

Hájek-Rényi Type Inequalities and Strong Law of Large Numbers for NOD Sequences

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Abstract: In the paper, we get the precise results of Hájek-Rényi type inequalities for the partial sums of negatively orthant dependent sequences, which improve the results of Theorem 3.1 and Corollary 3.2 in Kim (2006). In addition, the Marcinkiewicz type strong law of large numbers is obtained. At last, the strong stability for weighted sums of negatively orthant dependent sequences is discussed.

Keywords: Hájek-Rényi inequality; negatively orthant dependent sequences; strong law of large numbers.

1 Introduction

We use the following notations. Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space. Denote $S_n \doteq \sum_{i=1}^n X_i$ and $T_n \doteq \sum_{i=1}^n (X_i - EX_i)$ for each $n \geq 1$ and $I(A)$ be the indicator function of the set A .

Hájek and Rényi (1955) proved the following important inequality. If $\{X_n, n \geq 1\}$ is a sequence of independent random variables with mean zero, and $\{b_n, n \geq 1\}$ is a nondecreasing sequence of positive real numbers, then for any $\varepsilon > 0$ and positive integer $m < n$,

$$P\left(\max_{m \leq k \leq n} \left| \frac{\sum_{i=1}^k X_i}{b_k} \right| \geq \varepsilon\right) \leq \varepsilon^{-2} \left(\sum_{j=1}^m \frac{EX_j^2}{b_j^2} + \sum_{j=m+1}^n \frac{EX_j^2}{b_j^2} \right) \quad (1.1)$$

In the paper, we will further study the Hájek-Rényi type inequality and the strong law of large numbers for negatively orthant dependent sequences.

Definition 1.1. A finite collection of random variables X_1, X_2, \dots, X_n is said to be negatively upper orthant dependent (NUOD), if for all real numbers x_1, x_2, \dots, x_n ,

$$P(X_i > x_i, i = 1, 2, \dots, n) \leq \prod_{i=1}^n P(X_i > x_i), \quad (1.2)$$

and negatively lower orthant dependent (NLOD), if for all real numbers x_1, x_2, \dots, x_n ,

$$P(X_i \leq x_i, i = 1, 2, \dots, n) \leq \prod_{i=1}^n P(X_i \leq x_i). \quad (1.3)$$

A finite collection of random variables X_1, X_2, \dots, X_n is said to be negatively orthant dependent (NOD) if they are both NUOD and NLOD.

An infinite sequence $\{X_n, n \geq 1\}$ is said to be NOD (NUOD Or NLOD), if every finite subcollection is NOD (NUOD Or NLOD).

Lemma 1.1 (cf. Bozorgnia et al., 1996). Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables, f_1, f_2, \dots be all nondecreasing (or all nonincreasing) functions, then $\{f_n(X_n), n \geq 1\}$ is still a sequence of NOD random variables.

Lemma 1.2 (cf. Kim, 2006). Let X_1, X_2, \dots, X_n be NOD random variables with $EX_n = 0$ and $EX_n^2 < \infty$ for all $n \geq 1$. Then we have

$$E\left(\sum_{i=1}^p X_{m+i}\right)^2 \leq \sum_{i=1}^p EX_{m+i}^2 \quad (1.4)$$

for all integers $m, p \geq 1$ and $m + p \leq n$. Moreover, we have

$$E\left(\max_{1 \leq k \leq n} \left| \sum_{i=1}^k X_i \right|^2\right) \leq (\log_3 n + 2)^2 \sum_{i=1}^n EX_i^2 \quad (1.5)$$

By Lemma 1.1 and Lemma 1.2, we can get the following Corollary.

Corollary 1.1 (Khintchine-Kolmogorov theorem). Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables. If

$$\sum_{n=1}^{\infty} \text{Var}(X_n) \log^2 n < \infty \quad (1.6)$$

then

$$\sum_{n=1}^{\infty} (X_n - EX_n) \text{ converges a.s..}$$

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2 Hájek-Rényi type inequalities for NOD sequences

In this section, we will give Hájek-Rényi type inequalities for NOD sequences, which improve the results of Kim (2006).

