250	15.2681	36.5362	38.0362	39.5234	37.1193	0.0963	0.9925
	0.6433	0.7362	0.2354							
EOWEx	7.8084	3.4516	0.4443	15.4847	36.9694	38.4694	39.9566	37.5525	0.0912	0.9963
	3.3650	2.3087	0.0427							
Pareto			1.6971	21.2071	44.4143	44.6365	45.4100	44.6087	0.2851	0.0775
			0.3795							
MOAPP	14.7117	4.7186	4.9261	15.4312	36.8624	38.3624	39.8496	37.4455	0.1008	0.9872
	50.5351	8.2701	1.4283							
MOP	8654.61	0.1022	0.1281	17.1729	40.3457	41.8457	43.3329	40.9288	0.1134	0.9592
	15983.96	0.0519	0.0494							

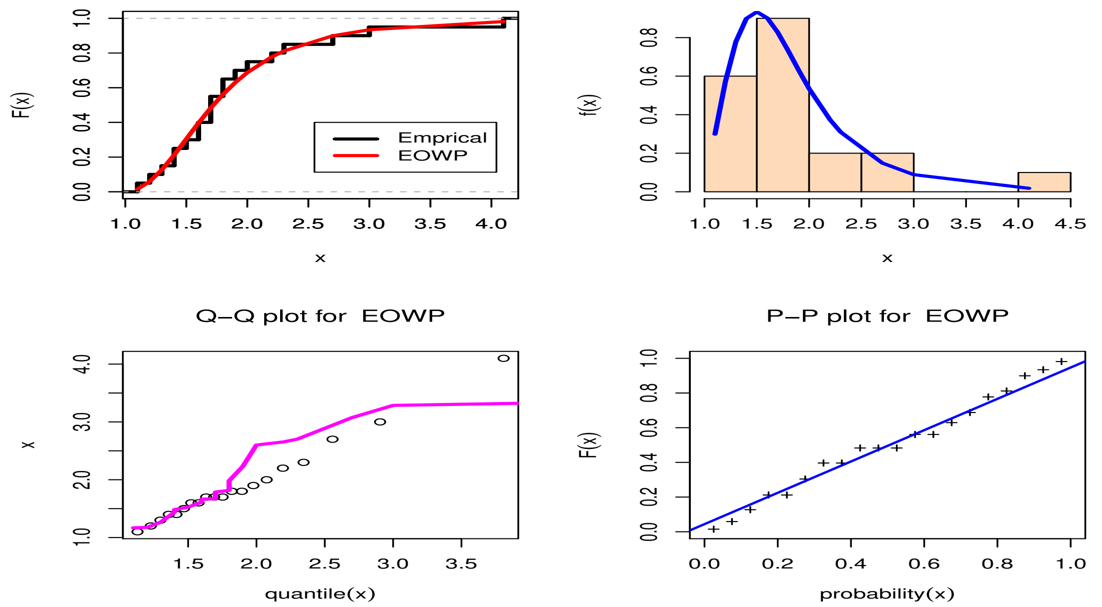

Fig. 5: Estimated PDF, CDF, PP-plot and QQ-plot of EOWP for medical data

The second real data set consider physical data, these refer to the fatigue 101 times of 6061-T6 aluminum coupons with maximum stress per cycle 26,000 psi are mentioned by Birnbaum and Saunders [31]. The data are given below: Physical data Set: 370, 706, 716, 746, 785, 797, 844, 855, 858, 886, 886, 930, 960 ,988, 990 ,1000, 1010, 1016, 1018, 1020, 1055, 1085, 1102, 1102, 1108, 1115 ,1120, 1134, 1140, 1199, 1200, 1200, 1203 ,1222, 1235, 1238, 1252, 1258, 1262, 1269,1270, 1290, 1293, 1300, 1310, 1313, 1315, 1330, 1355, 1390, 1416, 1419, 1420, 1420, 1450, 1452, 1475, 1478, 1481, 1485, 1502, 1505, 1513, 1522, 1522, 1530, 1540, 1560, 1567, 1578, 1594, 1602, 1604, 1608, 1630, 1642, 1674, 1730, 1750, 1750,1763, 1768, 1781, 1782, 1792, 1820, 1868, 1881, 1890, 1893, 1895, 1910, 1923, 1940 ,1945, 2023, 2100, 2130, 2215, 2268 and 2440.