Theorem 2.1. Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables and $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive numbers. Then for any $\varepsilon > 0$ and any integer $n \geq 1$,

$$P \left\{ \max_{1 \leq k \leq n} \left| \frac{1}{b_k} \sum_{j=1}^k (X_j - EX_j) \right| \geq \varepsilon \right\} \leq \frac{4}{\varepsilon^2} (\log_3 n + 2)^2 \sum_{j=1}^n \frac{\text{Var}(X_j)}{b_j^2}. \tag{2.1}$$

Proof. Without loss of generality, we assume that $b_n \geq 1$ for all $n \geq 1$. Let $\alpha = \sqrt{2}$. For $i \geq 0$, define

$$A_i = \{1 \leq k \leq n : \alpha^i \leq b_k < \alpha^{i+1}\}.$$

For $A_i \neq \emptyset$, we let $v(i) = \max\{k : k \in A_i\}$ and t_n be the index of the last nonempty set A_i . Obviously, $A_i A_j = \emptyset$ if $i \neq j$ and $\sum_{i=0}^{t_n} A_i = \{1, 2, \dots, n\}$. It is easily seen that $\alpha^i \leq b_k \leq b_{v(i)} < \alpha^{i+1}$ if $k \in A_i$ and $\{X_n - EX_n, n \geq 1\}$ is also a sequence of NOD random variables by Lemma 1.1. By Markov's inequality and (1.6) in Lemma 1.2, we have

$$\begin{aligned} & P \left\{ \max_{1 \leq k \leq n} \left| \frac{1}{b_k} \sum_{j=1}^k (X_j - EX_j) \right| \geq \varepsilon \right\} \\ &= P \left\{ \max_{0 \leq i \leq t_n} \max_{k \in A_i} \left| \frac{1}{b_k} \sum_{j=1}^k (X_j - EX_j) \right| \geq \varepsilon \right\} \\ &\leq \sum_{i=0, A_i \neq \emptyset}^{t_n} P \left\{ \frac{1}{\alpha^i} \max_{1 \leq k \leq v(i)} \left| \sum_{j=1}^k (X_j - EX_j) \right| \geq \varepsilon \right\} \\ &\leq \frac{1}{\varepsilon^2} \sum_{i=0, A_i \neq \emptyset}^{t_n} \frac{1}{\alpha^{2i}} E \left\{ \max_{1 \leq k \leq v(i)} \left| \sum_{j=1}^k (X_j - EX_j) \right|^2 \right\} \\ &\leq \frac{1}{\varepsilon^2} \sum_{i=0, A_i \neq \emptyset}^{t_n} \frac{1}{\alpha^{2i}} (\log_3 v(i) + 2)^2 \sum_{j=1}^{v(i)} \text{Var}(X_j) \\ &\leq \frac{1}{\varepsilon^2} (\log_3 n + 2)^2 \sum_{j=1}^n \text{Var}(X_j) \sum_{i=0, A_i \neq \emptyset, v(i) \geq j} \frac{1}{\alpha^{2i}}. \tag{2.2} \end{aligned}$$

Now we estimate $\sum_{i=0, A_i \neq \emptyset, v(i) \geq j} \frac{1}{\alpha^{2i}}$.

Let $i_0 = \min\{i : A_i \neq \emptyset, v(i) \geq j\}$, then $b_j \leq b_{v(i_0)} < \alpha^{i_0+1}$ follows from the definition of $v(i)$. Therefore,

$$\begin{aligned} \sum_{i=0, A_i \neq \emptyset, v(i) \geq j} \frac{1}{\alpha^{2i}} &< \sum_{i=i_0}^{\infty} \frac{1}{\alpha^{2i}} = \frac{1}{1 - \frac{1}{\alpha^2}} \frac{1}{\alpha^{2i_0}} \\ &< \frac{\alpha^2}{1 - \frac{1}{\alpha^2}} \frac{1}{b_j^2} = \frac{4}{b_j^2}. \tag{2.3} \end{aligned}$$

Thus (2.1) follows from (2.2) and (2.3) immediately.

Theorem 2.2. Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables and $\{b_n, n \geq 1\}$ be a nondecreasing sequence of positive numbers. Then for any $\varepsilon > 0$ and any positive integers $m < n$,

$$P \left(\max_{m \leq k \leq n} \left| \frac{1}{b_k} \sum_{j=1}^k (X_j - EX_j) \right| \geq \varepsilon \right) \leq \frac{4}{\varepsilon^2} \left(\sum_{j=1}^m \frac{\text{Var}(X_j)}{b_m^2} + 4 [\log_3(n - m) + 2]^2 \sum_{j=m+1}^n \frac{\text{Var}(X_j)}{b_j^2} \right) \tag{2.4}$$

Proof. Observe that

$$\begin{aligned} \max_{m \leq k \leq n} \left| \frac{1}{b_k} \sum_{j=1}^k (X_j - EX_j) \right| &\leq \left| \frac{1}{b_m} \sum_{j=1}^m (X_j - EX_j) \right| \\ &+ \max_{m+1 \leq k \leq n} \left| \frac{1}{b_k} \sum_{j=m+1}^k (X_j - EX_j) \right|, \end{aligned}$$

thus

$$\begin{aligned} & P \left(\max_{m \leq k \leq n} \left| \frac{1}{b_k} \sum_{j=1}^k (X_j - EX_j) \right| \geq \varepsilon \right) \\ &\leq P \left(\left| \frac{1}{b_m} \sum_{j=1}^m (X_j - EX_j) \right| \geq \frac{\varepsilon}{2} \right) \\ &+ P \left(\max_{m+1 \leq k \leq n} \left| \frac{1}{b_k} \sum_{j=m+1}^k (X_j - EX_j) \right| \geq \frac{\varepsilon}{2} \right) \end{aligned}$$

By Markov's inequality, Lemma 1.2 and Theorem 2.1,(2.4) can be obtained.

3 Marcinkiewicz type strong law of large numbers for NOD sequences

Theorem 3.1. Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed NOD random variables with $\sum_{n=1}^{\infty} P(|X_1| \geq n^{1/p}) \log^2 n < \infty$ for $0 < p < 2$.

Assume that $EX_1 = 0$ if $1 \leq p < 2$, then

$$\frac{1}{n^{1/p}} \sum_{k=1}^n X_k \rightarrow 0 \text{ a.s., } n \rightarrow \infty. \tag{3.1}$$

Proof.

Denote $Y_n = -n^{1/p}I(X_n \leq -n^{1/p}) + X_n I(|X_n| < n^{1/p}) + n^{1/p}I(X_n \geq n^{1/p})$ then

$$\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P(|X_n| \geq n^{1/p}) < \infty,$$

which implies that $P(X_n \neq Y_n, i.o.) = 0$ by the Borel-Cantelli lemma. Thus $\frac{1}{n^{1/p}} \sum_{k=1}^n X_k \rightarrow 0$ a.s. if and

only if $\frac{1}{n^{1/p}} \sum_{k=1}^n Y_k \rightarrow 0$ a.s. So we need to show that

$$\frac{1}{n^{1/p}} \sum_{k=1}^n (Y_k - EY_k) \rightarrow 0 \text{ a.s.}, \quad n \rightarrow \infty, \quad (3.2)$$

and

$$\frac{1}{n^{1/p}} \sum_{k=1}^n EY_k \rightarrow 0, \quad n \rightarrow \infty. \quad (3.3)$$

By Corollary 1.1 and Kronecker's lemma, to prove (3.2), it suffices to show that

$$\sum_{n=1}^{\infty} \text{Var} \left(\frac{Y_n}{n^{1/p}} \right) \log^2 n < \infty. \quad (3.4)$$