Table 5: MLE estimates, MLE estimates, LL, AIC, CAIC, BIC HQIC, K-S and P-values for physical data

model	α	β	δ	LL	AIC	CAIC	BIC	HQIC	KS	p-value
EOWP	21.742	0.0748	0.0946	745.672	1497.345	1497.592	1505.190	1500.521	0.0498	0.9636
	3.092	0.2219	0.0006							
EOWEx	1.132	194.808	0.1230	835.466	1676.932	1677.179	1684.777	1680.108	0.4078	0.0000
	10.441	98.676	1.1338							
Pareto			0.1388	1027.825	2057.650	2057.690	2060.265	2058.708	0.5879	0.0000
			0.0138							
MOAPP	471.06	1756.66	1.3189	844.493	1694.986	1695.234	1702.832	1698.162	0.3409	0.0000
	1380.4	946.85	0.1335							
MOP	750.83	194.16	0.0232	747.079	1500.158	1500.405	1508.003	1503.334	0.0543	0.9266
	1390.0	108.66	0.0761							

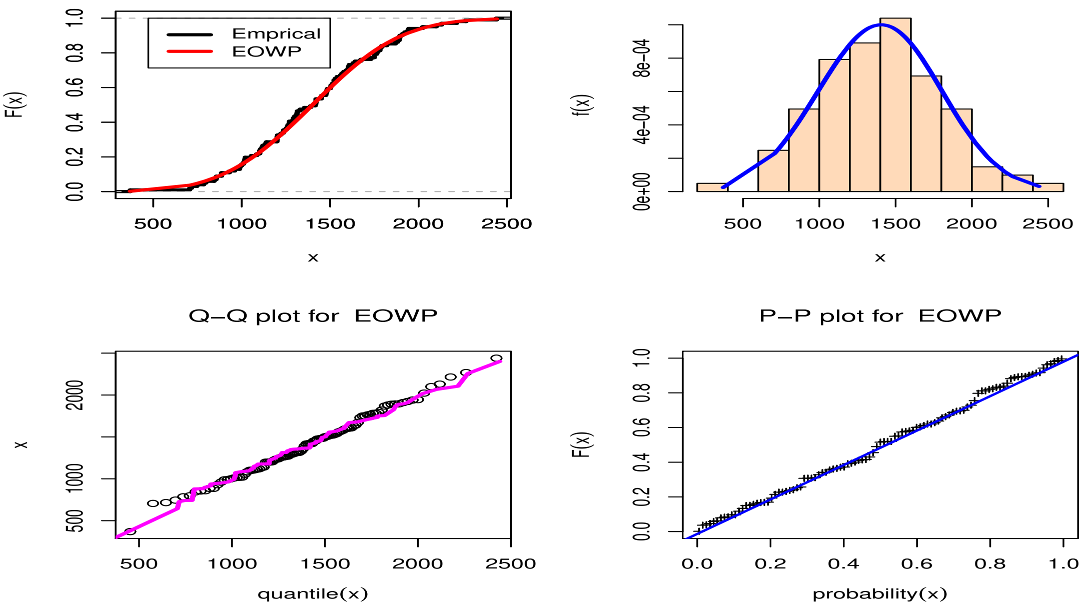

Fig. 6: Estimated PDF, CDF, PP-plot and QQ-plot of EOWP for physical data

From Tables 4 and 5 it is obvious that EOWP distribution has minimum values of all information criteria compared with other distributions. Also the P-value for KS has its highest value when the life time is EOWP distribution. This leads us to conclude that EOWP better fit the two real sets of data. Q-Q and P-P plots shown in figures 5 and 6, indicate that our distribution is a good choice for modeling the above real data.