In fact,

$$\begin{aligned} & \sum_{n=1}^{\infty} \text{Var} \left(\frac{Y_n}{n^{1/p}} \right) \log^2 n \leq \sum_{n=1}^{\infty} \frac{EY_n^2}{n^{2/p}} \log^2 n \\ & \leq C \sum_{n=1}^{\infty} P(|X_1| \geq n^{1/p}) \log^2 n + C \sum_{n=1}^{\infty} \frac{EX_1^2 I(|X_1| < n^{1/p})}{n^{2/p}} \log^2 n \\ & \leq C + C \sum_{n=1}^{\infty} \frac{\log^2 n}{n^{2/p}} \sum_{k=1}^n EX_1^2 I(k-1 \leq |X_1|^p < k) \\ & = C + C \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} \frac{\log^2 n}{n^{2/p}} E|X_1|^p |X_1|^{2-p} I(k-1 \leq |X_1|^p < k) \\ & \leq C + C \sum_{k=1}^{\infty} k^{1-2/p} E|X_1|^{pk(2-p)/p} I(k-1 \leq |X_1|^p < k) \\ & < \infty. \end{aligned}$$

Hence (3.2) holds. Next, we will prove (3.3). It will be divided into two cases:

(i) If $p = 1$, by $E|X_1|^p < \infty$ and Lebesgue dominated convergence theorem, we have

$$\lim_{n \rightarrow \infty} n^{1/p} P(|X_n| \geq n^{1/p}) = 0, \quad (3.5)$$

$$\begin{aligned} & \lim_{n \rightarrow \infty} EX_n I(|X_n| < n^{1/p}) \\ & = \lim_{n \rightarrow \infty} \int_{\Omega} X_1(\omega) I(|X_1(\omega)| < n^{1/p}) P(d\omega) \\ & = EX_1 = 0. \end{aligned} \quad (3.6)$$

Thus,

$$\begin{aligned} & |EY_n| \leq n^{1/p} P(|X_n| \geq n^{1/p}) \\ & + |EX_n I(|X_n| < n^{1/p})| \rightarrow 0, \text{ a.s. } n \rightarrow \infty. \end{aligned}$$

By the Toeplitz lemma, we obtain

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n EY_k = 0.$$

(ii) If $p \neq 1$, by the Kronecker's lemma, to prove (3.3), it suffices to show that

$$\sum_{n=1}^{\infty} \frac{|EY_n|}{n^{1/p}} < \infty. \quad (3.7)$$

For $0 < p < 1$,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|EY_n|}{n^{1/p}} & \leq \sum_{n=1}^{\infty} P(|X_1| \geq n^{1/p}) + \sum_{n=1}^{\infty} \frac{E|X_n| I(|X_n| < n^{1/p})}{n^{1/p}} \\ & \leq C + \sum_{n=1}^{\infty} \sum_{j=1}^n n^{-1/p} E|X_1| I(j-1 \leq |X_1|^p < j) \\ & = C + \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} n^{-1/p} E|X_1| I(j-1 \leq |X_1|^p < j) \\ & \leq C + C \sum_{j=1}^{\infty} j^{1-1/p} E|X_1|^p j^{(1-p)/p} I(j-1 \leq |X_1|^p < j) < \infty. \end{aligned}$$

For $1 \leq p < 2$, by $EX_n = 0$, we can see that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{|EY_n|}{n^{1/p}} & \leq \sum_{n=1}^{\infty} P(|X_1| \geq n^{1/p}) + \sum_{n=1}^{\infty} \frac{|EX_n I(|X_n| < n^{1/p})|}{n^{1/p}} \\ & \leq C + \sum_{n=1}^{\infty} n^{-1/p} E|X_n| I(|X_n| \geq n^{1/p}) \\ & = C + \sum_{n=1}^{\infty} \sum_{j=n}^{\infty} n^{-1/p} E|X_1| I(j \leq |X_1|^p < j+1) \\ & = C + \sum_{j=1}^{\infty} \sum_{n=1}^j n^{-1/p} E|X_1| I(j \leq |X_1|^p < j+1) \\ & \leq C + C \sum_{j=1}^{\infty} j^{1-1/p} E|X_1|^p j^{(1-p)/p} I(j \leq |X_1|^p < j+1) \\ & < \infty. \end{aligned}$$

Thus (3.7) holds, which implies (3.3) by Kronecker's lemma. We get the desired result.

4 Strong stability for weighted sums of NOD sequences

In this section, we will study the strong stability for weighted sums of NOD random variables. Firstly, we will give some definitions as follows:

Definition 4.1. A random variable sequence $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a constant C , such that

$$P(|X_n| > x) \leq CP(|X| > x) \quad (4.1)$$

for all $x \geq 0$ and $n \geq 1$.

Definition 4.2. A random variable sequence $\{Y_n, n \geq 1\}$ is said to be strongly stable if there exist two constant sequences $\{b_n, n \geq 1\}$ and $\{d_n, n \geq 1\}$ with $0 < b_n \uparrow \infty$, such that

$$b_n^{-1} Y_n - d_n \rightarrow 0 \text{ a.s.} \quad (4.2)$$

Lemma 4.1. Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X . For any $\alpha > 0$ and $b > 0$, the following statement holds:

$$E|X_n|^\alpha I(|X_n| \leq b) \leq C \{E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)\}.$$

Where C is a positive constant.

Theorem 4.1. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be two sequences of positive numbers with $c_1 = b_1/a_1$, $c_n = b_n/(a_n \log n)$ for $n \geq 2$ and $b_n \uparrow \infty$. Let $\{X_n, n \geq 1\}$ be a sequence of NOD random variables which is stochastically dominated by a random variable X . Define $N(x) = \text{Card}\{n : c_n \leq x\}$, $R(x) = \int_x^\infty N(y)y^{-3}dy$, $x > 0$.

If the following conditions are satisfied:

- (i) $N(x) < \infty$ for any $x > 0$,
- (ii) $R(1) = \int_1^\infty N(y)y^{-3}dy < \infty$,
- (iii) $EX^2R(|X|) < \infty$,

then there exist $d_n \in R$, $n = 1, 2, \dots$ such that

$$b_n^{-1} \sum_{i=1}^n a_i X_i - d_n \rightarrow 0 \text{ a.s.} \quad (4.3)$$

Proof. Denote

$X_k^{c_k} = -c_k I(X_k < -c_k) + X_k I(|X_k| \leq c_k) + c_k I(X_k > c_k), k \geq 1,$
 then $\{X_k^{c_k}, k \geq 1\}$ and $\{a_k X_k^{c_k}/b_k, k \geq 1\}$ are still NOD from Lemma 1.1.

Since $N(x)$ is nondecreasing, then for any $x > 0$
 $R(x) \geq N(x) \int_x^\infty y^{-3} dy = \frac{1}{2} x^{-2} N(x),$ (4.4)
 which implies that

$$EN(|X|) \leq 2EX^2R(|X|) < \infty.$$

Therefore

$$\begin{aligned} \sum_{i=1}^\infty P(X_i \neq X_i^{c_i}) &= \sum_{i=1}^\infty P(|X_i| > c_i) \\ &\leq C \sum_{i=1}^\infty P(|X| > c_i) \leq CEN(|X|) < \infty. \end{aligned} \quad (4.5)$$

By Borel-Cantelli lemma for any sequence $\{d_n, n \geq 1\} \subset R$, the sequences $\{b_n^{-1} \sum_{i=1}^n a_i X_i - d_n\}$ and $\{b_n^{-1} \sum_{i=1}^n a_i X_i^{c_i} - d_n\}$ converge to the same limit on the same set.

We will show that $b_n^{-1} \sum_{i=1}^n a_i (X_i^{c_i} - EX_i^{c_i}) \rightarrow 0$ a.s.,

which gives the theorem with $d_n = b_n^{-1} \sum_{i=1}^n a_i EX_i^{c_i}$.

It follows from Lemma 4.1 that

$$\begin{aligned} \sum_{n=1}^\infty \frac{\text{Var}(a_n X_n^{c_n})}{b_n^2} \log^2 n &\leq \sum_{n=1}^\infty c_n^{-2} E(X_n^{c_n})^2 \\ &\leq 3 \sum_{n=1}^\infty P(|X_n| > c_n) + 3 \sum_{n=1}^\infty c_n^{-2} EX_n^2 I(|X_n| \leq c_n) \\ &\leq C \sum_{n=1}^\infty P(|X| > c_n) + C \sum_{n=1}^\infty c_n^{-2} EX^2 I(|X| \leq c_n) \\ &\leq CEN(|X|) + C \sum_{n=1}^\infty c_n^{-2} EX^2 I(|X| \leq c_n). \end{aligned} \quad (4.6)$$

$$\begin{aligned} \sum_{n=1}^\infty c_n^{-2} EX^2 I(|X| \leq c_n) &= \sum_{n: c_n \leq 1} c_n^{-2} EX^2 I(|X| \leq c_n) \\ &+ \sum_{n: c_n > 1} c_n^{-2} EX^2 I(|X| \leq c_n) \doteq I_1 + I_2. \end{aligned} \quad (4.7)$$