Furthermore, the suggested methods of estimation (see Section 4) for the EOWP parameters are considered based on the previous data. Tables 6 and 7 displays different estimates of the EOWP parameters for the two data sets. In these data, we cannot use MPS because there are equal observations in the data, so the spacing will be zero, hence the product will also be zero and to compute the MPS estimator, we take the log of product spacing, as we know the log(0) equal -Inf. Despite the effectiveness of the MPS method in the estimate, but this problem hinders their use in the estimation process. So we add 0.00001 to equal observations, to solve this problem. The convergence of MCMC estimation of α, β and δ can be showed in Figures 7, 8 for the two data examples, respectively.

Table 6: Different estimators for EOWP parameters with respect to medical real data set

		α	β	δ
MLE	estimate	2.00710	0.69774	1.22503
	S.E	0.64326	0.73624	0.23538
LS	estimate	2.15732	0.99552	1.28241
	S.E	5.68367	8.13187	1.55593
MPS	estimate	1.33010	0.39325	1.07357
	S.E	2.23690	3.96574	1.60879
WLS	estimate	1.98972	0.92162	1.26866
	S.E	0.39305	0.56513	0.12520
Bayesian	estimate	2.072872	1.026373	1.291105
	S.E	0.35847	0.565325	0.123684

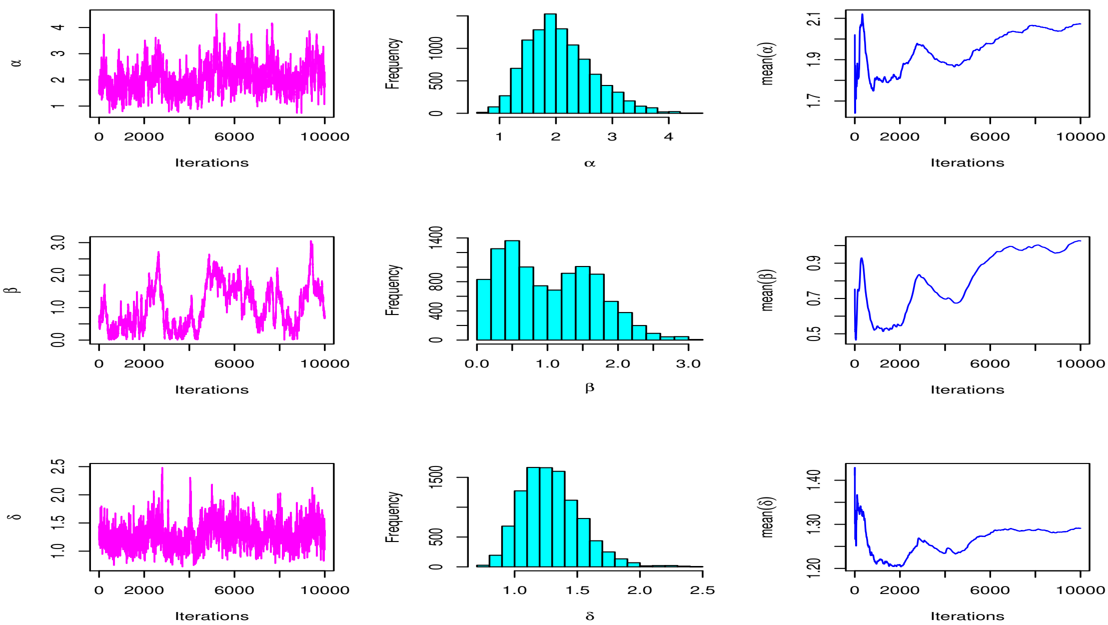

Fig. 7: Convergence of MCMC estimation for medical data

Table 7: Different estimators for EOWP parameters with respect to physical data set

		α	β	δ
MLE	estimate	21.74180	0.07480	0.09460
	S.E	3.09180	0.22190	0.00060
LS	estimate	21.86567	0.18761	0.09485
	S.E	0.09220	0.93604	0.00170
MPS	estimate	21.72633	0.15677	0.09470
	S.E	0.07646	1.30353	0.00511
WLS	estimate	23.09613	0.25338	0.09499
	S.E	0.00244	0.02940	0.00064
Bayesian	estimate	21.75317	0.070017	0.09455
	S.E	0.071072	0.026161	0.000432

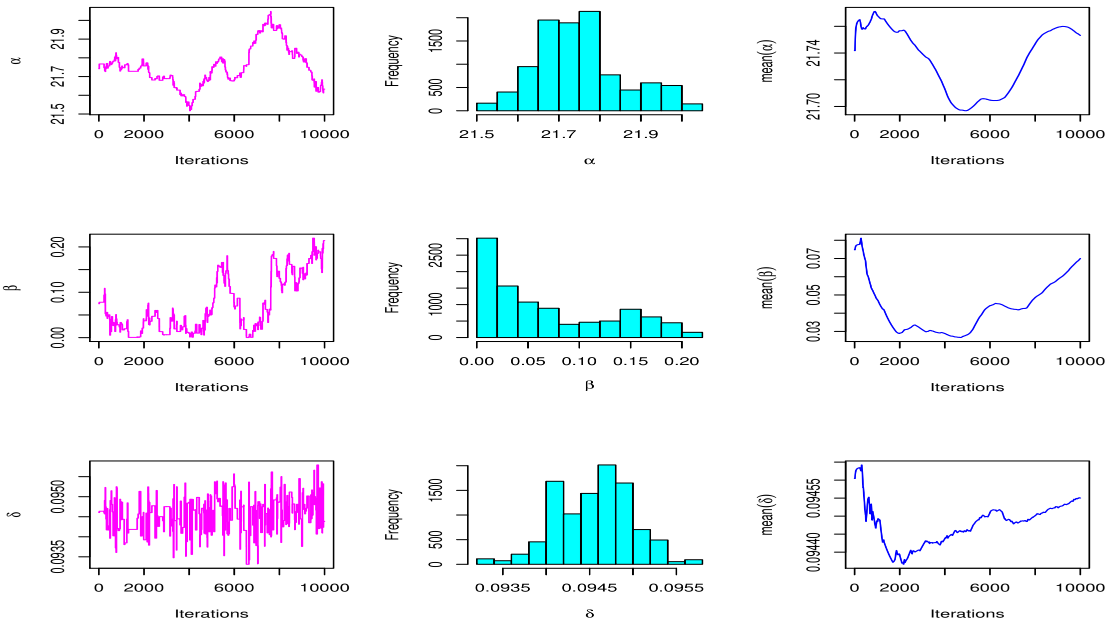

Fig. 8: Convergence of MCMC estimation for physical data

7 Conclusion

In this paper we formulate a new generalization of Pareto and Weibull distributions which is called EOWP distribution. We studied its statistical properties and obtained a linear representation for its pdf which was efficient in finding moments, moment generating function, mean residual and others. Different classical and Bayes estimation methods were considered to find point estimation of EOWP unknown parameters α, β and δ. A comparison was conducted via simulation analysis using R package to distinguish the performance of different estimation method. MCMC method was used for that purpose, also real data sets were considered and they showed that EOWP model better fit the real data and superior compared with other competitive distributions. Still there is space for more work in this area, one can chose different baseline functions and apply different point and interval estimation method, then using numerical methods and simulation to assess the performance of new models and different estimation methods.

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Availability of data and material: The data is included in Section 6. Application of Real Data.

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