Since $N(1) = \text{Card}\{n : c_n \leq 1\} \leq 2R(1) < \infty$ from (4.4) and condition (ii), then $I_1 < \infty$.

$$\begin{aligned} I_2 &= \sum_{n: c_n > 1} c_n^{-2} EX^2 I(|X| \leq c_n) \\ &= \sum_{k=2}^\infty \sum_{k-1 < c_n \leq k} c_n^{-2} EX^2 I(|X| \leq c_n) \\ &\leq \sum_{k=2}^\infty (N(k) - N(k-1)) (k-1)^{-2} EX^2 I(|X| \leq k) \\ &\leq \sum_{k=2}^\infty (N(k) - N(k-1)) (k-1)^{-2} EX^2 I(|X| \leq 1) \\ &+ \sum_{k=2}^\infty (N(k) - N(k-1)) (k-1)^{-2} EX^2 I(1 < |X| \leq k) \\ &\doteq I_{21} + I_{22}. \\ I_{21} &\leq C \sum_{k=2}^\infty (N(k) - N(k-1)) \sum_{j=k-1}^\infty j^{-3} \\ &= C \sum_{j=1}^\infty j^{-3} \sum_{k=2}^{j+1} (N(k) - N(k-1)) \\ &\leq C \sum_{j=1}^\infty (j+1)^{-3} N(j+1) \leq C \int_1^\infty y^{-3} N(y) dy < \infty. \end{aligned}$$

Since $N(x)$ is nondecreasing and $R(x)$ is nonincreasing, then

$$\begin{aligned} I_{22} &= \sum_{k=2}^\infty (N(k) - N(k-1)) (k-1)^{-2} EX^2 I(1 < |X| \leq k) \\ &= \sum_{k=2}^\infty (N(k) - N(k-1)) (k-1)^{-2} \sum_{m=2}^k EX^2 I(m-1 < |X| \leq m) \\ &= \sum_{m=2}^\infty EX^2 I(m-1 < |X| \leq m) \sum_{k=m}^\infty (N(k) - N(k-1)) (k-1)^{-2} \\ &\leq \sum_{m=2}^\infty EX^2 I(m-1 < |X| \leq m) \sum_{k=m}^\infty N(k) ((k-1)^{-2} - k^{-2}) \\ &\leq C \sum_{m=2}^\infty EX^2 I(m-1 < |X| \leq m) \sum_{k=m}^\infty \int_k^{k+1} N(x) x^{-3} dx \\ &= C \sum_{m=2}^\infty R(m) EX^2 I(m-1 < |X| \leq m) \\ &\leq C \sum_{m=2}^\infty EX^2 R(|X|) I(m-1 < |X| \leq m) \leq CEX^2 R(|X|) < \infty. \end{aligned}$$

Therefore

$$\sum_{n=1}^\infty \frac{\text{Var}(a_n X_n^{c_n})}{b_n^2} \log^2 n < \infty \quad (4.8)$$

following from the above statements. By Corollary 1.1 and Kronecker's Lemma, it follows that

$$b_n^{-1} \sum_{i=1}^n a_i (X_i^{c_i} - EX_i^{c_i}) \rightarrow 0 \text{ a.s.} \quad (4.9)$$

Taking

$d_n = b_n^{-1} \sum_{i=1}^n a_i EX_i^{c_i}, n \geq 1$, then $b_n^{-1} \sum_{i=1}^n a_i X_i^{c_i} - d_n \rightarrow 0$ a.s.

We complete the proof of the theorem.

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